

ON THE BASE FIELD CHANGE OF P -RINGS AND P -2 RINGS

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One finds the following example in [3, (34, B)]:

Let k be a field of characteristic p and $\underline{X} = \{X_1, \dots, X_n\}$ be n -variables over k . Then if $p > 0$ and $[k : k^p] = \infty$, $k^p[[\underline{X}]]/k$ is an n -dimensional regular local ring but not a Nagata ring. In particular it is not an excellent ring.

On the other hand, according to [1, Corollary 4.3], $k[[\underline{X}]]/l$ is an excellent ring if l is a separably algebraic field extension of k .

In Section 1 we study when a property such as being excellent ascends by a base field extension.

Conversely in Section 2 we study when such a property as in Section 1 descends by a base field reduction.

§ 1. Notation and definitions

We use the following notation and definitions, following [6]:

Let P be a property meaningful for a noetherian ring and satisfying the following four axioms:

- Axioms:
1. If A is regular, then A has P .
 2. P is a local property.
 3. If A is a complete local ring, then $P(A) = P$ -locus of A is Zariski open.
 4. Let $(A, \mathfrak{m}) \rightarrow (B, \mathfrak{n})$ be a faithfully flat local homomorphism. Then P descends from B to A ; if both A and $B/\mathfrak{m}B$ have P , then P ascends from A to B .

We say that a noetherian ring is a P -ring if its formal fibers are geometrically P . Then we have:

LEMMA 1. ([6, n. 3, Lemma]). *A noetherian local ring A is a P -ring iff, for any finite A -algebra B which is a domain, and for any prime ideal Q of \hat{B} with $Q \cap B = (0)$, the local ring \hat{B}_Q is P .*

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Moreover, imitating the proof of [3, Theorem 77], we have the following theorem (cf. [2, Theorem (7.4.4)]):

THEOREM 2. *Let A be a P -ring and B an A -algebra of finite type. Then B is a P -ring.*

We define P - i ($i = 0, 1, 2$) as follows:

DEFINITION 1. A ring A is P -0 iff $P(A)$ contains a non-empty open set.

2. A is P -1 iff $P(A)$ is an open set (maybe empty).
3. A is P -2 iff every A -algebra of finite type is P -1.

Next we define NC :

DEFINITION. A property P satisfying four axioms above has NC iff the following property holds for P ; a noetherian ring A is P -1 if, for every $\mathfrak{p} \in \text{Spec}(A)$, A/\mathfrak{p} is P -0.

Remark. If $P =$ regular, CI , Gorenstein or CM , then P has the four axioms and NC . ([6, n.2, Remark 1])

Then we have the following proposition:

PROPOSITION 3 ([6, n.2, Proposition 1]). *Let P satisfy NC . Then, for a noetherian ring A , the following are equivalent:*

1. A is P -2;
2. any finite A -algebra is P -1;
3. for any $\mathfrak{p} \in \text{Spec}(A)$, and for any finite radical extension L of the fraction field of A/\mathfrak{p} there is a finite A -algebra B , containing A/\mathfrak{p} and having L as fraction field, such that B is P -0.

Besides the above, we freely use the notation and the definitions of [3].

§ 2. Base field extension

In this section we fix the notation as follows: A , k and l are noetherian rings, and A and l are k -algebras. We assume that there is a multiplicatively closed subset S of $A \otimes_k l$ such that $S^{-1}(A \otimes_k l)$ is a noetherian ring.

Our first result is about P -rings:

PROPOSITION 4. *If A is a P -ring, and k and l are fields such that l is separably generated over k , then $S^{-1}(A \otimes_k l)$ is a P -ring.*

Proof. Let B be an essentially finite $S^{-1}(A \otimes_k l)$ -algebra which is a local domain. By Lemma 1, we have only to show that, for any $Q \in \text{Spec}(\hat{B})$ such that $Q \cap B = (0)$, the local ring \hat{B}_Q is P .

We can put $B = (S^{-1}(A \otimes_k l)[\underline{X}])_M/\mathfrak{p}$, where $\underline{X} = \{X_1, \dots, X_n\}$ are n -variables over $S^{-1}(A \otimes_k l)$ and $M \in \text{Spec}(S^{-1}(A \otimes_k l)[\underline{X}])$, $\mathfrak{p} \in \text{Spec}((S^{-1}(A \otimes_k l)[\underline{X}])_M)$. Since \mathfrak{p} is finitely generated, there is an intermediate field K between k and l which satisfies the following three conditions:

- (a) K is finitely generated over k ;
- (b) l is separably generated over K ;
- (c) if we put $\mathfrak{m} = M \cap (A \otimes_k K)[\underline{X}]$, $((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}$ contains a certain system of generators of \mathfrak{p} .

Put $\mathfrak{q} = \mathfrak{p} \cap ((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}$ and $C = ((A \otimes_k K)[\underline{X}])_{\mathfrak{m}}/\mathfrak{q}$, then we have the following commutative diagram:

$$\begin{array}{ccc} \hat{C} & \xrightarrow{\psi} & \hat{B} \\ \uparrow & & \uparrow \\ C & \xrightarrow{\varphi} & B \end{array}$$

where ψ is induced by φ . By (b) and (c), φ is formally smooth, hence so is ψ . Since \hat{C} is excellent, ψ is regular by André's Theorem. Put $Q \cap \hat{C} = P$, then $P \cap C = (Q \cap B) \cap C = (0)$. Now C is an essentially finite type over A , therefore C is a P -ring by Theorem 2. Therefore \hat{C}_P is P . Since ψ is regular, \hat{B}_Q is P . Q.E.D.

Now if P = regular, we have a stronger result as follows:

PROPOSITION 5. *Let A be a G -ring, and k and l be noetherian rings such that l is smooth over k . Then $S^{-1}(A \otimes_k l)$ is a G -ring.*

This proposition is a special case of the following theorem:

THEOREM ([7, Theorem 17] or [1]). *Let $u : A \rightarrow B$ and $v : B \rightarrow C$ be formally smooth homomorphisms of noetherian local rings. Suppose that A is a G -ring and $\Omega_{B/A} \otimes_B (C/Q)$ is a separated C/Q -module for any $Q \in \text{Spec}(C)$. Then v is regular.*

In fact, let $P \in \text{Spec}(S^{-1}(A \otimes_k l))$ and $P \cap A = \mathfrak{p}$. We put $A = A_{\mathfrak{p}}$, $B = S^{-1}(A \otimes_k l)_P$ and $C = \hat{B}$. Then, by our assumption, A is a G -ring, and $A \rightarrow B$ and $B \rightarrow C$ are formally smooth homomorphisms. Moreover $A \rightarrow B$ is smooth, so $\Omega_{B/A}$ is a projective B -module. Therefore $\Omega_{B/A} \otimes_B (C/Q)$ is a separated C/Q -module for any $Q \in \text{Spec}(C)$. Thus the assumption of the

above theorem is fulfilled. So $B \rightarrow C$ is regular, and $S^{-1}(A \otimes_k l)$ is a G -ring.

Remark. By Proposition 5, we have [1, Corollary 4.3]. We will generalize this result later.

Next proposition is about P -2:

PROPOSITION 6. *Let P have NC. Then if A is P -2 and k and l are fields such that l is separably generated over k , $S^{-1}(A \otimes_k l)$ is P -2.*

Proof. Let B be a separating basis of l over k and $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ be a finite extension domain over $S^{-1}(A \otimes_k l)$, where $\underline{X} = \{X_1, \dots, X_n\}$ are n -variables over $S^{-1}(A \otimes_k l)$ and $\mathfrak{p} \in \text{Spec}(S^{-1}(A \otimes_k l)[\underline{X}])$. By Proposition 3, it is enough to prove that $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ is P -0.

Since \mathfrak{p} is finitely generated, there is an intermediate field K between k and l which satisfies the following three conditions:

- (a) K is finitely generated over k ;
- (b) l is separably generated over K ;
- (c) $(A \otimes_k K)[\underline{X}]$ contains a system of generators of \mathfrak{p} .

Then $S^{-1}(A \otimes_k l)[\underline{X}]$ is smooth over $(A \otimes_k K)[\underline{X}]$. Put $\mathfrak{q} = \mathfrak{p} \cap (A \otimes_k K)[\underline{X}]$. Then from the definition of the field K , $\mathfrak{p} = \mathfrak{q}(S^{-1}(A \otimes_k l))[\underline{X}]$. So $S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p}$ is smooth over $(A \otimes_k K)[\underline{X}]/\mathfrak{q}$. Now since A is P -2 and K is finitely generated over k , $(A \otimes_k K)[\underline{X}]/\mathfrak{q}$ is P -2. Therefore $U = P((A \otimes_k K)[\underline{X}]/\mathfrak{q})$ is a non-empty open set in $\text{Spec}((A \otimes_k K)[\underline{X}]/\mathfrak{q})$. So the inverse image of U in $\text{Spec}(S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p})$ is a non-empty open set contained in $P(S^{-1}(A \otimes_k l)[\underline{X}]/\mathfrak{p})$. Thus $S^{-1}(A \otimes_k l)$ is P -2. Q.E.D.

Remark. We have not yet succeeded in proving the proposition under the weaker assumption that l is separable over k .

PROPOSITION 7. *If A is a universally catenary noetherian ring and l and k are fields, then $S^{-1}(A \otimes_k l)$ is universally catenary.*

Proof. Similar to the proof of [2, Lemma (18.7.5.1)].

Summing up, if we assume that $P = \text{regular}$ in particular, we get the following theorem:

THEOREM 8. *If A is a (quasi-)excellent ring and k and l are fields such that l is separably generated over k , then $S^{-1}(A \otimes_k l)$ is a (resp. quasi-) excellent ring.*

Next, we make examples which satisfy the assumptions of the above Theorem.

LEMMA 9. *Let A be a noetherian ring, k be a field contained in A , and l be a (not necessarily finite) algebraic field extension of k . Then if $A \otimes_k l$ is noetherian, $A[[\underline{X}]] \otimes_k l$ is noetherian where $\underline{X} = \{X_1, \dots, X_n\}$ is n -variables over A .*

Proof. Similar to that of [4, (E3.1)].

PROPOSITION 10. *Let A be a noetherian ring, k be a field contained in A , and l be a field extension of k satisfying $\text{tr. deg}_k l < \infty$. Let B be a transcendence base of l over k . Then if $A \otimes_k l$ is noetherian, so is $S^{-1}(A[[\underline{X}]] \otimes_k l)$, where $\underline{X} = \{X_1, \dots, X_n\}$ is n -variables and $S = 1 + \underline{X}(A[[\underline{X}]] \otimes_k k(B))$.*

Proof. Since $S^{-1}(A[[\underline{X}]] \otimes_k k(B))$ is a Zariski ring in the (\underline{X}) -adic topology by the definition of S , and $(A \otimes_k k(B))[[\underline{X}]]$ is the (\underline{X}) -adic completion of $S^{-1}(A[[\underline{X}]] \otimes_k k(B))$, the ring $(A \otimes_k k(B))[[\underline{X}]]$ is faithfully flat over $S^{-1}(A[[\underline{X}]] \otimes_k k(B))$. Therefore $(A \otimes_k k(B))[[\underline{X}]] \otimes_{k(B)} l$ is faithfully flat over $S^{-1}(A[[\underline{X}]] \otimes_k k(B)) \otimes_{k(B)} l \cong S^{-1}(A[[\underline{X}]] \otimes_k l)$.

Now since $A \otimes_k l$ is noetherian and l is algebraic over $k(B)$, by Lemma 9, $(A \otimes_k k(B))[[\underline{X}]] \otimes_{k(B)} l$ is noetherian. Therefore $S^{-1}(A[[\underline{X}]] \otimes_k l)$ is noetherian. Q.E.D.

Remark. If $B = \phi$, each element of S is unit in $A[[\underline{X}]] \otimes_k l$. So $S^{-1}(A[[\underline{X}]] \otimes_k l) = A[[\underline{X}]] \otimes_k l$. This is the case of Lemma 9.

COROLLARY 11. *Let k be a field and l be a field separably generated over k such that $\text{tr. deg}_k l < \infty$. Denote a transcendence base of l over k by B . Put $S = 1 + \underline{Y}(k[\underline{X}][[\underline{Y}]] \otimes_k k(B))$ where $\underline{X} = \{X_1, \dots, X_m\}$ ($\underline{Y} = \{Y_1, \dots, Y_n\}$) are m -(resp. n -)variables over k . Then $S^{-1}(k[\underline{X}][[\underline{Y}]] \otimes_k l)$ is an excellent ring.*

Proof. By Proposition 6, $S^{-1}(k[\underline{X}][[\underline{Y}]] \otimes_k l)$ is noetherian. By [5], $k[\underline{X}][[\underline{Y}]]$ is an excellent ring. Thus by Theorem 8, $S^{-1}(k[\underline{X}][[\underline{Y}]] \otimes_k l)$ is an excellent ring. Q.E.D.

Remark. This is the generalization of [1, Corollary 4.3].

§ 3. Base field reduction

In this section we consider a base field reduction.

PROPOSITION 12. *Let A be a noetherian ring containing a field k , and l be a field separable over k . We assume that \mathcal{P} satisfies the four axioms (but not NC). Then if $A \otimes_k l$ is a \mathcal{P} -ring, so is A .*

Proof. Let \mathfrak{m} be a maximal ideal of A . Since $A \otimes_k l$ is faithfully flat over A , there is a prime ideal M of $A \otimes_k l$ such that $M \cap A = \mathfrak{m}$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
 \widehat{A}_{\mathfrak{m}} & \xrightarrow{\psi} & (\widehat{A \otimes_k l})_M \\
 \varphi \uparrow & & \uparrow v \\
 A_{\mathfrak{m}} & \xrightarrow{u} & (A \otimes_k l)_M
 \end{array}$$

where u is the natural map and v, φ and ψ are induced by completions. Then since l is separable over k , u is regular. By the assumption, v is a \mathcal{P} -homomorphism. Therefore $v \circ u = \psi \circ \varphi$ is a \mathcal{P} -homomorphism. Now ψ is faithfully flat. Thus φ is a \mathcal{P} -homomorphism by [6, n.1, Remark 3].

Q.E.D.

PROPOSITION 13. *Let A be a noetherian ring containing a field k and l be a field extension of k . We assume that \mathcal{P} satisfies NC. Then if $A \otimes_k l$ is \mathcal{P} -2, so is A .*

Proof. By Proposition 3, we assume that A is a domain, and we have only to show that $\mathcal{P}(A)$ contains a non-empty open set.

Let \mathfrak{p} be an element of $\text{Ass}(A \otimes_k l)$. Then $\mathfrak{p} \cap A = (0)$. Now $A \otimes_k l/\mathfrak{p}$ is \mathcal{P} -2 by the assumption, so there is an element $t \in A \otimes_k l$ such that $\mathcal{P}(A \otimes_k l/\mathfrak{p}) \supseteq D(t) \not\subseteq \phi$. Since \mathfrak{p} is finitely generated, there is an intermediate field K between k and l such that K is finitely generated over k and $A \otimes_k K$ contains the element t and a system of generators of \mathfrak{p} . Put $\mathfrak{q} = \mathfrak{p} \cap (A \otimes_k K)$. Then $A \otimes_k K/\mathfrak{q} \rightarrow A \otimes_k l/\mathfrak{p}$ is faithfully flat, therefore $\mathcal{P}(A \otimes_k K/\mathfrak{q}) \supseteq D(t) \not\subseteq \phi$.

Now we denote a transcendence base of K over k by $\underline{X} = \{X_1, \dots, X_n\}$. Since A is a domain, so is $A \otimes_k k(\underline{X})$. Hence $\mathfrak{q} \cap A \otimes_k k(\underline{X}) = (0)$, and $A \otimes_k k(\underline{X}) \rightarrow A \otimes_k K/\mathfrak{q}$ is a finite injective homomorphism. Therefore, by the generic flatness and the fact that a flat morphism of finite type is an open map, there is an element $s \in A \otimes_k k[\underline{X}]$ such that $\mathcal{P}(A \otimes_k k(\underline{X})) \supseteq D(s) \not\subseteq \phi$. We put $B = A \otimes_k k[\underline{X}] \cong A[\underline{X}]$ and $C = A \otimes_k k(\underline{X})$.

Now for any $P \in D(s) \subseteq \text{Spec}(B)$, put $Q = P \cap A$. Then $QC \in \text{Spec}(C)$. Since $QC \cap B = QB \subseteq P$, the prime ideal $QC \in D(s) \subseteq \text{Spec}(C)$. So C_{QC} is

P . Therefore A_Q is P , because $A \rightarrow C$ is flat and $QC \cap A = Q$. Since $A_Q \rightarrow A_Q[\underline{X}]$ is regular, $A_Q[\underline{X}]$ is P . So its localization $B_P \cong (A_Q[\underline{X}])_{P_A Q[\underline{X}]}$ is also P . Thus we have $P(B) \supseteq D(s) \neq \emptyset$. Now $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is an open map, therefore $P(A)$ contains a non-empty open set. This is what we want. Q.E.D.

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