SOME GENERALIZED HARDY SPACES

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Introduction and notation. This paper is concerned with generalizations of the classical Hardy spaces (**8**, p. 39) and the question of boundary values for functions of these various spaces. The general setting is the "big disk" Δ discussed by Arens and Singer in (**1**, **2**) and by Hoffman in (**7**). Analytic functions are defined in (**1**). Classes of such functions corresponding to the Hardy H^p spaces are considered and shown to possess boundary values in (**2**). Contrary to the classical case, such functions do not form a Banach space; hence they are not the functional analytic analogue of the classical spaces. In (**3**) quasi-analytic functions are defined while in (**4**) Hardy spaces of such functions are considered and are shown to have boundary values and to form a Banach space.

The space Δ is defined as follows: Let G be a compact abelian group (all groups are written additively with 0 as a neutral element) whose dual group Γ is isomorphic to a discrete subgroup of **R** (the real numbers) under the isomorphism ψ. Let $\Gamma_+ = \psi^{-1}([0, \infty))$. Γ_+ orders Γ with an archimedian order in the obvious manner. Δ is the space of homomorphisms of the semigroup Γ_+ into the multiplicative semigroup of the closed unit disk in the complex plane. Δ is given the topology of uniform convergence on compact subsets of Γ_+.

The group G is naturally embedded as a closed subset of Δ. This permits an identification of the non-vanishing homomorphisms of Δ with the topological product $(0, 1] \times G$. The unique singular element ω vanishes on all non-zero members of Γ_+. The value of the homomorphism $\zeta = (r, x) \neq \omega$ at σ in Γ_+ is $r^{\psi(\sigma)}\langle x, \sigma\rangle$, where $\langle \cdot, \cdot \rangle$ is the pairing between the dual groups G and Γ. The isomorphism ψ: $\Gamma \to \mathbf{R}$ induces a dual mapping φ: $\mathbf{R} \to G$ with $\langle \varphi(t), \sigma\rangle = e^{-i\psi(\sigma)t}$ for all t in **R** and σ in Γ. The one-parameter subgroup $\varphi(\mathbf{R})$ is dense in G and, corresponding to each coset $x + \varphi(\mathbf{R})$, there is embedded in Δ an image Π_x of the right half-plane $\mathbf{C}_+$ defined by

$$\tau_x\colon \mathbf{C}_+ \to \Delta, \quad \tau_x(s + it) = (e^{-s}, x + \varphi(t)), \quad s \geqslant 0, \quad -\infty < t < \infty.$$

The Cauchy measures μ_r, $0 < r < 1$, are the regular Borel measures on G induced by φ and the Cauchy densities

$$C(s, t) = \frac{1}{\pi}\frac{s}{s^2 + t^2} \qquad (s = -\log r, -\infty < t < \infty).$$

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The Fourier–Stieltjes transform of μ_r at σ in Γ is $\tilde{\mu}_r(\sigma) = r^{|\psi(\sigma)|}$. We define μ_0 to be the Haar measure m of G and μ_1 to be the measure with unit mass on the identity of G. The various L^p-spaces of G and m, with norm $||\cdot||_p$, will be denoted by $L^p(G)$.

1. Harmonic and analytic functions. For each σ in Γ_+ the *monimial* Z^σ is defined on Δ by $Z^\sigma(\zeta) = \zeta(\sigma)$ $(\zeta \in \Delta)$. An *analytic polynomial* is a finite linear combination of monomials and a *harmonic polynomial* is the sum of an analytic polynomial and the conjugate of an analytic polynomial. A function defined on $\Delta^0 = \Delta \diagdown G$, the *interior* of Δ, is said to be *analytic* (to be *harmonic*) if it can be uniformly approximated on each compact subset of Δ^0 by analytic (by harmonic) polynomials. If F is harmonic on Δ^0, then it is also continuous and the functions F_r $(0 < r < 1)$, defined on G by $F_r(x) = F(r, x)$, are in $C(G)$, (the continuous functions on G). Conversely, if f is in $C(G)$, then the function F defined on Δ by

$$(1.1) \qquad F(r, x) = \int_G f(x + y)d\mu_r(y) = f * \mu_r(x) \qquad (0 \leqslant r \leqslant 1,\ x \in G)$$

is continuous on Δ and harmonic on the interior **(7**, Theorem 4.1**)**. These results applied to the monomial Z^σ yield $(Z^\sigma)_r * \mu_s = (Z^\sigma)_{rs}$ for $0 < r, s < 1$; hence, for harmonic F,

$$(1.2) \qquad F_r * \mu_s = F_{rs} \qquad (0 < r,\ s < 1).$$

If $1 \leqslant p \leqslant \infty$, then (1.2) and the Holder inequality imply that for harmonic F and $0 < r < s < 1$

$$(1.3) \qquad ||F_r||_p = ||F_s * \mu_{r/s}||_p \leqslant ||F_s||_p.$$

A function f in $L^p(G)$ is said to be of *analytic type* if its Fourier transform, $\tilde{f}$, vanishes off Γ_+.

2. Hardy spaces on Δ. We seek to construct on Δ the analogue of the classical Hardy spaces. Hoffman **(7)** has considered for $0 < p$ the class of functions F which are analytic on Δ^0 and, for some r $(0 < r < 1)$, satisfy

$$\int_G |F(s, x + y)|^p \, d\mu_r(y) \leqslant M(r) \qquad (0 < s < 1,\ x \in G)$$

for some constant $M(r)$. For such functions the limit, as $s \to 1$, of $F(s, x)$ exists except on a set having μ_r-measure zero for each r $(0 \leqslant r < 1)$ and the limit function is in the L^p space associated with each such measure **(7**, Theorem 5.11**)**. At this point the analogy with the classical case fails as there exist functions of analytic type in $L^p(G)$ which do not occur as boundary functions in this sense.

A more direct translation of the defining conditions for the Hardy spaces yields the following result.

(2.1) Theorem. *Let F be analytic on Δ^0 and satisfy, for some p $(1 \leqslant p < \infty)$,*

$$(2.1.1) \qquad \int_G |F(r, x)|^p \, dm(x) \leqslant M < \infty \qquad (0 < r < 1).$$

Then there exists a p-integrable function h of analytic type defined on G such that

(1) $F_r = h * \mu_r$ *m-almost everywhere for each r,* $0 < r < 1$,

(2) $\lim_{r\to 1} F(r, x) = h(x)$ *m-almost everywhere,*

(3) $\lim_{r\to 1} ||F_r - h||_p = 0$.

If F is bounded on Δ^0, *then* (1) *and* (2) *hold for some bounded measurable h of analytic type and* F_r *converges to h in the weak-star topology of* $L^\infty(G)$ *as r converges to one.*

Proof. For $1 < p < \infty$ or for bounded F the proof of (1) is parallel to that of **(8)** in the classical case. If $p = 1$, then we again follow the weak-star compactness argument of **(8)** to find a finite regular Borel measure ν on G which satisfies $\tilde{F}_r = (\nu * \mu_r)^\sim$, $0 < r < 1$. Thus ν is a φ-analytic measure in the sense of **(5)** and hence translates continuously in the direction of φ **(5**, pp. 179–186**)**. This, with the absolute continuity (with respect to m) of $\nu * \mu_r$, is sufficient to prove that ν is absolutely continuous. This defines h for the case $p = 1$. Equality of the Fourier transforms for each r implies (1).

To prove (2) we note that, except for a set of Haar measure zero, equality in (1) holds on a set independent of r and invariant under translation by $\varphi(\mathbf{R})$, i.e., equality holds on the image Π_x of the right half-plane for almost every x in G. The conclusion (2) then follows from known properties of the Poisson extension of $f(t) = h(x + \varphi(t))$ $(-\infty < t < \infty)$ to the right half-plane **(8**, p. 123**)** and from the easily established result that those x for which the limit of (2) does not exist form a subset of G which is null in the direction of φ **(5**, p. 181**)**.

The proof of (3) (or the weak-star convergence in the case of bounded F) follows directly from the properties of the approximate identity $\{\mu_r: 0 < r < 1\}$ **(7**, Theorem 4.4**)**. This completes the proof of (2.1).

For fixed p, $1 \leqslant p < \infty$, the analytic functions which satisfy (2.1.1) form a normed linear space with the norm given by

$$\text{(2.1.2)} \qquad \sup_r \left[\int_G |F_r|^p \, dm \right]^{1/p} = \lim_{r\to 1} \left[\int_G |F_r|^p \, dm \right]^{1/p}.$$

Completeness of this space is equivalent to such functions being in 1–1 correspondence with functions of analytic type in $L^p(G)$ through their boundary values. Since F_r is a continuous function of analytic type, we find, using the results of **(6**, Theorem **7)**, the inequality of Malliavin **(6**, (49)**)**, and the Fubini theorem, that if h is the boundary value function of (2.1), $\log |h|$ is Haar-integrable. There exist bounded measurable functions h of analytic type for which $\log |h|$ is not integrable, (if G is not the circle group) **(6)**. Thus the space of analytic functions considered above is not complete.

From a function theoretic point of view one cannot but consider the functions of (2.2) as the proper generalization of the classical H^p-spaces; nevertheless, since the corresponding normed linear space is not complete nor

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isometrically isomorphic to the subspace of $L^p(G)$ consisting of those functions of analytic type, this generalization is not satisfactory from the standpoint of functional analysis. (This is not a new observation as most functional analysts have taken the subspace of $L^p(G)$ as the proper analogue of the classical H^p-spaces; see (**9**, Chapter 8).

The question of defining a suitable space of the Hardy type consisting of functions on Δ will be considered in the following.

3. Quasi-harmonic and quasi-analytic functions. In view of the identification of the non-vanishing elements of Δ and the space $I \times G$ ($I = (0, 1]$), there exists on Δ a unique regular Borel measure Λ which extends the product Baire measure $\lambda \times m$ (λ is Lebesque measure on I) and satisfies $\Lambda(\{\omega\}) = 0$ (**3**, Theorem 4, p. 232). We shall now consider certain subspaces of $L^p(\Delta, \Lambda) = L^p(\Delta)$ $(1 \leqslant p \leqslant \infty)$.

There are many properties of harmonic functions subject to generalization; here we choose the property exemplified by (1.2). This concept must be made meaningful for the elements of $L^p(G)$ (which we continue to refer to as functions).

If F is in $C(\Delta)$ (the continuous functions on Δ) and $0 < r < 1$, then the functions $P_r F$ and $S_r F$ are well-defined on Δ by $P_r F(s, x) = F_s * \mu_r(x)$ and $S_r F(s, x) = F(rs, x)$. P_r and S_r, considered as linear operators defined on the dense subspace $C(\Delta)$ of $L^1(\Delta)$, have continuous extensions (with the same name) mapping $L^1(\Delta)$ into $L^1(\Delta)$. If F is harmonic on Δ^0, then $P_r F = S_r F$ for each r in $(0, 1)$. This motivates the following definition.

(3.1) *Definition.* A function F in $L^1(\Delta)$ is called quasi-harmonic if $P_r F = S_r F$ for each r, $0 < r < 1$, equality being considered in $L^1(\Delta)$.

The Fourier transform on $L^1(G)$ extends to $L^1(\Delta)$ in the following way.

If F is in $C(\Delta)$, the Fourier transform of F_r is, for fixed σ in Γ, a λ-integrable function on the interval $I = (0, 1]$. The mapping $F \to \hat{F}_r(\sigma)$ has a continuous extension $\mathfrak{F}_\sigma$, mapping $L^1(\Delta)$ into $L^1(I, \lambda) = L^1(I)$.

(3.2) Proposition. *Let F be in $L^1(\Delta)$ and suppose $\mathfrak{F}_\sigma F = 0$ for each $\sigma \in \Gamma$. Then $F = 0$.*

Proof. The algebra generated on Δ by the constant functions and functions of the form $\phi\langle \cdot, \sigma\rangle$, where σ is in Γ and ϕ is continuous on $[0, 1]$ vanishing at 0, are dense in $C(\Delta)$. Since $\int_\Delta \phi\langle\cdot, \sigma\rangle H\, d\Lambda = \int_I \phi \mathfrak{F}_\sigma(H) d\lambda$ for $H \in C(\Delta)$, continuity shows that $\int_\Delta hF\, d\Lambda = 0$ for each h in $C(\Delta)$. Thus $F = 0$ in $L^1(\Delta)$.

For a quasi-harmonic function F we find the following.

(3.3) Proposition. *Let F be quasi-harmonic. For each σ in Γ the function $\mathfrak{F}_\sigma F$ is continuous on I and*

$$\mathfrak{F}_\sigma F(r) = r^{|\psi(\sigma)|} a_\sigma = \tilde{\mu}_r(\sigma) a_\sigma \qquad (r \in I) \tag{3.3.1}$$

for some constant a_σ.

Proof. For σ in Γ and $0 < r < 1$, $\mathfrak{F}_\sigma(P_r F) = \mathfrak{F}_\sigma(S_r F)$. It is clear that $\mathfrak{F}_\sigma(S_r F) = (\mathfrak{F}_\sigma F) \circ \beta_r$, where β_r is the mapping $s \to sr$ of I into I. Calculation for continuous functions and a passage to the limit utilizing the continuity of $\mathfrak{F}_\sigma$ and P_r shows that $\mathfrak{F}_\sigma(P_r F) = \tilde{\mu}_r(\sigma)\mathfrak{F}_\sigma F$. If g is a representing function for $\mathfrak{F}_\sigma F$, then $g \circ \beta_r(s) = g(sr) = \tilde{\mu}_r(\sigma)g(s)$ for almost every s in I. Clearly g may be assumed continuous and

$$a_\sigma = \lim_{s\to 1} g(s).$$

As remarked earlier, if f is in $C(G)$, $F(r, x) = f * \mu_r(x)$ is harmonic, hence quasi-harmonic, and $||F||_1 \leqslant ||f||_1$. Since the quasi-harmonic functions form a closed linear subspace of $L^1(\Delta)$ we may extend the mapping $f \to F$ (defined above) to a continuous operator Ψ defined on $L^1(G)$ with values in the quasi-harmonic subspace of $L^1(\Delta)$. Calculations show that

$$\mathfrak{F}_\sigma(\Psi f)(r) = \tilde{\mu}_r(\sigma)\tilde{f}(\sigma) \qquad (f \in L^1(G),\ \sigma \in \Gamma). \tag{3.3.2}$$

If f is in $L^p(G)$ $(1 \leqslant p \leqslant \infty)$, Ψf is in $L^p(\Delta)$ and

$$||\Psi f||_p \leqslant ||f||_p. \tag{3.3.3}$$

A quasi-harmonic function F is called quasi-analytic if $\mathfrak{F}_\sigma F = 0$ for all σ not in Γ_+. Clearly continuous quasi-analytic functions are analytic on Δ^0 and all integrable functions analytic on Δ are quasi-analtyic. In particular the functions considered in (2.2) are quasi-analytic. The quasi-analytic functions also form a closed linear subspace of $L^1(\Delta)$.

4. The spaces $\mathbf{H}^p(\Delta)$. Those quasi-analytic functions which are also in $L^\infty(\Delta)$ are denoted by $\mathbf{H}^\infty$. $\mathbf{H}^\infty$ is a closed subspace of $L^\infty(\Delta)$.

To construct other Hardy spaces it is necessary to impose some growth requirement, similar to that of (2.1.1), on the quasi-analytic functions. This is accomplished by the continuous mappings N_p: $L^p(\Delta) \to L^1(I)$ $(1 \leqslant p < \infty)$ which are extensions by continuity of the mapping $F \to \int_G |F_r|^p\, dm$ defined for the continuous subspace of $L^p(\Delta)$.

(4.1) *Definition.* Let $1 \leqslant p < \infty$ and F be a quasi-analytic element of $L^p(\Delta)$. F is said to be of class $\mathbf{H}^p$ if $N_p(F)$ is in $L^\infty(I)$.

$\mathbf{H}^p$ $(1 \leqslant p < \infty)$ is a normed space under the norm $|||\cdot|||_p$ defined for F in $\mathbf{H}^p$ by $|||F|||_p = ||(N_p(F))^{1/p}||_\infty = \text{ess sup}\ (N_p(F))^{1/p}$.

(4.2) PROPOSITION. *Let f be of analytic type in $L^p(G)$ for some p, $1 \leqslant p \leqslant \infty$. Then Ψf (see* **(3)***) is in $\mathbf{H}^p$ and*

$$|||\Psi f|||_p \leqslant ||f||_p. \tag{4.2.1}$$

Proof. Let $\{f_n\colon n = 1, 2, \ldots\}$ be a sequence of continuous functions of analytic type with $f_n \to f$ in $L^p(G)$ and $||f_n||_p \leqslant ||f||_p$ for all n. Then $\Psi f_n \to \Psi f$ in $L^p(\Delta)$ and, for $1 \leqslant p < \infty$, $|\Psi f_n|^p \to |\Psi f|^p$ in $L^1(\Delta)$, which implies that

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$N_p(\Psi f_n) \to N_p(\Psi f)$ in $L^1(I)$. For the continuous functions f_n the Fubini theorem is applicable and yields $|||\Psi f_n|||_p \leqslant ||f_n||_p$; thus the functions $N_p(\Psi f_n)$ form a bounded sequence in $L^\infty(I)$. Hence $N_p(\Psi f)$ is in $L^\infty(I)$ and Ψf is in $\mathbf{H}^p$ if $1 \leqslant p < \infty$. The case $p = \infty$ is obvious.

From (4.2) it is seen that Ψ embeds the functions of analytic type in $L^p(G)$ into $\mathbf{H}^p$. The remaining portion of this section is devoted to showing that this embedding is an isometric isomorphism.

(4.3) PROPOSITION. *Let F be in $\mathbf{H}^p$, $1 \leqslant p < \infty$. Then there exists a function f in $L^p(G)$ such that $F = \Psi f$ and*

$$|||F|||_p = ||f||_p. \tag{4.3.1}$$

Proof. Consider first the case $1 < p \leqslant \infty$. Let λ_r be a λ-absolutely continuous measure with unit mass on the interval $[r, 1]$ $(0 < r < 1)$. The mapping $F \to A_r F = \int_I F(s, \cdot)\, d\lambda_r$ is well defined into $L^p(G)$ (consider for continuous functions and extend by continuity) and $||A_r F||_p \leqslant |||F|||_p$. From the weak-star compactness of bounded sets in $L^p(G)$ we find an f in $L^p(G)$, with $||f||_p \leqslant |||F|||_p$, which is a weak-star cluster point of the net $\{A_r F: 0 < r < 1\}$. Straightforward calculations utilizing the Fubini theorem and (3.3) lead to the uniqueness of f and the equalities $\mathfrak{F}_\sigma F = \mathfrak{F}_\sigma(\Psi f)$ for each σ in Γ. Hence $F = \Psi f$ by (3.2). The inequality $||f||_p \leqslant |||F|||_p$ combined with the inverse inequality from (4.2.1) finishes the proof in this instance.

If $p = 1$ we must again utilize the weak-star compactness of bounded sets in $M(G)$ (the space of finite regular Borel measures on G). Proceeding as above we find a measure ν in $M(G)$ with $||\nu|| \leqslant |||F|||_1$ and $\mathfrak{F}_\sigma F(r) = (\nu * \mu_r)^\sim(\sigma)$ for each σ in Γ. If we were certain that the measures $\nu * \mu_r$ were absolutely continuous with respect to the Haar measure, the proof would follows as in (2.2).

(4.3.1) LEMMA. *Let F be in $L^1(\Delta)$ and ν be in $M(G)$. Suppose, for each $\sigma \in \Gamma$, that $\mathfrak{F}_\sigma F(r) = (\nu * \mu_r)^\sim(\sigma)$ for λ-almost every r in the interval $(0, 1)$. Then $\nu * \mu_r$ is absolutely continuous with respect to the Haar measure of G for each r of the interval.*

Proof. Let E be a Borel subset of the open interval $(0, 1)$ and define the linear functions L and L' on $C(G)$ by

$$L(\phi) = \int_E \int_G \phi\, d(\nu * \mu_r)\, d\lambda(r), \qquad L'(\phi) = \int_{E \times G} \phi F\, d\Lambda \qquad (\phi \in C(G)).$$

(F is Λ-integrable and the function $r \to \nu * \mu_r$ into the weak-star topology of $M(G)$ is continuous so that L and L' are well-defined). There exist measures τ and τ' in $M(G)$ satisfying $L(\phi) = \int \phi\, d\tau$ and $L'(\phi) = \int \phi\, d\tau'$. For $\phi = \langle \cdot, \sigma \rangle$ $(\sigma \in \Gamma)$ we find that $\tilde{\tau}(\sigma) = \tilde{\tau}'(\sigma)$, by hypothesis; hence $\tau(S) = \tau'(S)$ for each Borel subset S of G. If S has Haar measure zero, then $\tau'(S) = 0$, which implies that $\int_E \nu * \mu_r(S)\, d\lambda(r) = 0$ for each Borel subset E of $(0, 1)$. Since

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the function $r \to \nu * \mu_r(S)$ is continuous, $\nu * \mu_r(S)$ vanishes identically for $0 < r < 1$. This proves the lemma and completes the proof of (4.3).

From (4.2) and (4.3) we find the following theorem.

(4.4) THEOREM. *For each p, $1 \leqslant p \leqslant \infty$, the space $\mathbf{H}^p$ with norm $|||\cdot|||_p$ is isometrically isomorphic to $H^p(G)$, the closed linear subspace of $L^p(G)$ consisting of those functions of analytic type.*

The space $H^p(G)$ has many special properties most of which are based upon certain factorization theorems. The following proposition shows that these properties are preserved under the mapping Ψ.

(4.5) PROPOSITION. *Let $1 \leqslant p \leqslant \infty$ and $f \in H^p(G)$.*
(1) *If $g \in H^q(G)$ (q the index dual to p), then $\Psi(f \cdot g) = \Psi f \cdot \Psi g$.*
(2) *If $\hat{f}(0) \neq 0$, then $\log |\Psi f| \leqslant \Psi(\log |f|)$,*
with equality if f is outer, see **(6**, p. 178; **7)**.

Proof. The proof of (1) is obvious, while the proof of (2) is essentially a rephrasing of Malliavin's inequality **(6**, (49)**)** and may be proved by using the methods of **(6)**. Alternatively one may show that for h, defined and integrable on G, $\mu_r * h(x)$ is Λ-measurable and a representative of Ψh (the proof of this statement is similar to that used in **(4)**) from which (2) follows as a consequence of Malliavin's theorem and **(6**, Theorem 4**)**.

Of the many special properties of $H^p(G)$ inherited by $\mathbf{H}^p$, we note the following: (1) the factorization of F in $\mathbf{H}^1$ into a product of (appropriately defined) inner and outer functions, (2) if F in $\mathbf{H}^1$ is an outer function, then $\log |F|$ is quasi-harmonic, and (3) the representations of invariant subspaces under the multiplication operators $M_\sigma F = Z^\sigma F$ (σ in Γ_+).

5. Non-Archimedian ordered groups. In this discussion we extend the results of the preceding to certain compact groups whose dual groups are ordered with a non-archimedian order. We shall consider here discrete totally ordered abelian groups Γ, with positive semigroups Γ_+ of two types: the first, Γ linearly ordered with Γ_+ the semigroup of elements not less than identity; the second, Γ homomorphic to a subgroup of the real numbers under the order-preserving homomorphism

$$\psi\colon \Gamma \to \mathbf{R}, \quad \Gamma_+ = \{\sigma \in \Gamma\colon \psi(\sigma) \geqslant 0\} \qquad (\psi \text{ not one–one}).$$

Investigation of the character semigroups Δ of Γ_+ under these conditions shows the non-vanishing characters are again homeomorphic to $(0, 1] \times G$ when given the topology of uniform convergence on compact subsets. (Note: For ordered groups of the second type there exist non-trivial characters if ψ is non-trivial. For Γ of the first type the existence of a non-trivial character is equivalent to the existence of an "archimedian" element, i.e., an element $\sigma \in \Gamma_+$ such that for each $\tau \in \Gamma_+$ there exists an integer n such that $n\sigma \geqslant \tau$.)

The results of **(2)** show that analytic functions are determined by their values on the non-singular elements of Δ. The results of **(2)** are valid in this context.

We may again define the measure Λ on Δ as in § 3. With this definition the subset of singular elements has zero Λ-measure. Examples show this to be expected since the set of singular elements may have dimension strictly less than the dimension of Δ (see **(1**, § 5**)**. Clearly, if Λ is so defined the results of **(4)** are valid here.

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