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ABELIAN IDEALS IN A COMPLEX SIMPLE LIE ALGEBRA

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Abstract. In this note, we give a new simple construction of all maximal abelian ideals in a Borel subalgebra of a complex simple Lie algebra. We also derive formulas for dimensions of certain maximal abelian ideals in terms of the theory of Borel de Siebenthal.

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1. Introduction. Let $\mathfrak{g}$ be a complex simple Lie algebra and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$ (i.e. a maximal solvable subalgebra of $\mathfrak{g}$). In this note, we give a new simple construction of all maximal abelian ideals in $\mathfrak{b}$, see Theorem 4.2. We also derive dimension formulas in terms of Borel de Siebenthal theory [**1**], see Corollary 3.4. There is already an extensive literature on this topic (see [**4**, **7**, **8**, **9**, **5**, **10**, **11**], and references therein), most of it concerning the affine Weyl group and coset enumeration. Our approach is based on root theoretic properties. The results obtained here comprised part of the author's Ph.D thesis and he would like to thank Dr J. Burns for his expert guidance.

2. Preliminaries. Throughout this note, $\mathfrak{g}$ will denote a finite dimensional complex simple Lie algebra, with fixed Cartan subalgebra $\mathfrak{h}$ and $\mathfrak{b}$ a fixed Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. All basic facts and definitions can be found in [**6**]. As usual, we have the root space decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Here, Φ is the set of roots, we partition this set into two sets of positive and negative roots denoted by Φ^+ and Φ^-. Let $\Pi := \{\alpha_1, \ldots, \alpha_r\}$ be a set of positive simple roots that span Φ, and r is the rank of $\mathfrak{g}$. We define $\mathfrak{h}^*_{\mathbb{R}}$ to be the real span of the roots. We define the height of a root α as $ht(\alpha) = \sum_{i=1}^r n_i^\alpha$, where $\alpha = \sum_{i=1}^r n_i^\alpha \alpha_i$. The fundamental weights $\{\omega_1, \ldots, \omega_r\}$ are defined by the condition that $(\omega_i, 2\alpha_j) = (\alpha_i, \alpha_j)\delta_{ij}$ for all i, j, where $(\cdot\,,\cdot)$ is an invariant inner product normalized so that the highest root, denoted by $\tilde{\alpha}$ has length (denoted $||\tilde{\alpha}||$) squared two. If $\lambda = \sum_{i=1}^r m_i^\lambda \omega_i$, where $m_i^\lambda \geq 0$ for all $1 \leq i \leq r$, then λ is said to be a dominant weight. There are at most two root lengths in Φ, called *long* and *short*. We denote the set of long and short roots by, Φ_l and Φ_s, respectively. If all roots in Φ have equal length we say Φ is simply laced and the root length is called *long*, otherwise we say Φ is non-simply laced. In the non-simply laced case, there exists a highest short root $\tilde{\beta}$, $\tilde{\beta}$ and $\tilde{\alpha}$ are the only positive roots that are also

dominant weights. We also note that $\tilde{\beta} = \alpha - \alpha_j$, where α is a highest long root such that $m_j^\alpha > 1$ and α_j a short simple root. Let $S_\alpha(\beta) := \beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ denote the reflection of a root β by a root α. We note that $S_\alpha(\Phi) = \Phi$ for any $\alpha \in \Phi$.
The Dynkin diagram of $\mathfrak{g}$ denoted $\Delta_\mathfrak{g}$, is the multi graph with r vertices, labelled by the simple roots and $c_{\alpha_i,\alpha_j}c_{\alpha_j,\alpha_i}$ edges connecting the α_i and α_j vertices, where c_{α_i,α_j} is the (i, j) entry in the Cartan matrix. The extended Dynkin diagram $\tilde{\Delta}_\mathfrak{g}$, is $\Delta_\mathfrak{g}$ with the extra node $\alpha_0 := -\tilde{\alpha}$, connected to any other vertex α_i by $c_{\alpha_i,-\tilde{\alpha}}c_{-\tilde{\alpha},\alpha_i}$ edges. We will denote by $\tilde{\Delta}_{\mathfrak{g}\setminus\{\alpha_k\}}$, the extended Dynkin Diagram $\tilde{\Delta}_\mathfrak{g}$, with the node α_k deleted. Similarly, $\Delta_{\mathfrak{g}\setminus\{\alpha_i\in I\}}$ denotes the Dynkin diagram $\Delta_\mathfrak{g}$ with the set of nodes $\{\alpha_i \in I\}$ for some set I deleted. Finally, let $|\tilde{\Delta}_{\mathfrak{g}\setminus\{\alpha_k\}}|$ and $|\Delta_{\mathfrak{g}\setminus\{\alpha_i\in I\}}|$ denote the number of positive roots in the root system with Dynkin diagram $\tilde{\Delta}_{\mathfrak{g}\setminus\{\alpha_k\}}$ and $\Delta_{\mathfrak{g}\setminus\{\alpha_i\in I\}}$, respectively.
Let $\mathfrak{a} \subset \mathfrak{b}$ be an abelian ideal of a Borel subalgebra $\mathfrak{b}$ in $\mathfrak{g}$. Since $\mathfrak{a}$ is an ideal, it is ad-$\mathfrak{h}$ stable and hence compatible with the root space decomposition. Since $\mathfrak{a}$ is abelian, it lies inside the nilpotent radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, hence $\mathfrak{a}$ is of the form $\mathfrak{a} = \bigoplus_{\psi\in\Psi} \mathfrak{g}_\psi$ for some subset $\Psi \subseteq \Phi^+$. Using the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$, we see:

(i) The ideal property for $\mathfrak{a}$ translates into the condition for Ψ that,

$$\Psi + \Phi^+ := \{\alpha + \beta | \alpha \in \Psi, \beta \in \Phi^+\} \cap \Phi^+ \subseteq \Psi.$$

(ii) The abelian condition becomes, $\Psi + \Psi := \{\alpha + \beta | \alpha, \beta \in \Psi\} \cap \Phi^+ = \emptyset$.

We now have a bijection between abelian ideals $\mathfrak{a} \subset \mathfrak{b}$ and subsets $\Psi \subset \Phi^+$ that satisfy (i) and (ii). We will require the following definition and lemmas. Following the notation used by Suter **[11]**,

DEFINITION 2.1. Following the notation used by Suter in **[11]**, given a set of roots I, we denote by $\langle I\rangle$ the sum $\sum_{\alpha\in I}\alpha$.

LEMMA 2.2 (**[2, 6]**). *Given two non-proportional roots α, β, then*

(*i*) *If $(\alpha, \alpha) \le (\beta, \beta)$, then $2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \{-1, 0, 1\}$.*
(*ii*) *If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.*
(*iii*) *If $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi$.*
(*iv*) *If $(\alpha, \beta) = 0$ and $\alpha + \beta \in \Phi$, then $\alpha - \beta \in \Phi$.*

LEMMA 2.3 (**[3]** – Theorem 1.1, **[12]**). *For a finite dimensional representation of $\mathfrak{g}$ with weights Λ,*

$$\sum_{\lambda\in\Lambda_{i_1,x_1}\cap\cdots\cap\Lambda_{i_p,x_p}} \lambda = \sum_{j=1}^{p} c_{i_j}\omega_{i_j} \qquad x, c_{i_j} \in \mathbb{R},$$

where $\Lambda_{i,x} := \{\lambda \in \Lambda | (\lambda, \omega_i) = x\}$ and $i \in \{1, \ldots, r\}$.

The most obvious way to obtain a maximal abelian ideal is given by the following theorem (various proofs can be found in **[4, 9, 11, 8]**).

THEOREM 2.4. *Given a root system of a Lie algebra such that $n_i^{\tilde{\alpha}} = 1$ for some i, let $\Psi := \{\alpha \in \Phi^+ | n_i^\alpha = 1\}$. Then, $\mathfrak{a}_\Psi := \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ is a maximal abelian ideal of dimension $g||\omega_i||$, where g is the dual Coxeter number.*

3. Maximal abelian ideals. In [9], Theorem 4.2 (page 242) yielded a maximal abelian ideal for every index i such that $n_i^{\tilde{\alpha}} = 2$ (where $n_i^{\tilde{\alpha}}$ denotes the coefficient of α_i in the expression $\tilde{\alpha} = \sum_{i=1}^{r} n_i^{\alpha}\alpha_i$). In this section, we prove a generalized version of this theorem, although we do not strictly need this result, it is of independent interest and also yields a dimension formula.

Definition 3.1. Let $\eta = \sum_{p\in P} c_p\omega_p + \sum_{q\in Q} d_q\omega_q$ be a weight with $c_p > 0$ for all $p \in P \subset \{1, \dots, r\}$ and $d_q < 0$ for all $q \in Q \subset \{1, \dots, r\}$. The η−height of a root α is denoted by $ht_\eta(\alpha) := \sum_{p\in P} n_p^{\alpha}$, and we call the elements of P the grading positions associated to η.

Lemma 3.2. *Let $\alpha_k \in \Pi$ be long and $n_k^{\tilde{\alpha}}$ be even. Let $\gamma \in \Phi^+ \setminus \{\tilde{\alpha}\}$ such that $n_k^{\gamma} = c$ (a positive integer) with $c \geq \frac{n_k^{\tilde{\alpha}}}{2}$ and $ht(\gamma) \geq ht(\alpha)$ for all α such that $n_k^{\alpha} = c$. Then, γ is long and $(\gamma, \tilde{\alpha}) = 1$.*

Proof. We first prove that $m_i^{\gamma} \geq 0$ for all $i \neq k$. If $m_i^{\gamma} < 0$ for any $i \neq k$, we have $(\gamma, \alpha_i) < 0$ thus $\gamma + \alpha_i \in \Phi^+$ with $n_k^{\gamma+\alpha_i} = c$, and $ht(\gamma + \alpha_i) > ht(\gamma)$, which is a contradiction.

Suppose γ is short. If $m_k^{\gamma} < 0$, then $S_\gamma(\alpha_k) = \alpha_k - \frac{2(\gamma,\alpha_k)}{(\gamma,\gamma)}\gamma = \alpha_k + 2\gamma \in \Phi^+$ (or $\alpha_k + 3\gamma$ in the case of G_2) but $n_k^{\alpha_k+2\gamma} > n_k^{\tilde{\alpha}}$, a contradiction. Therefore, $m_k^{\gamma} \geq 0$ forcing $\gamma = \tilde{\beta}$ (since we already know that $m_i^{\gamma} \geq 0$ for any $i \neq k$), and there exists a long root α such that $\alpha - \alpha_i = \gamma$. If $i \neq k$, then γ is not a highest root such that $n_k^{\gamma} = c$, thus $i = k$. Observe that

$$\begin{aligned}(\gamma, \gamma) &= (\alpha - \alpha_k, \alpha - \alpha_k)\\ &= (\alpha, \alpha) + (\alpha_k, \alpha_k) - 2(\alpha, \alpha_k).\end{aligned}$$

Now, since α and α_k are long roots, $(\alpha, \alpha_k) \in \mathbb{Z}$ and hence (γ, γ) is an even integer, which is impossible if γ is a short root. So γ is long.
Since both $\tilde{\alpha}$ and γ are long, we will show that $(\gamma, \tilde{\alpha}) > 0$ and use Lemma 2.2 to conclude that $(\gamma, \tilde{\alpha}) = 1$. We know that $n_k^{\gamma} > 0$. If $n_i^{\gamma} = 0$ for some index $i \neq k$, then $(\gamma, \alpha_i) < 0$, hence $\gamma + \alpha_i \in \Phi^+$, contradicting the definition of γ. Thus, $n_i^{\gamma} > 0$ for any index i. Finally, $m_i^{\tilde{\alpha}} > 0$ for some index i, hence $(\gamma, \tilde{\alpha}) > 0$. □

The following observation will prove useful. Let γ and k be as in Lemma 3.2 and $\alpha \in \Phi^+$, then $(\gamma, \alpha_k) = -1$ and $(\gamma, \alpha_i) = 1$ if and only if $m_i^{\gamma} > 0$ (by Lemma 2.2). Then,

$$(\gamma, \alpha) = \left(\gamma, \sum_{j\neq k}^{r} n_j^{\alpha}\alpha_j\right) - n_k^{\alpha} = \sum_{j\neq k}^{r} n_j^{\alpha}(\gamma, \alpha_j) - n_k^{\alpha} = \sum_{\{j|m_j^{\gamma}>0\}} n_j^{\alpha} - n_k^{\alpha} = ht_\gamma(\alpha) - n_k^{\alpha}. \quad (1)$$

Theorem 3.3. *Let α_k be a long simple root such that $n_k^{\tilde{\alpha}} = 2n$, where $n \in \mathbb{Z}_{>0}$. Let γ be a root such that $n_k^{\gamma} = n$ and $ht(\gamma) \geq ht(\alpha)$ for all α such that $n_k^{\alpha} = n$, and let P be the set of grading positions associated to γ. Let*

$$\Lambda := \{\alpha \in \Phi^+ | n_k^{\alpha} \geq n + 1\} \cup \{\gamma\} \cup \{\gamma - \alpha | (\gamma, \alpha) > 0 \text{ and } n_k^{\alpha} = 0\}.$$

Then, $\mathfrak{a}_\Lambda$ is a maximal abelian ideal. Moreover, for $n \geq 2$,
$|P| \leq 2$ and $\langle\{\alpha \in \Lambda | ht_\gamma(\alpha) = ht_\gamma(\tilde{\alpha})\}\rangle = k_p\omega_p + k_q\omega_q$ with $k_p > 0$ and $k_q \geq 0$, where α_p is the node closest to $-\tilde{\alpha}$ in the extended Dynkin diagram.

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Proof. We first show that $\Lambda \subseteq \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq n+1\}$. Using Lemma 3.2 and equation (1) above, $(\gamma, \gamma) = 2 = ht_\gamma(\gamma) - n$, hence $ht_\gamma(\gamma) = n+2$. Also $(\gamma, \tilde{\alpha}) = 1 = ht_\gamma(\tilde{\alpha}) - 2n$, thus $ht_\gamma(\tilde{\alpha}) = 2n+1$. Let $\alpha \in \Phi^+$ such that $n_k^\alpha \geq n+1$, if $(\gamma, \alpha) < 0$, then $\gamma + \alpha \in \Phi^+$ and $n_k^{\gamma+\alpha} > n_k^{\tilde{\alpha}}$ a contradiction, thus $(\gamma, \alpha) \geq 0$. Hence , $(\gamma, \alpha) = ht_\gamma(\alpha) - n_k^\alpha \geq 0$, and $ht_\gamma(\alpha) \geq n+1$. Lastly, let $\alpha \in \Phi^+$ such that $n_k^\alpha = 0$ and $(\gamma, \alpha) > 0$, then by Lemma 2.2 $(\gamma, \alpha) = 1 = ht_\gamma(\alpha) - n_k^\alpha$ so $ht_\gamma(\alpha) = 1$ and $ht_\gamma(\gamma - \alpha) = n+1$.

Now to show that $\{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq n+1\} \subseteq \Lambda$. Let $\alpha \in \Phi^+$ such that $ht_\gamma(\alpha) \geq n+1$ and $\alpha \notin \Lambda$. If $(\gamma, \alpha) < 0$, then $\alpha + \gamma \in \Phi^+$ and $ht_\gamma(\alpha + \gamma) \geq 2n+3$ a contradiction (since $ht_\gamma(\tilde{\alpha}) = 2n+1$) , therefore $(\gamma, \alpha) \geq 0$. Also since $ht_\gamma(\alpha) \geq n+1$ and $n_k^\alpha \leq n$, we have $(\gamma, \alpha) \neq 0$, hence $(\gamma, \alpha) > 0$. Since $\alpha \neq \gamma$, $(\gamma, \alpha) = 1$, thus $\gamma - \alpha \in \Phi^+$ and $n_k^{\gamma-\alpha} = 0$. Furthermore, $(\gamma, \gamma - \alpha) = 2 - 1 > 0$. Since $n_k^{\gamma-\alpha} = 0$ and $(\gamma, \gamma - \alpha) > 0$, we know that $\gamma - (\gamma - \alpha) = \alpha \in \Lambda$ a contradiction. Hence, $\Lambda = \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq n+1\}$. Since $\Lambda = \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq n+1\}$, the abelian and ideal conditions are satisfied.

Before we address the question of maximality, we note that if $n_k^{\tilde{\alpha}} = 2$, the statement of the theorem is essentially that found in **[9]** and we do not cover the proof here. Since maximality for the remaining cases (i.e. $n_k^{\tilde{\alpha}} = 4, 6$) has already been verified in **[9]** (by computer), we do not include our proof here, the interested reader should consult the appendix for our proof of maximality which makes use of the following observations on the grading positions associated to γ. Henceforth, $n_k^{\tilde{\alpha}} = 4$ or 6 so $\mathfrak{g}$ is of type E_7 or E_8.

Since the Dynkin diagram is connected, $\mu := \sum_{i=1}^r \alpha_i \in \Phi^+$, now $(\gamma, \mu) = -1 + x$, where $x = |P|$ (the number of grading positions associated to γ). If $x = 3 \Rightarrow (\gamma, \mu) = 2$, hence $\gamma = \mu$ and this only arises if $n_k^{\tilde{\alpha}} = 2$. Since $|(\gamma, \mu)| \not> 2$, $x > 3$ is a contradiction, also $x \neq 0$ since $m_i^\gamma > 0$ for at least one i. Therefore, $x = 1$ or 2, and $|P| \leq 2$.

Now, let p be the grading position (associated to γ) such that the α_p node is closest to the α_{l_0} node in the Dynkin diagram, where $\tilde{\alpha} = \omega_{l_0}$. Let the set of nodes on the shortest path from α_p to α_{l_0} be $\{l_0, \ldots, l_{p-1}, l_p\}$, then $\nu := \tilde{\alpha} - \alpha_{l_0} - \cdots - \alpha_{l_{p-1}} \in \Phi^+$, $ht_\gamma(\nu) = ht_\gamma(\tilde{\alpha})$ and $m_p^\nu > 0$.

Let $\alpha \in \Phi^+$ such that $ht_\gamma(\alpha) = ht_\gamma(\tilde{\alpha})$, then $m_p^\alpha \not< 0$ (otherwise $\alpha + \alpha_p \in \Phi^+$ and $ht_\gamma(\alpha + \alpha_p) > ht_\gamma(\tilde{\alpha})$, a contradiction). Using Lemma 2.3, we conclude that

$$\langle \{\alpha | ht_\gamma(\alpha) = ht_\gamma(\tilde{\alpha})\} \rangle = k_p \omega_p + k_q \omega_q,$$

where p and q are the grading positions and $k_p > 0$ and $k_q \geq 0$, since $m_p^\nu \geq 0$ for all $p \in P$ and $ht_\gamma(\nu) = ht_\gamma(\tilde{\alpha})$.

In summary:

- $|P| \leq 2$.
- $\langle \{\alpha | ht_\gamma(\alpha) = ht_\gamma(\tilde{\alpha})\} \rangle = k_p \omega_p + k_q \omega_q$, where $k_p > 0$ and p is defined above.

The above conditions on the set P, reduces the maximality arguments to a handful of case by case arguments. These arguments are covered in the appendix. □

Corollary 3.4. *With Λ, P, k and γ as in Theorem 3.3.*

(*i*) *If $n_k^{\tilde{\alpha}} = 2$, then* $|\Lambda| = |\tilde{\Delta}_{\mathfrak{g} \setminus \alpha_k}| - |\Delta_{\mathfrak{g} \setminus \{\alpha_i | i \in P \cup \{k\}\}}| + 1$.

(*ii*) *For all other cases,* $|\Lambda| = |\tilde{\Delta}_{\mathfrak{g} \setminus \alpha_k}| - |\Delta_{\mathfrak{g} \setminus \{\alpha_i | i \in P \cup \{k\}\}}| + 1 + \sum_{i=n+1}^{2n-1} |\{\alpha \in \Phi^+ | n_k^\alpha = i\}|$.

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Proof.

(i): Let $\alpha \in \Phi^+$ such that $n_k^\alpha = 0$, then $(\gamma, \alpha) \geq 0$ since $m_i^\gamma < 0$ if and only if $i = k$. If $(\gamma, \alpha) = 0$, then $n_p^\alpha = 0$ for all $p \in P$. Now,

$$\Lambda = \{\alpha \in \Phi^+ | n_k^\alpha \geq 2\} \cup \{\gamma\} \cup \{\gamma - \alpha | (\gamma, \alpha) > 0 \text{ and } n_k^\alpha = 0\}.$$

Hence,

$$|\Lambda| = |\{\alpha \in \Phi^+ | n_k^\alpha = 2, 0\}| - |\{\alpha \in \Phi^+ | n_k^\alpha = 0, (\alpha, \gamma) = 0\}| + 1.$$

Using the theory of Borel de Siebenthal, or Theorem 8.10.9 (page 280) in [**13**],

$$|\Lambda| = |\tilde{\Delta}_{\mathfrak{g} \backslash \alpha_k}| - |\Delta_{\mathfrak{g} \backslash \{\alpha_i | i \in P \cup \{k\}\}}| + 1.$$

(ii): The proof is once again by argument along the lines of Borel de Siebenthal. □

We illustrate the above results for the exceptional Lie algebra E_7, here $\tilde{\alpha} = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. In the statement of Theorem 3.3, choose $k = 2$ (i.e. $n_2^{\tilde{\alpha}} = 2$), then $\gamma = -\omega_2 + \omega_5$ (i.e. $ht_\gamma(\tilde{\alpha}) = 3$ and $P = \{5\}$), then $\Lambda = \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq 2\}$. To compute $|\Lambda|$, we use Corollary 3.4, and see that $|\tilde{\Delta}_{E_7 \backslash \alpha_2}| = |\Delta_{A_7}| = 28$ and $|\Delta_{E_7 \backslash \{\alpha_2, \alpha_5\}}| = |\Delta_{A_3 \times A_2}| = 9$. Thus, $|\Lambda| = 28 - 9 + 1 = 20$.

Finally, we give an example with multiple grading positions. In the case of E_8, $n_4^{\tilde{\alpha}} = 6$ and $\gamma = \omega_2 - \omega_4 + \omega_6$. So, $ht_\gamma(\tilde{\alpha}) = 7$, $P = \{2, 6\}$ and $\Lambda = \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq 4\}$. Lastly, $|\Lambda| = |\tilde{\Delta}_{E_8 \backslash \alpha_4}| - |\Delta_{E_8 \backslash \{\alpha_4, \alpha_2, \alpha_6\}}| + 1 + \sum_{i=4}^{5} |\{\alpha \in \Phi^+ | n_4^\alpha = i\}| = 19 - 7 + 1 + 21 = 34$.

4. A uniform method to find all maximal abelian ideals. In this final section, we present a uniform method for finding all maximal abelian ideals in $\mathfrak{b}$. Furthermore, this method highlights the natural one to one correspondence between long simple roots and maximal abelian ideals for classical $\mathfrak{g}$. This was first observed by Panyushev and Röhrle [**9**] for any $\mathfrak{g}$ and later proved by Suter in [**11**]. We first introduce some notation. Let $\Phi_{(0)} := \{\alpha \in \Phi^+ | (\alpha, \tilde{\alpha}) = 0\}$ and $\Phi_{(1)} := \{\alpha \in \Phi^+ | (\alpha, \tilde{\alpha}) = 1\}$.

LEMMA 4.1. *Let $\mathfrak{a}_\Psi$ be a maximal abelian ideal in $\mathfrak{b}$. Then, for all $\alpha \in \Phi_{(1)}$, either $\alpha \in \Psi$ or $\tilde{\alpha} - \alpha \in \Psi$.*

Proof. Let $\alpha \in \Phi_{(1)}$ be such that $\{\alpha, \tilde{\alpha} - \alpha\} \not\subseteq \Psi$. Then, there exists such a root of maximal height, henceforth denoted α.

Let $\beta \in \Psi$. If $\beta \in \Phi_{(1)}$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Phi^+$, so $\alpha + \beta \in \Psi$ by the ideal condition. Since $\alpha, \beta \in \Phi_{(1)}$, $\alpha + \beta = \tilde{\alpha}$ and $\beta = \tilde{\alpha} - \alpha \in \Psi$, a contradiction. If $\beta \in \Phi_{(0)}$ and $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Psi \cap \Phi_{(1)}$. Thus, $\tilde{\alpha} - (\alpha + \beta) \in \Phi^+$. Using the ideal condition $\beta + (\tilde{\alpha} - (\alpha + \beta)) = \tilde{\alpha} - \alpha \in \Psi$, a contradiction. Also, $(\alpha, \tilde{\alpha}) \geq 0$ by definition of $\tilde{\alpha}$. Hence, $(\alpha, \beta) \geq 0$ for all $\beta \in \Psi$ and $\alpha \in \Phi_{(1)}$.

We will show that $\Psi \cup \{\alpha\}$ is an abelian ideal, thus giving a contradiction. Let $\alpha + \beta \in \Phi^+$ for some $\beta \in \Psi$. If $\beta \in \Phi_{(1)}$, then $\alpha + \beta = \tilde{\alpha}$ and hence $\beta = \tilde{\alpha} - \alpha$, a contradiction as before. Let $\beta \in \Phi_{(0)}$, if $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi^+$ (since $\alpha \in \Phi_{(1)}$ $\beta \in \Phi_{(0)}$) and $(\alpha - \beta) + \beta = \alpha \in \Psi$ by the ideal condition, a contradiction. If $(\alpha, \beta) = 0$ and $\alpha + \beta \in \Phi^+$, then $\alpha - \beta \in \Phi^+$ by Lemma 2.2, hence $\beta + (\alpha - \beta) = \alpha \in \Psi$ by the ideal condition, a contradiction. Hence, we have shown that $\alpha + \beta \notin \Phi^+$, thus $\Psi \cup \{\alpha\}$ is abelian.

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We now show that $\Psi \cup \{\alpha\}$ is an ideal. Let $\gamma \in \Phi^+$ such that $\alpha + \gamma \in \Phi^+$. If $\gamma \in \Phi_{(1)}$, then $\alpha + \gamma = \tilde{\alpha} \in \Psi$, otherwise $\gamma \in \Phi_{(0)}$ and $\alpha + \gamma \in \Phi_{(1)}$. By definition of $\alpha, \alpha + \gamma \in \Psi$. Hence, $\Psi \cup \{\alpha\}$ is an abelian ideal. $\square$

The above lemma prompts us to partition $\Phi_{(1)}$ into two (top and bottom) halves, $\Phi^T_{(1)}$ and $\Phi^B_{(1)}$. Let $\Phi^T_{(1)} := \{\alpha \in \Phi_{(1)} | ht(\alpha) \geq \frac{ht(\tilde{\alpha})+1}{2}\}$ (in the case of A_{even}, where $ht(\tilde{\alpha})$ is even , we let $\Phi^T_{(1)} := \{\alpha \in \Phi_{(1)} | ht(\alpha) \geq \frac{ht(\tilde{\alpha})}{2}\}$), and $\Phi^B_{(1)} := \{\Phi_{(1)} \setminus \Phi^T_{(1)}\}$. Now if $\alpha \in \Phi^T_{(1)}$, then $\tilde{\alpha} - \alpha \in \Phi^B_{(1)}$.

By Lemma 4.1, we must firstly determine, for every maximal abelian ideal, which roots from $\Phi^T_{(1)}$ are to be excluded from the ideal. We observe that if a root $\alpha \in \Phi^T_{(1)}$ is excluded (so that $\tilde{\alpha} - \alpha \in \Phi^B_{(1)}$ is included), then $2n^{\alpha}_k < n^{\tilde{\alpha}}_k$ for some k (since $\tilde{\alpha} - \alpha$ cannot be an ancestor of α). To obtain all possibilities therefore, for each k with $n^{\tilde{\alpha}}_k = 2n + 1,\ n \geq 0$, we choose $\hat{\alpha} \in \Phi^T_{(1)}$ of maximal height such that $n^{\hat{\alpha}}_k = n$. Thus, we obtain (see Theorem 4.2 below) a maximal abelian ideal $I^{\hat{\alpha}} = \{\alpha \in \Phi^+ | ht_{-\hat{\alpha}}(\alpha) > ht_{-\hat{\alpha}}(\hat{\alpha})\}$. We now repeat the process with the next highest $\hat{\alpha} \in \Phi^T_{(1)}$ such that $n^{\hat{\alpha}}_k = n$, and we continue until all possibilities are exhausted for all choices of k with $n^{\tilde{\alpha}}_k = 2n + 1,\ n \geq 0$. For example, in the case where $n^{\tilde{\alpha}}_k = 1$, the first root $\hat{\alpha} \in \Phi^T_{(1)}$ we exclude is the highest root such that $n^{\hat{\alpha}}_k = 0$. This root is of the form $-\omega_k + \sum_{m \in M} \omega_M$ for some set of indices M, where $\sum_{m \in M} \omega_m$ is the highest root of the embedded root system obtained by deletion of α_k from the Dynkin diagram. Now, if we consider roots α such that $ht_{-\hat{\alpha}}(\alpha) > ht_{-\hat{\alpha}}(\hat{\alpha})$ (i.e. using the grading positions associated to $-\hat{\alpha}$ and choosing all roots such that $n^{\alpha}_k = 1$), we recover the maximal abelian ideal ($\{\alpha \in \Phi^+ | n^{\alpha}_k = 1\}$) as described by Theorem 2.4.

For any subsequent $\hat{\alpha}$ of lower height, there will be multiple indices i such that $m^{\hat{\alpha}}_i < 0$, we will use the grading positions associated to the root $-\hat{\alpha}$ (i.e. $\{i | m^{\hat{\alpha}}_i < 0\}$). When no root is removed from $\Phi^T_{(1)}$, we use the grading positions associated to $\langle \Phi^T_{(1)} \rangle$ excluding the index p, where $\tilde{\alpha} = \omega_p$, in the case of A_n we exclude both 1 and n. For convenience, let $\langle \Phi^T_{(1)} \rangle = \sum_{i \in I} m_i \omega_i$, and let $\delta := \sum_{\substack{i \in I \\ i \neq p}} m_i \omega_i$. We exclude the index p since $\tilde{\alpha} \notin \Phi_{(1)} \cup \Phi_{(0)}$. The following theorem provides a simple uniform description of all maximal abelian ideals. The proof relies on determination of grading positions, which unavoidably will introduce some case by case analysis.

THEOREM 4.2. *Let $\mathfrak{g}$ be a complex simple Lie algebra. Every maximal abelian ideal in $\mathfrak{b}$ is of the form $\mathfrak{a}_\Upsilon$, where Υ is either*

(*i*) $\{\alpha \in \Phi^+ | ht_{-\hat{\alpha}}(\alpha) > ht_{-\hat{\alpha}}(\hat{\alpha})\}$, *where* $\hat{\alpha} \in \Phi^T_{(1)}$, $n^{\hat{\alpha}}_k = n$ *and* k *is such that* $n^{\tilde{\alpha}}_k = 2n + 1$, *where* $n \in \mathbb{Z}_{\geq 0}$.
Or,

(*ii*) $\{\alpha \in \Phi^+ | ht_\delta(\alpha) > \frac{ht_\delta(\tilde{\alpha})-1}{2}\}$.

Proof. For simplicity, we will not cover the case $\mathfrak{g}$ being of type A_l (Theorem 2.4 already recovers all maximal abelian ideals).

For each k such that $n^{\tilde{\alpha}}_k = 2n + 1$, where $n \in \mathbb{Z}_{\geq 0}$ let $\hat{\alpha}_0,\ \hat{\alpha}_1,\ \ldots \hat{\alpha}_j \ldots$ be the highest, next highest, ... root in $\Phi^T_{(1)}$ with $n^{\hat{\alpha}_j}_k = n$. We observe that for all classical cases, the only odd coefficient of $\tilde{\alpha}$ is one and when $n^{\tilde{\alpha}}_k = 1$ we have $n^{\hat{\alpha}_0}_k = 0$. Clearly, $\hat{\alpha}_0$ is the highest root of the embedded root system with Dynkin diagram a component of that of $\mathfrak{g}$ with the α_k node removed. When $\mathfrak{g}$ is of type B_l, then $k = 1$ is the only choice for k. Since $\tilde{\alpha} = \omega_2$ and $\hat{\alpha}_0$ is the highest root of the embedded B_{l-1} root system, obtained by

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deletion of the α_1 node, we see that $\hat{\alpha}_0 = -\omega_1 + \omega_3$. The grading position associated to $-\hat{\alpha}_0$ is $\{1\}$, and $n_1^{\hat{\alpha}_0} = 0$. So, $\Upsilon = \{\alpha \in \Phi^+ | ht_{-\hat{\alpha}_0}(\alpha) > 0\}$ (i.e. $\{\alpha \in \Phi^+ | n_1^{\alpha} = 1\}$). This ideal was described in Theorem 2.4, and so is a maximal abelian ideal. Now, $\hat{\alpha}_1$ is $S_{\alpha_3}(\hat{\alpha}_0)$, so that $\hat{\alpha}_1 = -\omega_1 + \omega_2 - \omega_3 + \omega_4$. The grading positions associated to $-\hat{\alpha}_1$ are $\{1, 3\}$, also $n_1^{\hat{\alpha}_1} = 0$ and $n_3^{\hat{\alpha}_1} = 1$. So, $\Upsilon = \{\alpha \in \Phi^+ | ht_{-\hat{\alpha}_1}(\alpha) > 1\}$. This ideal was already described in Theorem 3.3 and hence is a maximal abelian ideal. We can continue in this fashion to recover a total of $(l-2)$ $\hat{\alpha}_i$ roots (any more reflections would force the height of $\hat{\alpha}_i$ included too low to be an element of $\Phi^T_{(1)}$). The grading positions associated to $-\hat{\alpha}_i$ are $\{1, i+1\}$ for $2 \leq i \leq l-2$. Each of these correspond to a maximal abelian ideal described in Theorem 3.3. Lastly, the grading positions associated to δ are $\{1, l\}$, once again these correspond to a maximal abelian ideal described in Theorem 3.3. Hence, we have a total of $l-1$ maximal abelian ideals.

Let $\mathfrak{g}$ be of type C_l, then $\tilde{\alpha} = 2\omega_1$ and $ht(\tilde{\alpha}) = 2l-1$. Here, $n_k^{\tilde{\alpha}}$ is odd for $k = l$. Hence, $\hat{\alpha}_0$ must belong to an embedded A_{l-1} root system, forcing $\hat{\alpha}_0 \notin \Phi^T_{(1)}$, a contradiction. So, there are no $\hat{\alpha}$ roots in this case. The grading position associated to δ is $\{l\}$, and $n_l^{\tilde{\alpha}} = 1$. Hence, $\Upsilon = \{\alpha \in \Phi^+ | ht_\delta(\alpha) > 0\}$. This is the only maximal abelian ideal, and already found in Theorem 2.4.

Let $\mathfrak{g}$ be of type D_l, then $\tilde{\alpha} = \omega_2$, $ht(\tilde{\alpha}) = 2l-3$ and $n_k^{\tilde{\alpha}}$ is odd if and only if $k \in \{1, l-1, l\}$. If $k = l$ or $l-1$, then $\hat{\alpha}_0$ is found by $l-2$ simple reflections from $\tilde{\alpha}$ (each reflection corresponding to those nodes in the path from α_2 to α_k in the Dynkin diagram), thus $ht(\hat{\alpha}_0) = ht(\tilde{\alpha}) - (l-2) = l-1$. Hence, we cannot further decrease in height and remain contained in $\Phi^T_{(1)}$. So each $k \in \{l-1, l\}$ gives one $\hat{\alpha}$, the grading positions associated to $-\hat{\alpha}_0$ are $\{l-1\}$ for $k = l-1$ and $\{l\}$ for $k = l$ (i.e. corresponding to the maximal abelian ideals found in Theorem 2.4). If $k = 1$, $\hat{\alpha}_0 = -\omega_1 + \omega_3$, the grading position associated to $-\hat{\alpha}_0$ is $\{1\}$ (found in Theorem 2.4). Using simple reflections, we can find $l-4$ more $\hat{\alpha}_i$ roots, where $\{1, i\}$ are the grading positions associated to $-\hat{\alpha}_i$ for $3 \leq i < l-2$. Once again these grading positions correspond to the maximal abelian ideals found in Theorem 3.3. Lastly, the grading positions associated to δ are $\{1, l-1, l\}$. This is our first example of three grading positions. Here, $ht_\delta(\tilde{\alpha}) = 3$, and $\Upsilon = \{\alpha \in \Phi^+ | ht_\delta(\alpha) > 1\}$. Once again this maximal abelian ideal was described by Theorem 3.3.

Let $\mathfrak{g}$ be of type E_6. Here, $n_k^{\tilde{\alpha}} = 1$ for $k = 1$ or $k = 6$. In each of these cases, the grading positions associated to $-\hat{\alpha}_0$ describe maximal abelian ideals found by Theorem 2.4. There is only one other $\hat{\alpha}$ root, the grading positions associated to $-\hat{\alpha}_1$ are $\{1, 5\}$ for $k = 1$ (and $\{3, 6\}$ for $k = 6$). Once again both of these correspond to maximal abelian ideals described in Theorem 3.3. For $k = 4$ (i.e. $n_4^{\tilde{\alpha}} = 3$) $\hat{\alpha}_0 = \omega_1 + \omega_2 - \omega_4 + \omega_6$, so the grading position associated to $-\hat{\alpha}_0$ is $\{4\}$, there are no more $\hat{\alpha}$ roots for $k = 4$. Once again this was described in Theorem 3.3. The grading positions associated to δ are $\{1, 4, 6\}$, and $ht_\delta(\tilde{\alpha}) = 5$. So, $\Upsilon = \{\alpha \in \Phi^+ | ht_\delta(\alpha) > 2\}$. To show that $\mathfrak{a}_\Upsilon$ is maximal, let $\beta \in \Phi^+$, such that $ht_\delta(\beta) = 2$, and suppose that $\beta \cup \Upsilon$ is an abelian ideal. Now, $\langle \{\alpha \in \Phi^+ | ht_\delta(\alpha) = 5\} \rangle = \omega_4$. If $(\beta, \omega_4) < 0$, then there exists an α such that $ht_\delta(\alpha) = 5$ and $\alpha + \beta \in \Phi^+$ a contradiction. If $(\beta, \omega_4) > 0$, then there exists an α such that $\alpha - \beta \in \Phi^+$ and since $ht_\delta(\alpha) = 5$, $\alpha - \beta \in \Upsilon$. This is a contradiction since $\beta + (\alpha - \beta) \in \Phi^+$ (violating the abelian condition). So $(\beta, \omega_4) = 0$, hence $n_4^\beta = 0$. This is another contradiction since $ht_\delta(\beta) = 2$ and $n_4^\beta = 0$, no such root exists in the E_6 root system (since if $n_4^\beta = 0$, β belongs to a product of embedded type A root systems). Hence, $\mathfrak{a}_\Upsilon$ is maximal. This gives a total of six maximal abelian ideals.

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Table 1. Maximal abelian ideals

Lie algebra	Grading positions	Theorems
A_n	$\{1\}, \ldots, \{r\}$.	2.4
$B_n, n \geq 3$	$\{1\}$.	2.4
	$\{1, 3\}, \ldots, \{1, n\}$.	3.3
C_n	$\{n\}$.	2.4
$D_n, n > 3$	$\{1\}, \{n-1\}, \{n\}$.	2.4
	$\{1, 3\}, \ldots, \{1, n-2\}, \{1, n-1, n\}$.	3.3
E_6	$\{1\}, \{6\}$.	2.4
	$\{1, 5\}, \{3, 6\}, \{4\}$.	3.3
	$\{1, 4, 6\}$.	4.2
E_7	$\{7\}$.	2.4
	$\{5\}, \{3\}, \{2, 7\}, \{3, 6\}$.	3.3
	$\{4, 7\}, \{3, 5, 7\}$.	4.2
E_8	$\{7\}, \{2\}, \{5\}, \{1, 7\}, \{2, 6\}$.	3.3
	$\{3, 7\}, \{4, 7\}, \{2, 5, 7\}$.	4.2
F_4	$\{2\}$.	3.3
	$\{2, 4\}$.	4.2
G_2	$\{1\}$.	3.3

The arguments for E_7 and E_8 are almost identical to those for E_6. In E_7, the only grading positions not found previously are $\{3, 5, 7\}$ associated to δ and $\{4, 7\}$ associated to $\hat{\alpha}$. To show maximality, we repeat a similar argument to the E_6 case. In E_8, the only grading positions not found previously are $\{2, 5, 7\}$, $\{3, 7\}$ and $\{4, 7\}$, maximality follows along the lines of previous arguments again. Giving a total of seven and eight maximal abelian ideals for E_7 and E_8, respectively.

For F_4, $\tilde{\alpha} = \omega_1$ and $n_k^{\tilde{\alpha}}$ is odd if and only if $k = 2$. Here, $\hat{\alpha}_0 = \omega_1 - \omega_2 + 2\omega_4$ and the grading position associated to $-\hat{\alpha}_0$ is $\{2\}$, there are no more $\hat{\alpha}$ roots. This corresponds to a maximal abelian ideal described in Theorem 3.3. The grading positions associated to δ are $\{2, 4\}$ so $ht_\delta(\tilde{\alpha}) = 5$ and $\langle\{\alpha | ht_\delta(\alpha) = 5\}\rangle = \omega_2$. A similar argument to that of the above will show maximality, giving a total of two maximal abelian ideals.

Lastly, we find one maximal abelian ideal in G_2, grading position $\{1\}$ associated to δ, which was already found by Theorem 3.3. We have exhausted all possible choices of $\alpha \in \Phi_{(1)}$ to produce maximal abelian ideals. Using the result that there are as many maximal abelian ideals as the number of long simple roots, proved in **[11]** and **[10]**, we have found every maximal abelian ideal. $\square$

We present our results in table form (see Table 1), showing grading positions associated to both $-\hat{\alpha}_i$ and δ in the case of Theorem 4.2 or γ for Theorem 3.3. The last column shows which theorems found these grading positions, we also include Theorem 2.4 when it found the same maximal abelian ideals, as our other theorems. Theorem 4.2 describes every maximal abelian ideal but we only reference it in the table when it is the only theorem that identifies the grading positions shown. We also note that Theorems 2.4 and 3.3 struggle for exceptional $\mathfrak{g}$.

A Appendix

Conclusion of the proof of Theorem 3.3. We know that $\Lambda = \{\alpha \in \Phi^+ | ht_\gamma(\alpha) \geq n+1\}$. Using the two observations in the proof of Theorem 3.3, we now show that

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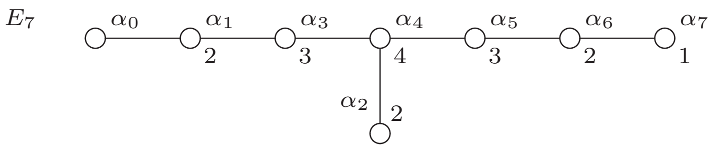

Figure 1. Extended dynkin diagram of E_7.

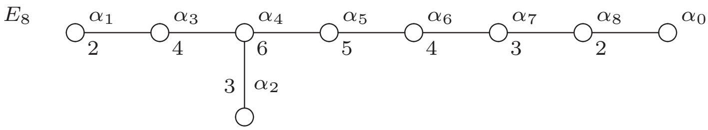

Figure 2. Extended dynkin diagram of E_8.

$\mathfrak{a}_\Lambda$ is maximal for $n \geq 2$. To prove maximality, we show that for any root α such that $ht_\gamma(\alpha) = n$, $\Lambda \cup \{\alpha\}$ is not an abelian ideal.

Let $\alpha \in \Phi^+$ such that $ht_\gamma(\alpha) = n$ and suppose $\Lambda \cup \{\alpha\}$ is an abelian ideal. If $(\langle\{\beta \in \Lambda | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle, \alpha) < 0$, then there exists a β such that $\beta + \alpha \in \Phi^+$, a contradiction. If $(\langle\{\beta | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle, \alpha) > 0$, then there exists a β such that $\beta - \alpha \in \Phi^+$ and since $ht_\gamma(\alpha) = n$, $\beta - \alpha \in \Lambda$. Hence, $\alpha + (\beta - \alpha) \in \Lambda$ a contradiction. So, $(\langle\{\beta | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle, \alpha) = 0$.

Suppose $|P| = 1$, then $\langle\{\beta | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle = k_p\omega_p$, where p is the grading position and by our observations $k_p > 0$. If $(\langle\{\beta | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle, \alpha) = 0$, then $(k_p\omega_p, \alpha) = k_p(\omega_p, n_p^\alpha \alpha_p) = 0$, so $n_p^\alpha = 0$ a contradiction since $ht_\gamma(\alpha) = n$. Since $|P| \leq 2$, there must be exactly two grading positions, henceforth let $P = \{p, q\}$, and let α_p be closer to the $-\tilde{\alpha}$ node in the extended Dynkin diagram.

If $(\langle\{\beta | ht_\gamma(\beta) = ht_\gamma(\tilde{\alpha})\}\rangle, \alpha) = 0$, then

$$(k_p\omega_p, \alpha) + (k_q\omega_q, \alpha) = k_p(\omega_p, n_p^\alpha \alpha_p) + k_q(\omega_q, n_q^\alpha \alpha_q) = 0.$$

Since $(\omega_p, \alpha_p) > 0$ and $k_p > 0$, $k_q \geq 0$, then $n_p^\alpha = 0$. If $k_q > 0$, then $n_q^\alpha = 0$, also a contradiction since $ht_\gamma(\alpha) = n$. Therefore, $k_q = 0$, and $n_q^\alpha = n$ since $ht_\gamma(\alpha) = n$. So, we require a root α such that $ht_\gamma(\alpha) = n$ and $n_q^\alpha = n$.

Since we are only concerned with the cases where $n_k^{\tilde{\alpha}} > 2$ (i.e. where $n_k^{\tilde{\alpha}} = 4$ or 6) and α_k is long, the only root systems left to consider are that of E_7 and E_8. Our arguments from now on we will make extensive use of Dynkin diagrams and embedded root systems. To aid the reader in seeing the embedded root systems for the various possibilities of grading positions associated to γ , we will provide the extended Dynkin diagrams for both E_7 and E_8 (see Figures 1 and 2). The nodes are denoted by α_l and the number adjacent to each node is $n_l^{\tilde{\alpha}}$. We start with E_7.

In E_7, the only case to consider is that of $k = 4$, here $ht_\gamma(\tilde{\alpha}) = 5$, $ht_\gamma(\gamma) = 4$ and $\tilde{\alpha} = \omega_1$. If $\{1\} \in P$, then, $n_1^\gamma = 1$ since $n_1^{\tilde{\alpha}} = 2$. Hence, $n_q^\gamma = 3$, since $ht_\gamma(\gamma) = 4$. Now, $m_p^\gamma > 0$ for all $p \in P$ by definition of P. So, $\eta := \gamma - \alpha_1$ is a root. Moreover, $n_1^\eta = 0$, hence η belongs to an embedded D_6 root system. This is a contradiction since $n_q^\eta = 3$ and $n_i^{\tilde{\alpha}} \leq 2$ for all i in D_6. Therefore, $\{1\} \notin P$.

Since $ht_\gamma(\tilde{\alpha}) = 5$, the following are the only possible choices for grading positions associated to γ, $\{\{3, 2\}, \{3, 6\}, \{5, 2\}, \{5, 6\}, \{4, 7\}\}$. We now use the observations that $n_p^\alpha = 0$ and $n_q^\alpha = n$, to exclude possible grading positions associated to γ. Suppose $\{3\} \in P$. Since α_3 is closer to the α_1 node in the Dynkin diagram, we know that $n_3^\alpha = 0$, hence α belongs to embedded $A_1 \times A_5$, a contradiction since we require $n_q^\alpha = 2$.

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Suppose $P = \{5, 2\}$, then $\alpha \in A_4 \times A_2$ such that $n_2^\alpha = 2$, a contradiction. If $P = \{5, 6\}$, then there exists $\alpha \in A_4 \times A_2$ such that $n_6^\alpha = 2$, a contradiction. Lastly, if $P = \{4, 7\}$, then there exists $\alpha \in A_2 \times A_1 \times A_3$ such that $n_6^\alpha = 2$ another contradiction.

For E_8 (see extended Dynkin diagram below), $\tilde{\alpha} = \omega_8$. First, $n_k^{\tilde{\alpha}} = 4$, for $k \in \{3, 6\}$. In either case, the set of possible grading positions associated to γ is $\{\{1, 2\}, \{1, 7\}, \{2, 8\}, \{7, 8\}\}$. Similar to the E_7 case, arguments along the lines of embedded root systems that contain some root η such that $n_p^\eta = 0$ and $n_q^\eta > 2$, exclude these as grading positions.

Finally, if $k = 4$, $n_k^{\tilde{\alpha}} = 6$, $ht_\gamma(\tilde{\alpha}) = 7$ and $ht_\gamma(\gamma) = 5$. The set of possible grading positions associated to γ is $\{\{5, 8\}, \{1, 5\}, \{3, 2\}, \{3, 7\}, \{2, 6\}, \{6, 7\}\}$.

Similar to the case of E_7, if $P = \{5, 8\}$, then $n_8^\gamma = 1$ (since $\tilde{\alpha} = \omega_8$) and hence $n_5^\gamma = 4$, so there exists an η such that $n_8^\eta = 0$ and $n_5^\eta = 4$ a contradiction since η belongs to an embedded E_7 root system.

If $P = \{1, 5\}$, then $\alpha \in A_4 \times A_3$, where $n_1^\alpha = 3$. If $P = \{3, 2\}$, then $\alpha \in A_7$, where $n_3^\alpha = 3$. If $P = \{3, 7\}$, then there exists $\alpha \in E_6$ such that $n_3^\alpha = 3$. If $P = \{2, 6\}$, then $\alpha \in D_5 \times A_2$ such that $n_2^\alpha = 3$. If $P = \{6, 7\}$, then there exists $\alpha \in E_6$, such that $n_6^\alpha = 3$, all of which a contradictions. Thus, $\Lambda \cup \{\alpha\}$ is not abelian when $ht_\gamma(\alpha) = n$.

Now, let β be a root of maximal height such that $ht_\gamma(\beta) = n - 1$ and $\Lambda \cup \{\beta\}$ is an abelian ideal. There exists an α_l such that $\beta + \alpha_l \in \Phi^+$ and $\beta + \alpha_l \in \Lambda$ by the ideal condition. If $l \in P$, then $ht_\gamma(\beta + \alpha_l) = n$, a contradiction. If $l \notin P$, then $ht_\gamma(\beta + \alpha_l) = n - 1$ and $ht(\beta + \alpha_l) > ht(\beta)$ a contradiction by definition of β. Hence, the theorem is proved. $\square$

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