# 14

# K3 orientifolds and compactification

In section 7.6, we constructed type IIA and IIB compactifications on K3 with pure geometrical orbifolds. In that same chapter, we met the prototype orientifold, which constructs the type I string theory from the type IIB string theory. Now that we have understood the behaviour of D-branes in the neighbourhood of orbifold fixed points, we are in a good position to revisit the orbifold construction of K3 compactification, by combining our ideas with those about the orientifold to make K3 compactifications of type I string theory. There are many models that can be made, and we will present only a small sample of them here[236], in order to illustrate the key ideas.

## 14.1 $\mathbb{Z}_N$ orientifolds and Chan–Paton factors

Let us consider constructing K3 in its orbifold limits using the spacetime symmetry group $\mathbb{Z}_N$, which acts as described in subsection 7.6.5, which the reader should consult. Recall that we denoted the generator of $\mathbb{Z}_N$ by $\alpha_N$, the group elements being the powers $\alpha_N^k$, for $k \in \{0, 1, 2, \dots, N-1\}$. There are spacetime symmetries, acting on $\mathbb{R}^4$ with coordinates $(x^6, x^7, x^8, x^9)$.

We now define what might be called the 'orientifold group', combining both spacetime and world-sheet symmetries, which contains the elements $\alpha_N^k$ and also $\Omega \cdot \alpha_N^k$ (which we shall sometimes denote $\Omega_k$), where $\Omega$ is world-sheet parity. Gauging the action of $\alpha_N^k$ (i.e. projecting onto states invariant under it) will require the introduction of the familiar closed string twisted sectors for the orbifold part, while gauging $\Omega_k$ will result in unoriented world-sheets, as described in section 2.6.

At this level, we have a choice as to the elements we wish to consider in our orientifold group, the only constraint being closure. There are two



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distinct choices of $\mathbb{Z}_N$ orientifold group, which we can denote* as $\mathbb{Z}_N^A$ and $\mathbb{Z}_N^B$. One choice is to have

$$\mathbb{Z}_N^A = \{1, \Omega, \alpha_N^k, \Omega_j\}, \quad k, j = 1, 2, \ldots, N-1. \tag{14.1}$$

A second choice (only available for $N$ even) is

$$\mathbb{Z}_N^B = \{1, \alpha_N^{2k-2}, \Omega_{2j-1}\}, \quad k, j = 1, 2, \ldots, \frac{N}{2}. \tag{14.2}$$

Both of these orientifold groups close under group multiplication since $\Omega^2 = 1$.

Let us consider the presence of D-branes in this situation, introducing Chan–Paton factors, $\lambda$. As in section 2.6, an open string state will be denoted $\lambda_{ij}|\psi, ij\rangle$, where $\psi$ is the state of the world-sheet fields and $i$ and $j$ label the endpoint states. Note that there should be no non-zero elements of the Chan–Paton matrices, $\lambda$, which connect D-branes which are at different points in spacetime.

The action of an orientifold group element $g \in \mathbb{Z}_N^{A,B}$ on this complete set will be represented by the unitary matrices denoted $\gamma_g$. We have for example, generalising expressions in section 2.6:

$$\alpha_N^k: \quad |\psi, ij\rangle \quad \rightarrow \quad (\gamma_k)_{ii'}|\alpha_N^k \cdot \psi, i'j'\rangle(\gamma_k^{-1})_{j'j} \tag{14.3}$$

while for $\Omega \cdot \alpha_N^k \equiv \Omega_k$,

$$\Omega_k: \quad |\psi, ij\rangle \quad \rightarrow \quad (\gamma_{\Omega_k})_{ii'}|\Omega_k \cdot \psi, j'i'\rangle(\gamma_{\Omega_k}^{-1})_{j'j}. \tag{14.4}$$

As before, when the group element includes $\Omega$, the ends of the string are transposed. Composing various actions of the group elements, we see that since $(\alpha_N^k)^N = 1$, then

$$(\alpha_N^k)^N: \quad |\psi, ij\rangle \quad \rightarrow \quad (\gamma_k^N)_{ii'}|\psi, i'j'\rangle(\gamma_k^{-N})_{j'j} \tag{14.5}$$

and so

$$\gamma_k^N = \pm 1.$$

Similarly, since $\Omega^2 = 1$

$$\Omega^2: \quad |\psi, ij\rangle \quad \rightarrow \quad (\gamma_\Omega \ (\gamma_\Omega^T)^{-1})_{ii'}|\psi, i'j'\rangle(\gamma_\Omega^T \gamma_\Omega^{-1})_{j'j}, \tag{14.6}$$

resulting in

$$\gamma_\Omega = \pm\gamma_\Omega^T. \tag{14.7}$$

Other examples of such conditions will be put to explicit use later.

* Here $A$ and $B$ have nothing to do with the $A$ and $B$ of the type II strings.

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## 14.2 Loops and tadpoles for ALE $\mathbb{Z}_M$ singularities

### *14.2.1 One-loop diagrams and tadpoles*

In open and/or unoriented string theory, certain divergences arise at the one-loop level, which may be interpreted[31] as inconsistencies in the field equations for the R–R potentials in the theory. They manifest themselves as 'tadpoles': amplitudes for emission of quanta from the vacuum. We must ensure that these are absent from the theory, and the way to do this is to cancel them against each other, possibly form different sectors of the theory. We saw this in the prototype orientifold, constructing the $SO(32)$ type I theory from the type IIB superstring. Converting the earlier language of chapter 7 to the one we use here, the tadpoles are of two types, disc tadpoles and $\mathbb{RP}^2$ tadpoles. They are perhaps best visualised as the process of emitting an R–R closed string state from a D$p$-brane (for the disc), or from an orientifold plane (for $\mathbb{RP}^2$). In that prototype case the disc and $\mathbb{RP}^2$ produce a divergence proportional to $(2n_9-32)^2$ for $SO(2n_9)$ Chan–Paton factors (i.e. there are $n_9$ D9-branes) and $(2n_9+32)^2$ for $USp(2n_9)$. Cancellation of the divergences therefore requires gauge group $SO(32)$ (i.e. 16 D9-branes). Here, the orientifold group is simply $\{1, \Omega\}$, as there are no spacetime symmetries to consider. The tadpole cancellation ensures consistency of the ten-form potential's field equation.

Just as in the example of computing the D-brane and orientifold tensions in chapter 7, the most efficient way of computing the divergent contribution of the tadpoles is to compute the one-loop diagrams (the Klein bottle (KB), Möbius strip (MS) and cylinder (C)) and then to take a limit which extracts the divergent pieces. The fact that these diagrams yield the disc and $\mathbb{RP}^2$ tadpoles in terms of the sums of three different products means that the requirement of factorisation of the final expression is a strong consistency check on the whole computation.

The three types of diagram which can be drawn, labelled by the possible elements of the orientifold group under consideration, are depicted in figure 14.1. In the figure, the crosscaps show the action of $\Omega_m$ as one goes half way around the open string channel (*around* the cylinder). (Recall that the the crosscap is a disc with the edges identified. See figure 2.13,

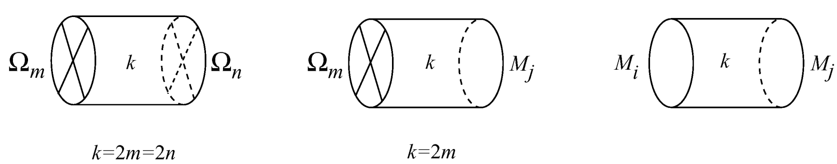


Fig. 14.1. One-loop diagrams from which we will extract the tadpoles.

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page 56.) Going around all the way picks up another action of $\Omega_m$, yielding the $\mathbb{Z}_N$ element $\Omega_m^2 = \alpha_N^{2m}$ which is the twist which propagates in the closed string channel (*along* the cylinder). If there is a crosscap with $\Omega_n$ at the other end, forming a Klein bottle, then $\Omega_n^2 = \alpha_N^{2n}$ should yield the same twist in the closed string channel, i.e. $2n = 2m$. For $\mathbb{Z}_N$ with $N$ odd, there is only one solution to this: $m = n \bmod N$. When $N$ is even however, we can have also the solution $n = m + N/2 \bmod N$.

### 14.2.2 Computing the one-loop diagrams

We may parametrise the surfaces as cylinders with length $2\pi l$ and circumference $2\pi$ with either boundaries or crosscaps on their ends with boundary conditions on a generic field $\phi(\sigma^1, \sigma^2)$ (and its derivatives):

$$\begin{aligned}
\mathrm{KB}: \quad & \phi(0, \pi + \sigma^2) = \Omega_m \cdot \phi(0, \sigma^2) \\
& \phi(2\pi l, \pi + \sigma^2) = \Omega_n \cdot \phi(2\pi l, \sigma^2) \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2); \quad k = 2m = 2n; \\
\mathrm{MS}: \quad & \phi(2\pi l, \sigma^2) \in M_j \\
& \phi(0, \pi + \sigma^2) = \Omega_m \cdot \phi(0, \sigma^2) \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2); \quad k = 2m; \\
\mathrm{C}: \quad & \phi(0, \sigma^2) \in M_i \\
& \phi(2\pi l, \pi + \sigma^2) \in M_j \\
& \phi(\sigma^1, 2\pi + \sigma^2) = \alpha_N^k \cdot \phi(\sigma^1, \sigma^2).
\end{aligned} \tag{14.8}$$

In computing the traces to yield the one-loop expressions, it is convenient to parametrise the Klein bottle and Möbius strip in the region $0 \leq \sigma^1 \leq 4\pi l, 0 \leq \sigma^2 \leq \pi$ as follows:

$$\begin{aligned}
\mathrm{KB}: \quad & \phi(\sigma^1, \pi + \sigma^2) = \Omega_m \cdot \phi(4\pi l - \sigma^1, \sigma^2) \\
& \phi(4\pi l, \sigma^2) = \alpha_N^{m-n} \cdot \phi(0, \sigma^2); \\
\mathrm{MS}: \quad & \phi(0, \sigma^2) \in M_j \\
& \phi(4\pi l, \sigma^2) \in M_j \\
& \phi(\sigma^1, \pi + \sigma^2) = \Omega_m \cdot \phi(4\pi l - \sigma^1, \sigma^2).
\end{aligned} \tag{14.9}$$

After the standard rescaling of the coordinates such that open strings are length $\pi$ while closed strings are length $2\pi$, the amplitudes are

$$\begin{aligned}
\mathrm{KB}: \quad & \mathrm{Tr}_{\mathrm{c},k} \left( \Omega_m (-1)^{F+\tilde{F}} e^{\pi(L_0+\bar{L}_0)/2l} \right), \\
\mathrm{MS}: \quad & \mathrm{Tr}_{\mathrm{o},jj} \left( \Omega_m (-1)^F e^{\pi L_0/4l} \right), \\
\mathrm{C}: \quad & \mathrm{Tr}_{\mathrm{o},ij} \left( \alpha_N^k (-1)^F e^{\pi L_0/l} \right).
\end{aligned} \tag{14.10}$$

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(In the above, 'o' and 'c' mean 'open' and 'closed', respectively.)

The complete one-loop amplitude is

$$\int_0^\infty \frac{dt}{t}\left\{\mathrm{Tr}_\mathrm{c}\left(\mathbf{P}(-1)^\mathbf{F}e^{-2\pi t(L_0+\bar{L}_0)}\right)+\mathrm{Tr}_\mathrm{o}\left(\mathbf{P}(-1)^\mathbf{F}e^{-2\pi tL_0}\right)\right\}. \quad (14.11)$$

The projector $\mathbf{P}$ includes the GSO and group projections and $\mathbf{F}$ is the fermion number. The traces are over transverse oscillator states and include sums over spacetime momenta. After we evaluate the traces, the $t \to 0$ limit will yield the divergences. Note also that the loop modulus $t$ is related to the cylinder length $l$ as $t = 1/4l, 1/8l$ and $1/2l$ for the Klein bottle, Möbius strip and cylinder, respectively. See figure 6.2, page 148.

Note that the elements $\alpha_N^k$ act as follows on the bosons and in the Neveu–Schwarz (NS) sector:

$$\alpha_N^k : \begin{cases} z_1 = X^6 + iX^7 \to e^{\frac{2\pi ik}{N}} z_1 \\ z_2 = X^8 + iX^9 \to e^{-\frac{2\pi ik}{N}} z_2 \end{cases}, \quad (14.12)$$

and it acts in the Ramond (R) sector as

$$\alpha_N^k = e^{\frac{2\pi ik}{N}(J_{67}-J_{89})}. \quad (14.13)$$

As a consequence of this latter convention, notice for example that $\alpha_N^k$ gives the result $4\cos^2\frac{\pi k}{N}$ when evaluated on the R ground states while $(-)^F\alpha_N^k$ gives $4\sin^2\frac{\pi k}{N}$.

For the closed string with integer of half-integer modes labelled by $r$, we may write the action of $\Omega$:

$$\Omega\alpha_r\Omega^{-1} = \tilde{\alpha}_r, \quad \Omega\psi_r\Omega^{-1} = \tilde{\psi}_r, \quad \Omega\tilde{\psi}_r\Omega^{-1} = -\psi_r. \quad (14.14)$$

For open strings with mode expansion

$$X^\mu(\sigma,0) = x^\mu + i\sqrt{\frac{\alpha'}{2}}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}\frac{\alpha_m}{m}(e^{im\sigma} \pm e^{-im\sigma}). \quad (14.15)$$

Being more explicit, we must compute the following amplitudes,

$$\begin{aligned}
&\mathrm{KB}: \quad \mathrm{Tr}^{U+T}_{\mathrm{NS-NS+R-R}}\left\{\frac{\Omega}{2}\sum_{k=0}^{N-1}\frac{\alpha_N^k}{N}\cdot\frac{1+(-1)^F}{2}\cdot e^{-2\pi t(L_0+\bar{L}_0)}\right\},\\
&\mathrm{MS}: \quad \mathrm{Tr}^{99+55}_{\mathrm{NS-R}}\left\{\frac{\Omega}{2}\sum_{k=0}^{N-1}\frac{\alpha_N^k}{N}\cdot\frac{1+(-1)^F}{2}\cdot e^{-2\pi tL_0}\right\},\\
&\mathrm{C}: \quad \mathrm{Tr}^{99+55+95+59}_{\mathrm{NS-R}}\left\{\frac{1}{2}\sum_{k=0}^{N-1}\frac{\alpha_N^k}{N}\cdot\frac{1+(-1)^F}{2}\cdot e^{-2\pi tL_0}\right\}, \quad (14.16)
\end{aligned}$$

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**Insert 14.1. Jacobi's $\vartheta$-functions**

There are key seventeenth-century special functions which play a role in organising one-loop amplitudes:

$$\begin{aligned}\vartheta_1(z|t) &= 2q^{1/4}\sin\pi z \\ &\quad\times\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1-q^{2n}e^{2\pi iz})\prod_{n=1}^{\infty}(1-q^{2n}e^{-2\pi iz}),\\ \vartheta_2(z|t) &= 2q^{1/4}\cos\pi z \\ &\quad\times\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n}e^{2\pi iz})\prod_{n=1}^{\infty}(1+q^{2n}e^{-2\pi iz}),\\ \vartheta_3(z|t) &= \prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n-1}e^{2\pi iz})\prod_{n=1}^{\infty}(1+q^{2n-1}e^{-2\pi iz}),\\ \vartheta_4(z|t) &= \prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1-q^{2n-1}e^{2\pi iz})\prod_{n=1}^{\infty}(1-q^{2n-1}e^{-2\pi iz}),\end{aligned} \tag{14.17}$$

where $q = e^{-\pi t}$. We will need their asymptotics at $t \to 0$. Since the asymptotics as $t \to \infty$ are straightforward we can obtain the $t \to 0$ asymptotia from the modular transformations ($\tau = it$)

$$\begin{aligned}\vartheta_1(z|\tau) &= \tau^{-1/2}e^{3i\pi/4}e^{-i\pi z^2/\tau}\vartheta_1\left(\frac{z}{\tau}|-\frac{1}{\tau}\right),\\ \vartheta_3(z|\tau) &= \tau^{-1/2}e^{i\pi/4}e^{-i\pi z^2/\tau}\vartheta_3\left(\frac{z}{\tau}|-\frac{1}{\tau}\right),\\ \vartheta_2(z|\tau) &= \tau^{-1/2}e^{i\pi/4}e^{-i\pi z^2/\tau}\vartheta_4\left(\frac{z}{\tau}|-\frac{1}{\tau}\right),\\ \vartheta_4(z|\tau) &= \tau^{-1/2}e^{i\pi/4}e^{-i\pi z^2/\tau}\vartheta_2\left(\frac{z}{\tau}|-\frac{1}{\tau}\right).\end{aligned} \tag{14.18}$$

where $U(T)$ refers to the untwisted (twisted) sector of the closed string. Since $\Omega$ forces the left- and right-moving sector to be identical, there is no need to include $\frac{1}{2}(1+(-1)^{\bar{F}})$ in the trace in the Klein bottle. The open string traces include a sum over Chan–Paton factors.

The results can be written in terms of Jacobi's elliptic functions (see insert 14.1), generalising what we had before with a new twist. In the various loop channels, there is a twist by $\alpha_N^k$, which introduces a $z = \frac{k}{N}$

**Insert 14.2. The abstruse indentity**

The useful abstruce identity (*'aequatio identico satis abstrusa'*) ensures the cancellation between the R–R and NS–NS sectors (see chapter 7):

$$f_3(q)^8 - f_4(q)^8 - f_2^8(q) = 0. \tag{14.19}$$

It follows from the more general identities

$$\begin{aligned} \vartheta_3^2(0|t)\vartheta_3^2(z|t) - \vartheta_4^2(0|t)\vartheta_4^2(z|t) - \vartheta_2^2(0|t)\vartheta_2^2(z|t) &= 0 \\ \vartheta_3^2(0|t)\vartheta_2^2(z|t) - \vartheta_4^2(0|t)\vartheta_1^2(z|t) - \vartheta_2^2(0|t)\vartheta_3^2(z|t) &= 0 \\ \vartheta_3^2(0|t)\vartheta_4^2(z|t) - \vartheta_4^2(0|t)\vartheta_3^2(z|t) - \vartheta_2^2(0|t)\vartheta_1^2(z|t) &= 0, \end{aligned} \tag{14.20}$$

for the full $\vartheta$-functions. The twisting in the loop channels by $\alpha_N^k$ introduces $z = k/N$ into the oscillator sums, and hence we find that supersymmetry of the models are ensured by these more general identities representing the cancellations between twisted sectors.

into the oscillator sums since there is a shift in the energy of the twisted states.

In the case of the Klein bottle, there is also a twist in the closed string loop channel by $\alpha_N^{n-m}$. Such a space twist will in general change the moding of the fermion and bosons. This should manifest itself as another type of twist of the $\vartheta$-functions. To write this relationship, we use the notation

$$\vartheta_1 = \vartheta[{}^1_1], \quad \vartheta_2 = \vartheta[{}^0_1], \quad \vartheta_3 = \vartheta[{}^0_0], \quad \vartheta_4 = \vartheta[{}^1_0], \tag{14.21}$$

in which we can succinctly write[227]:

$$\vartheta[{}^{\epsilon}_{\epsilon'}](z - \zeta t|t) = e^{i\pi\{-\tau\zeta^2+\zeta\epsilon'+2\zeta z\}}\vartheta[{}^{\epsilon-2\zeta}_{\epsilon'}](z|t), \tag{14.22}$$

where $(\epsilon, \epsilon') \in \{0, 1\}$ for the familiar $\vartheta$-functions. In evaluating the Klein bottle amplitude, the identities (14.22) are used to rewrite twisted expressions in terms of $\vartheta$-functions.

For the twisted 99 cylinders the one-loop amplitudes are ($z{=}k/N$):

$$\begin{aligned} &\frac{V_6}{2^3N}\sum_{k=1}^{N-1}\frac{(\mathrm{Tr}(\gamma_{k,9}))^2}{(4\sin^2\pi z)^2}\int_0^\infty\frac{dt}{t}(8\pi^2\alpha' t)^{-3}\ 4\sin^2\pi z\ f_1^{-6}(t)\vartheta_1^{-2}(z|t) \\ &\times\left\{\vartheta_3^2(0|t)\vartheta_3^2(z|t) - \vartheta_4^2(0|t)\vartheta_4^2(z|t) - \vartheta_2^2(0|t)\vartheta_2^2(z|t)\right\}, \end{aligned} \tag{14.23}$$

while for the twisted 55 cylinders they are:

$$\frac{V_6}{2^3 N}\sum_{k=1}^{N-1}(\mathrm{Tr}(\gamma_{k,5}))^2\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-3}\ 4\sin^2\pi z\ f_1^{-6}(t)\vartheta_1^{-2}(z|t)$$
$$\times\left\{\vartheta_3^2(0|t)\vartheta_3^2(z|t)-\vartheta_4^2(0|t)\vartheta_4^2(z|t)-\vartheta_2^2(0|t)\vartheta_2^2(z|t)\right\}. \qquad (14.24)$$

The 95 cylinders give:

$$2\frac{V_6}{2^3 N}\sum_{k=1}^{N-1}\mathrm{Tr}(\gamma_{k,9})\mathrm{Tr}(\gamma_{k,5})\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-3}\ f_1^{-6}(t)\vartheta_4^{-2}(z|t)$$
$$\times\left\{\vartheta_3^2(0|t)\vartheta_2^2(z|t)-\vartheta_4^2(0|t)\vartheta_1^2(z|t)-\vartheta_2^2(0|t)\vartheta_3^2(z|t)\right\}. \qquad (14.25)$$

The twisted Möbius strip amplitudes are, for the D5-branes ($z{=}m/N$):

$$-\frac{V_6}{2^3 N}\sum_{m=1}^{N-1}\mathrm{Tr}(\gamma^{-1}_{\Omega_m,5}\gamma^{T}_{\Omega_m,5})\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-3}$$
$$\times 4\cos^2\pi z\, f_1^{-6}(iq)\vartheta_2^{-2}(iq,z)$$
$$\times\left\{\vartheta_3^2(iq,0)\vartheta_4^2(iq,z)-\vartheta_4^2(iq,0)\vartheta_3^2(iq,z)-\vartheta_2^2(iq,0)\vartheta_1^2(iq,z)\right\}, \quad (14.26)$$

and for the D9-branes:

$$-\frac{V_6}{2^3 N}\sum_{m=1}^{N-1}\frac{\mathrm{Tr}(\gamma^{-1}_{\Omega_m,9}\gamma^{T}_{\Omega_m,9})}{(4\sin^2\pi z)^2}\int_0^\infty \frac{dt}{t}(8\pi^2\alpha' t)^{-3}$$
$$\times 4\sin^2\pi z\ f_1^{-6}(iq)\vartheta_1^{-2}(iq,z)$$
$$\times\left\{\vartheta_3^2(iq,0)\vartheta_3^2(iq,z)-\vartheta_4^2(iq,0)\vartheta_4^2(iq,z)-\vartheta_2^2(iq,0)\vartheta_2^2(iq,z)\right\}. \quad (14.27)$$

Finally, the Klein bottle gives ($t^+{=}t+\xi t$, $t^-{=}t-\xi t$):

$$\frac{V_6}{2^3 N}\sum_{m,n=1}^{N-1}\frac{1}{(4\sin^2\pi z)^2}\int_0^\infty \frac{dt}{t}(4\pi^2\alpha' t)^{-3}\ 4\sin^2 2\pi(z-\zeta t)$$
$$\times f_1^{-6}(2t)\vartheta_1^{-1}(z|2t^-)\vartheta_1^{-1}(z|2t^+)$$
$$\times\left\{-\vartheta_4^2(0|2t)\vartheta_4(z|2t^-)\vartheta_4(z|2t^+)+\vartheta_3^2(0|2t)\vartheta_3(z|2t^-)\vartheta_3(z|2t^+)\right.$$
$$\left.-\ \vartheta_2^2(0|2t)\vartheta_2(z|2t^-)\vartheta_2(z|2t^+)\right\}. \qquad (14.28)$$

In the Klein bottle amplitudes, we have the twist $\zeta{=}(m-n)/N$ in the closed string channel, resulting in a zero point energy shift for the bosons and fermions which contribute. $V_6$ is the regularised six dimensional space-time volume.

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N.B. The $f$-functions we met in computations in chapters 4 and 7 are a special case of the functions (14.17), at $z = 0$, as shown in equation (4.44) (here, a prime denotes $\partial/\partial z$):

$$\begin{aligned} f_1(q) &= q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n}) = (2\pi)^{-1/3}\vartheta_1'(0|t)^{1/3} \\ f_2(q) &= \sqrt{2}q^{1/12}\prod_{n=1}^{\infty}(1+q^{2n}) = (2\pi)^{1/6}\vartheta_2'(0|t)^{1/2}\vartheta_1'(0|t)^{-1/6} \\ f_3(q) &= q^{-1/24}\prod_{n=1}^{\infty}(1+q^{2n-1}) = (2\pi)^{1/6}\vartheta_3'(0|t)^{1/2}\vartheta_1'(0|t)^{-1/6} \\ f_4(q) &= q^{-1/24}\prod_{n=1}^{\infty}(1-q^{2n-1}) = (2\pi)^{1/6}\vartheta_4'(0|t)^{1/2}\vartheta_1'(0|t)^{-1/6}. \end{aligned} \tag{14.29}$$

The factor of $(4\sin^2\pi z)^{-2}$ is a non-trivial contribution from evaluating the trace of the operator $\mathcal{O}$ in the $z^1$ and $z^2$ complex planes in the NN sector. The operator $\mathcal{O}$ is the rotation

$$\mathcal{O}: \quad z^{1,2} \to e^{\pm\frac{2\pi ik}{N}} z^{1,2}. \tag{14.30}$$

We have

$$\mathrm{Tr}[e^{\frac{2\pi ik}{N}}] = \int dz^1 dz^2 \langle z^1, z^2|\mathcal{O}|z^{1\prime}, z^{2\prime}\rangle = \left(4\sin^2\frac{\pi k}{N}\right)^{-2}, \tag{14.31}$$

where we have used the basis

$$\langle z^1, z^2|z^{1\prime}, z^{2\prime}\rangle = \frac{1}{V_{T^4}}\delta(z^1 - z^{1\prime})\delta(z^2 - z^{2\prime}). \tag{14.32}$$

Supersymmetry is manifest here, as due to the identities (14.20) each of these amplitudes vanishes identically. However, we wish to extract the tadpoles for closed string massless NS–NS fields from these amplitudes, and we do so by identifying the contribution of this sector from each of these amplitudes.

### *14.2.3 Extracting the tadpoles*

The next step is to extract the asymptotics as $t \to 0$ of the amplitudes, relating this limit to the $l \to \infty$ limit for each surface, (using the relation between $l$ and $t$ for each surface given earlier). Here, the asymptotic

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behaviour of the $\vartheta$-functions given in equations (14.18) are used. This extracts the (divergent) contribution of the massless closed string R–R fields, which we shall list below. In what follows, we shall neglect the overall factors of $1/N$ and powers of two which accompany all of the amplitudes.

First, we list the tadpoles for the untwisted R–R potentials. For the ten-form we have the following expression (proportional to $(1-1)v_6v_4\int_0^\infty dl$):

$$\mathrm{Tr}(\gamma_{0,9})^2 - 64\mathrm{Tr}(\gamma_{\Omega,9}^{-1}\gamma_{\Omega,9}^T) + 32^2, \qquad (14.33)$$

corresponding to the diagrams in figure 14.2

Here, $v_D = V_D(4\pi^2\alpha')^{-D/2}$, where $V_D$ is a $D$ dimensional volume. The limit where we focus upon the neighbourhood of one ALE point is equivalent to taking the non-compact limit $v_4 \to \infty$ while keeping $v_{10} = v_6v_4$ finite.

For the six-form we have (proportional to $(1-1)\frac{v_6}{v_4}\int_0^\infty dl$):

$$\mathrm{Tr}(\gamma_{0,5})^2 - 64\mathrm{Tr}(\gamma_{\Omega_{\frac{N}{2}},5}^{-1}\gamma_{\Omega_{\frac{N}{2}},5}^T) + 32^2, \qquad (14.34)$$

which arise from the diagrams in figure 14.3.

In the non-compact limit we are considering here, this last contribution does not survive, as it is proportional to $v_6/v_4$. The fact that it vanishes is consistent with the fact that if space is not compact, there is no restriction from charge conservation on the number of D5-branes which may be present: the analogue of Gauss's Law for the six-form potential's field strength does not apply, as the flux lines can stretch to infinity. In the compact case, they must begin and end all within the compact volume. So this equation will be relevant only when we return to the study of global six-form charge cancellation in the compact K3 examples.

Notice also in this case that the last two diagrams obviously vanish in the case when $N$ is odd. An immediate consequence of this is that $\mathbb{Z}_3$ fixed points have no six-form charge.

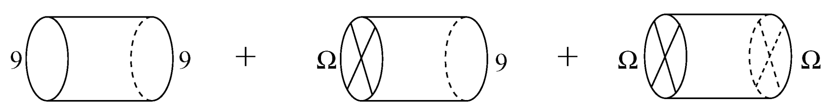


Fig. 14.2. The tadpoles for D9-branes.

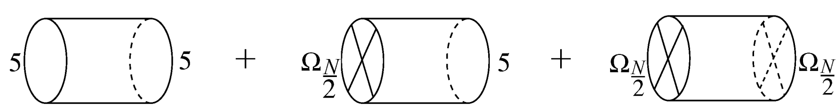


Fig. 14.3. The tadpoles for D5-branes.

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The twisted sector tadpoles are (proportional to $(1-1)v_6 \int_0^\infty dl$):

$$\sum_{\substack{k=1\\ k\neq\frac{N}{2}}}^{N-1}\left[\frac{1}{4\sin^2\frac{\pi k}{N}}\mathrm{Tr}(\gamma_{k,9})^2 - 2\mathrm{Tr}(\gamma_{k,9})\mathrm{Tr}(\gamma_{k,5}) + 4\sin^2\frac{\pi k}{N}\,\mathrm{Tr}(\gamma_{k,5})^2\right]$$
$$-16\sum_{\substack{k=1\\ k\neq\frac{N}{2}}}^{N-1}\left[4\cos^2\frac{\pi k}{N}\,\mathrm{Tr}(\gamma^{-1}_{\Omega_k,5}\gamma^T_{\Omega_k,5}) + \frac{1}{4\sin^2\frac{\pi k}{N}}\mathrm{Tr}(\gamma^{-1}_{\Omega_k,9}\gamma^T_{\Omega_k,9})\right]$$
$$+64\sum_{\substack{k=1\\ k\neq\frac{N}{2}}}^{N-1}\left[\frac{\cos^2\frac{\pi k}{N}}{\sin^2\frac{\pi k}{N}} - \delta_{N\,\mathrm{mod}\,2,0}\right]. \tag{14.35}$$

These tadpoles correspond to the diagrams in figure 14.4.

Notice that since $\alpha_N^k$ and $\alpha_N^{k+N/2}$ both square to the same element, $\alpha_N^{2k}$, we can make opposite phase choices in the composition algebra of the $\gamma_{\Omega_k}$ matrices:

$$\begin{aligned}\mathrm{Tr}[\gamma^{-1}_{\Omega_k,9}\gamma^T_{\Omega_k,9}] &= \mathrm{Tr}[\gamma_{2k,9}]\\ \mathrm{Tr}[\gamma^{-1}_{\Omega_{k+\frac{N}{2}},9}\gamma^T_{\Omega_{k+\frac{N}{2}},9}] &= -\mathrm{Tr}[\gamma_{2k,9}]\end{aligned} \tag{14.36}$$

for D9-branes and

$$\begin{aligned}\mathrm{Tr}[\gamma^{-1}_{\Omega_k,5}\gamma^T_{\Omega_k,5}] &= -\mathrm{Tr}[\gamma_{2k,5}]\\ \mathrm{Tr}[\gamma^{-1}_{\Omega_{k+\frac{N}{2}},5}\gamma^T_{\Omega_{k+\frac{N}{2}},5}] &= \mathrm{Tr}[\gamma_{2k,5}]\end{aligned} \tag{14.37}$$

for D5-branes. This is more than an aesthetic choice, as the first line of each of these conditions is simply the crucial result derived in section 8.7

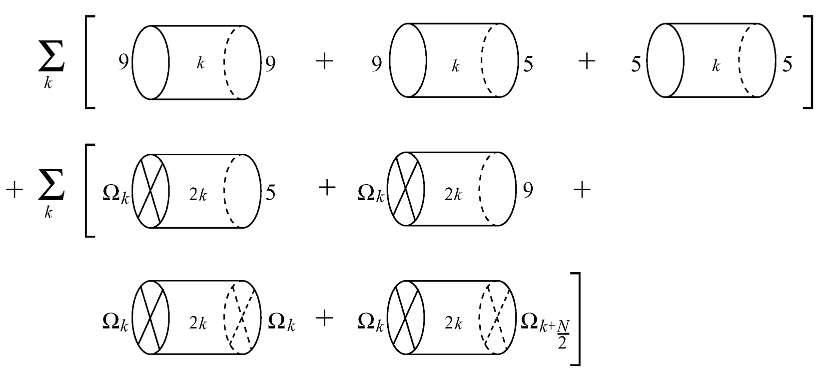

Fig. 14.4. The tadpoles for twisted sectors.

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that $\Omega^2$=1 in the 99 sector, but $-1$ in the 55 sector[132]. The second line in each is the statement that $\gamma^2_{N/2}=-1$ in each sector.

With (14.36) and (14.37), the expression (14.35) can be factorised, for even $N$:

$$\sum_{k=1}^{\frac{N}{2}} \frac{1}{4\sin^2\frac{(2k-1)\pi}{N}} \left[\mathrm{Tr}(\gamma_{2k-1,9}) - 4\sin^2\frac{(2k-1)\pi}{N}\,\mathrm{Tr}(\gamma_{2k-1,5})\right]^2$$
$$\sum_{k=1}^{\frac{N}{2}} \frac{1}{4\sin^2\left(\frac{2\pi k}{N}\right)} \left[\mathrm{Tr}(\gamma_{2k,9}) - 4\sin^2\left(\frac{2\pi k}{N}\right)\mathrm{Tr}(\gamma_{2k,5}) - 32\cos\left(\frac{2\pi k}{N}\right)\right]^2 \tag{14.38}$$

and for odd $N$:

$$\sum_{k=1}^{M-1} \frac{1}{4\sin^2\frac{\pi k}{N}} \left[\mathrm{Tr}(\gamma_{2k,9}) - 4\sin^2\left(\frac{2\pi k}{N}\right)\mathrm{Tr}(\gamma_{2k,5}) - 32\cos^2\frac{\pi k}{N}\right]^2. \tag{14.39}$$

Having extracted the divergences and factorised them, revealing the tadpole equations (which may be also interpreted as charge cancellation equations, as discussed earlier) we are ready to find ways of solving these equations for the various orientifold groups.

## 14.3 Solving the tadpole equations

### *14.3.1 T-duality relations*

Compact manifolds which can be constructed as $T^4/\mathbb{Z}_N$ (as described in section 7.6.5) exist only for $N=2,3,4$ and 6. From the discussion in section 14.1, we can therefore construct orientifolds of type $A$ for all these values of $N$, but of type $B$ only for $N=2,4$ and 6. We list below explicitly the orientifold groups:

$$\begin{aligned}
&\mathbb{Z}_2^A=\{1,\alpha_2^1,\Omega,\Omega\alpha_2^1\}, \quad \mathbb{Z}_2^B=\{1,\Omega\dot{\alpha_2}\},\\
&\mathbb{Z}_3^A=\{1,\alpha_3^1,\alpha_3^2,\Omega,\Omega\alpha_3^1,\Omega\alpha_3^2\},\\
&\mathbb{Z}_4^A=\{1,\alpha_4^1,\alpha_4^2,\alpha_4^3,\Omega,\Omega\alpha_4^1,\Omega\alpha_4^2,\Omega\alpha_4^3,\}, \quad \mathbb{Z}_4^B=\{1,\alpha_4^2,\Omega\alpha_4^1,\Omega\alpha_4^3,\},\\
&\mathbb{Z}_6^A=\{1,\alpha_6^1,\dots,\alpha_6^5,\Omega,\Omega\alpha_6^1,\dots,\Omega\alpha_6^5\},\\
&\mathbb{Z}_6^B=\{1,\alpha_6^2,\alpha_6^4,\Omega\alpha_6^1,\Omega\alpha_6^3,\Omega\alpha_6^5\},
\end{aligned} \tag{14.40}$$

where $\alpha_N^{\frac{N}{2}}\equiv R$. In equation (14.33) for the untwisted ten-form potential, $\mathrm{Tr}(\gamma_{0,9})=2n_9$, where $n_9$ is the number of D9-branes. All of the orientifold groups of type $A$ contain the element $\Omega$, and therefore there will be an equation of the form (14.33), telling us that there are 16 D9-branes.

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Similarly, all type $A$ models except $\mathbb{Z}_3^A$ will contain 16 D5-branes also, as the presence of an element $\Omega R$ means that there will be an equation of the form (14.34).

In contrast, the models of type $B$ all lack the element $\Omega$ and therefore have only the first term of equation (14.33). The only solution is that the number of D9-branes in these models is zero. All type $B$ models except $\mathbb{Z}_4^B$ have the element $\Omega R$, and therefore have 16 D5-branes. So $\mathbb{Z}_4^B$ has the distinction of having no open string sectors at all: it is a consistent unoriented closed string theory

Now T-duality in the $(x^6, x^7, x^8, x^9)$ directions exchanges the elements $\Omega$ and $\Omega R$. This also exchanges D9-branes with D5-branes. So models $\mathbb{Z}_2^A$, $\mathbb{Z}_4^A$, $\mathbb{Z}_6^A$ and $\mathbb{Z}_4^B$ are self $\mathrm{T}_{6789}$-dual. Meanwhile $\mathbb{Z}_3^A$, which has only D9-branes, is dual to $Z_6^B$ which has only D5-branes. $\mathbb{Z}_2^B$, which has only $\Omega R$ as a non-trivial element of its orientifold group, is dual to ordinary type I string theory (which we may denote as $\mathbb{Z}_1^A$), whose orientifold group has only $\Omega$ as its non-trivial element. This is summarised in figure 14.5.

### *14.3.2 Explicit solutions*

Let us write out the tadpole equations explicitly in each case. For the $\mathbb{Z}_2^A$ case, for which there is one twisted tadpole equation (recall $\alpha_2^1 \equiv R$):

$$\mathrm{Tr}[\gamma_{1,9}] - 4\mathrm{Tr}[\gamma_{1,5}] = 0. \tag{14.41}$$

A solution is

$$\begin{aligned} \gamma_{\Omega,9} &= \gamma_{\Omega R,5} = I_{32} \\ \gamma_{R,9} &= \gamma_{\Omega,5} = \begin{pmatrix} 0 & I_{16} \\ -I_{16} & 0 \end{pmatrix}. \end{aligned}$$

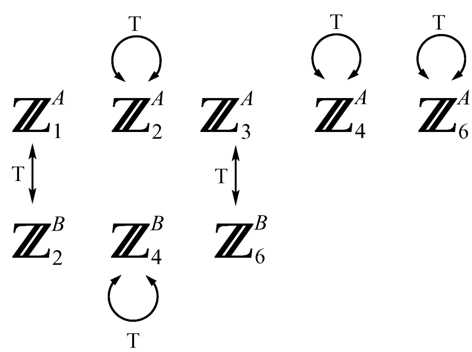


Fig. 14.5. T-duality relations between the models.

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For the $\mathbb{Z}_3^A$ case we have

$$\begin{aligned} \mathrm{Tr}[\gamma_{1,9}] - 3\mathrm{Tr}[\gamma_{1,5}] &= 8 \\ \mathrm{Tr}[\gamma_{2,9}] - 3\mathrm{Tr}[\gamma_{2,5}] &= 8, \end{aligned} \tag{14.42}$$

and since we have already learned that the number of D5-branes is zero in this case, we have solution $\gamma_5 = 0$ for all orientifold elements in the D5-brane sector, and we can write

$$\begin{aligned} \gamma_{\Omega,9} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \otimes I_8, \\ \gamma_{1,9} &= \mathrm{diag}\{e^{\frac{2\pi i}{3}}\,(8 \text{ times}), e^{-\frac{2\pi i}{3}}\,(8 \text{ times}), 1\,(16 \text{ times})\}, \end{aligned}$$

from which it is trivially verified that (14.42) is satisfied.

Notice that $\gamma_\Omega$ acts by exchanging the roots and their complex conjugates: $e^{\frac{2\pi i}{3}} \leftrightarrow e^{-\frac{2\pi i}{3}}$. This will be the case in all of the later models, and so we will no longer list it explicitly in the later solutions. Note also that in the other type $A$ orientifolds, $\gamma_{\Omega,9} = \gamma_{\Omega R,5}$, and $\gamma_{\Omega R,9} = \gamma_{\Omega,5}$. We can always choose a phase such that we can always write $\gamma_{1,9} = e^{\frac{2\pi i m}{N}}\gamma_{1,5}$, for $m$ any odd integer. This simple relationship between the $\gamma$ matrices in the D5- and D9-brane sectors is a manifestation of $\mathrm{T}_{6789}$ duality.

For the $\mathbb{Z}_4^A$ case we have:

$$\begin{aligned} \mathrm{Tr}[\gamma_{1,9}] - 2\mathrm{Tr}[\gamma_{1,5}] &= 0 \\ \mathrm{Tr}[\gamma_{2,9}] - 4\mathrm{Tr}[\gamma_{2,5}] &= 0 \\ \mathrm{Tr}[\gamma_{3,9}] - 2\mathrm{Tr}[\gamma_{3,5}] &= 0. \end{aligned} \tag{14.43}$$

Note that the middle case correctly reproduces the $\mathbb{Z}_2^A$ example, and therefore the $\mathbb{Z}_2^A$ example appears as a substructure. This will be true for $\mathbb{Z}_6^A$ also.

The solution is ($\alpha_4^2 \equiv R$):

$$\gamma_{1,9} = \mathrm{diag}\{e^{\frac{\pi i}{4}}\,(8\,\text{times}), e^{-\frac{\pi i}{4}}\,(8\,\text{times}), e^{\frac{3\pi i}{4}}\,(8\,\text{times}), e^{-\frac{3\pi i}{4}}\,(8\,\text{times})\}.$$

For the $\mathbb{Z}_6^A$ case we have:

$$\begin{aligned} \mathrm{Tr}[\gamma_{1,9}] - \mathrm{Tr}[\gamma_{1,5}] &= 0 \\ \mathrm{Tr}[\gamma_{2,9}] - 3\mathrm{Tr}[\gamma_{2,5}] &= 16 \\ \mathrm{Tr}[\gamma_{3,9}] - 4\mathrm{Tr}[\gamma_{3,5}] &= 0 \\ \mathrm{Tr}[\gamma_{4,9}] - 3\mathrm{Tr}[\gamma_{4,5}] &= -16 \\ \mathrm{Tr}[\gamma_{5,9}] - \mathrm{Tr}[\gamma_{5,5}] &= 0, \end{aligned} \tag{14.44}$$

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for which we have ($\alpha_4^2 \equiv R$):

$$\mathrm{Tr}[\gamma_{1,9}] = \mathrm{Tr}[\gamma_{1,5}] = 0,\ \ \mathrm{Tr}[\gamma_{3,9}] = \mathrm{Tr}[\gamma_{3,5}] = 0,\ \ \mathrm{Tr}[\gamma_{5,9}] = \mathrm{Tr}[\gamma_{5,5}] = 0,$$
$$\mathrm{Tr}[\gamma_{2,9}] = \mathrm{Tr}[\gamma_{2,5}] = -8,\ \ \mathrm{Tr}[\gamma_{4,9}] = \mathrm{Tr}[\gamma_{4,5}] = 8,$$
$$\gamma_{1,9} = \mathrm{diag}\{e^{\frac{\pi i}{6}}\,(4\text{ times}), e^{-\frac{\pi i}{6}}\,(4\text{ times}),$$
$$e^{\frac{5\pi i}{6}}\,(4\text{ times}), e^{-\frac{5\pi i}{6}}\,(4\text{ times}), -i\,(8\text{ times}), i\,(8\text{ times})\}.$$

Note here that:

$$\gamma_{1,9} \equiv \mathrm{diag}\{e^{\frac{2\pi i}{3}}\,(4\text{ times}), e^{-\frac{2\pi i}{3}}\,(4\text{ times}), 1\,(8\text{ times})\} \otimes \mathrm{diag}\{-i, i\},$$

which shows a $\mathbb{Z}_3 \times \mathbb{Z}_2$ structure, using the solutions previously obtained for the $\mathbb{Z}_2^A$ and $\mathbb{Z}_3^A$ models.

Also notice that in all cases above, the coefficient of the $\gamma_{k,5}$ trace is the square root of the number of fixed points invariant under $\alpha_N^k$. Also interesting is that (generalising the $\mathbb{Z}_2$ case,) the same choice made for D9-branes can be made for D5-branes, up to a phase.

The tadpoles for the case $\mathbb{Z}_6^B$ will turn out to be isomorphic to those listed above for $\mathbb{Z}_3^A$, while there are no tadpoles to list for the $\mathbb{Z}_4^B$ model, as there are no D-branes required.

Let us now turn to the closed string spectra.

## 14.4 Closed string spectra

Just as we did in section 7.6, we should construct the closed string spectrum for each model, which we will combine with the open string spectrum later. The procedure is much the same as we did for the K3 orbifold, except that we will apply the orientifold projection, which will throw out more states.

The right-moving untwisted sector has the massless states given below.

| Sector | State | $\alpha_N^k$ | $SO(4)$ charge |
|---|---|---|---|
| NS | $\psi^{\mu}_{-1/2}\|0\rangle$ | $1$ | $(\mathbf{2},\mathbf{2})$ |
| | $\psi^{1\pm}_{-1/2}\|0\rangle$ | $e^{\pm\frac{2\pi ik}{N}}$ | $2(\mathbf{1},\mathbf{1})$ |
| | $\psi^{2\pm}_{-1/2}\|0\rangle$ | $e^{\mp\frac{2\pi ik}{N}}$ | $2(\mathbf{1},\mathbf{1})$ |
| R | $\|s_1s_2s_3s_4\rangle$ | | |
| | $s_1 = +s_2,\ s_3 = +s_4$ | $1$ | $2(\mathbf{2},\mathbf{1})$ |
| | $s_1 = -s_2,\ s_3 = -s_4$ | $e^{\pm\frac{2\pi ik}{N}}$ | $2(\mathbf{1},\mathbf{2})$ |

Meanwhile, the right-moving sector twisted by $\frac{m}{N} \neq \frac{1}{2}$ has the following states.

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| Sector | State | $\alpha_N^k$ | $SO(4)$ charge |
|---|---|---|---|
| NS | $\psi^{1+}_{-1/2+m/N}\|0\rangle$ | $e^{\frac{2\pi ik}{N}(1-2\frac{m}{N})}$ | **(1,1)** |
| | $\psi^{2+}_{-1/2+m/N}\|0\rangle$ | $e^{\frac{2\pi ik}{N}(1-2\frac{m}{N})}$ | **(1,1)** |
| R | $\|s_1s_2\rangle,\ s_1=-s_2$ | $e^{\frac{2\pi ik}{N}(1-2\frac{m}{N})}$ | **(1,2)** |

The exception to the situation depicted there is when we have an $\frac{m}{N}=\frac{1}{2}$ twist.

| Sector | State | $\alpha_N^k$ | $SO(4)$ charge |
|---|---|---|---|
| NS | $\|s_3s_4\rangle,\ s_3=+s_4$ | 1 | 2**(1,1)** |
| R | $\|s_1s_2\rangle,\ s_1=-s_2$ | 1 | **(1,2)** |

We have imposed the GSO projection, and decomposed the little group of the spacetime Lorentz group as $SO(4)=SU(2)\times SU(2)$, just as in section 7.6. We form the spectrum for orientifold group of type $A$ by taking products of states from the left and right sectors (to give states invariant under $\alpha_N^k$), symmetrised by the $\Omega$ projection in the NS–NS sector, while antisymmetrising in the R–R sector, since we have fermions there.

In this way, we have from the untwisted closed string sector of the type $A$ $\mathbb{Z}_N$ orientifold ($N\neq 2$).

| Sector | $SO(4)$ charge |
|---|---|
| NS–NS | **(3,3)+5(1,1)** |
| R–R | **(3,1)+(1,3)+4(1,1)** |

This is the content of the $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and two hypermultiplets (see table 14.3, p. 341, for a list of the types of multiplet). In the case of $\mathbb{Z}_2$ it is as follows[132].

| Sector | $SO(4)$ charge |
|---|---|
| NS–NS | **(3,3)+11(1,1)** |
| R–R | **(3,1)+(1,3)+6(1,1)** |

This is the $D=6$, $\mathcal{N}=1$ supergravity multiplet in six dimensions, accompanied by one tensor multiplet and four hypermultiplets

The twisted sectors will produce additional multiplets. The bosonic content of a hypermultiplet is four scalars $4(\mathbf{1},\mathbf{1})$, while that of a tensor multiplet is $(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1})$. By combining on the left and right the sectors twisted by $\frac{m}{N}$ and $(1-\frac{m}{N})$, we find that the NS–NS sector produces one hypermultiplet while the R–R sector produces a tensor multiplet. A sector

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twisted by $\frac{1}{2}$ simply produces one hypermultiplet: one quarter coming from the R–R sector and three quarters from the NS–NS sector, just as in section 7.6.

To evaluate the number of hypermultiplets (and, as we shall see, tensor multiplets) coming from the twisted sectors of a K3 orbifold, we need to recall the structure of the fixed points and their transformation properties, done in section 7.6.5. Using the data above, we see that in the case of $\mathbb{Z}_2^A$, we simply multiply by the number of $\mathbb{Z}_2$ fixed points, and we find that there are 16 hypermultiplets from the twisted sectors, giving a total of 20 hypermultiplets when combined with the four from the untwisted sector. For $\mathbb{Z}_3^A$ there are nine fixed points, each supplying a hypermultiplet and a tensor multiplet (for twists by $(\frac{1}{3}, \frac{2}{3})$), giving a total of 11 hypermultiplets and 10 tensor multiplets when added to those arising in the untwisted sector.

For $\mathbb{Z}_4^A$ the four $\mathbb{Z}_4$ invariant fixed points give four hypermultiplets and four tensor multiplets. They are also $\mathbb{Z}_2$ fixed points and so supply an additional four hypermultiplets. The other 12 $\mathbb{Z}_2$ fixed points form six $\mathbb{Z}_4$ invariant pairs, from which arise six hypermultiplets. This gives a total of 16 hypermultiplets and five tensor multiplets for the complete model.

Finally, for the model $\mathbb{Z}_6^A$, the $\mathbb{Z}_6$ fixed point gives two hypermultiplets and two tensor multiplets from $(\frac{1}{6}, \frac{5}{6})$ and $(\frac{1}{3}, \frac{2}{3})$ twists. It also gives six hypermultiplets from the $\frac{1}{2}$ twisted sector. The four pairs of $\mathbb{Z}_3$ points give four tensor multiplets and four hypermultiplets for $(\frac{1}{3}, \frac{2}{3})$ twists, while the five $\mathbb{Z}_2$ triplets of fixed points supply five hypermultiplets. This gives 14 hypermultiplets and seven tensor multiplets in all.

For $\mathbb{Z}_N$ orientifolds of type $B$ the situation is as follows. For closed string states, prior to making the theory unorientable, the relevant orbifold states to consider are those for the group formed by the remaining pure $\mathbb{Z}_N$ elements in the orientifold group, which is therefore $\mathbb{Z}_{\frac{N}{2}}$. The possible left and right states are evaluated as before, and then they are projected to the unoriented theory invariant under $\Omega \cdot \alpha_N^1$.

It is thus easy to see that the closed string spectra for $\mathbb{Z}_6^B$ and $\mathbb{Z}_3^A$ are isomorphic, as are those of $\mathbb{Z}_2^B$ and $\mathbb{Z}_1^A$, (the latter being simply ten dimensional type I string theory: there is no orbifold to perform for $\mathbb{Z}_2^B$).

There remains only the spectrum of $\mathbb{Z}_4^B$ to compute, which is self T-dual. The pure orbifold states to consider are those of $\mathbb{Z}_2$. Tensoring left and right to form the $\Omega$ invariant spectrum, we obtain 12 hypermultiplets and nine tensor multiplets in total.

In summary, we have (in addition to the usual gravity and tensor multiplet) a spectrum of hypermultiplets and tensor multiplets from the closed string sector for each model, given in table 14.1.

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Table 14.1. *The spectrum of hypermultiplets and tensor multiplets in the various orientifold models*

| Model | Neutral hypermultiplets | Extra tensor multiplets |
|---|---|---|
| $\mathbb{Z}_2^A$ | 20 | 0 |
| $\mathbb{Z}_3^A$ | 11 | 9 |
| $\mathbb{Z}_4^A$ | 16 | 4 |
| $\mathbb{Z}_6^A$ | 14 | 6 |
| $\mathbb{Z}_4^B$ | 12 | 8 |
| $\mathbb{Z}_6^B$ | 11 | 9 |

Notice that in the $\mathbb{Z}_2^A$ case we have a total of 20 hypermultiplets. Looking at table 14.3 we see that this is a total of 80 scalar fields: the 80 moduli of the K3 surface[228]. However, in the other other orientifold examples, some of the potential scalars have instead combine into tensor multiplets, leaving us with fewer hypermultiplets in the final model. This is a reduction in the dimension of the moduli space of K3 deformations available to these models: some of the K3 moduli are frozen.

To complete the story, let us turn to the open string sector, computing what the allowed gauge groups are.

## 14.5 Open string spectra

Let us study first the 99 open string sector. The massless bosonic spectrum arises as follows.

For the 55 states at a fixed point we have the following.

| state | $\alpha_N^k = +$ | $\Omega = +$ | $SO(4)$ charge |
|---|---|---|---|
| $\psi^{\mu}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = \gamma_{k,9}\lambda\gamma_{k,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ | **(2,2)** |
| $\psi^{1\pm}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = e^{\pm\frac{2\pi k}{N}}\gamma_{k,9}\lambda\gamma_{k,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ | 2**(1,1)** |
| $\psi^{2\pm}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = e^{\mp\frac{2\pi k}{N}}\gamma_{k,9}\lambda\gamma_{k,9}^{-1}$ | $\lambda = -\gamma_{\Omega,9}\lambda^T\gamma_{\Omega,9}^{-1}$ | 2**(1,1)** |

| state | $\alpha_N^k = +$ | $\Omega = +$ | $SO(4)$ charge |
|---|---|---|---|
| $\psi^{\mu}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = \gamma_{k,5}\lambda\gamma_{k,5}^{-1}$ | $\lambda = -\gamma_{\Omega,5}\lambda^T\gamma_{\Omega,5}^{-1}$ | **(2,2)** |
| $\psi^{1\pm}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = e^{\pm\frac{2\pi k}{N}}\gamma_{k,5}\lambda\gamma_{k,5}^{-1}$ | $\lambda = \gamma_{\Omega,5}\lambda^T\gamma_{\Omega,5}^{-1}$ | 2**(1,1)** |
| $\psi^{2\pm}_{-1/2}\|0,ij\rangle\lambda_{ij}$ | $\lambda = e^{\mp\frac{2\pi k}{N}}\gamma_{k,5}\lambda\gamma_{k,5}^{-1}$ | $\lambda = \gamma_{\Omega,5}\lambda^T\gamma_{\Omega,5}^{-1}$ | 2**(1,1)** |

For the 55 states away from a fixed point we have the following.

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| state | $\Omega = +$ | $SO(4)$ charge |
|---|---|---|
| $\psi^{\mu}_{-1/2}\|0, ij\rangle\lambda_{ij}$ | $\lambda = -\gamma_{\Omega,5}\lambda^T\gamma^{-1}_{\Omega,5}$ | **(2,2)** |
| $\psi^{1\pm}_{-1/2}\|0, ij\rangle\lambda_{ij}$ | $\lambda = \gamma_{\Omega,5}\lambda^T\gamma^{-1}_{\Omega,5}$ | 2**(1,1)** |
| $\psi^{2\pm}_{-1/2}\|0, ij\rangle\lambda_{ij}$ | $\lambda = \gamma_{\Omega,5}\lambda^T\gamma^{-1}_{\Omega,5}$ | 2**(1,1)** |

For the 59 states we have the following at a fixed point.

| state | $\alpha^k_N = +$ | $SO(4)$ charge |
|---|---|---|
| $\|s_3 s_4, ij\rangle\lambda_{ij}, s_3 = s_4$ | $\lambda = \gamma_{k,5}\lambda\gamma^{-1}_{k,9}$ | **2(1,1)** |

Away from a fixed point we have the following.

| state | $SO(4)$ charge |
|---|---|
| $\|s_3 s_4, ij\rangle\lambda_{ij},\ s_3 = s_4$ | 2**(1,1)** |

Using the solution presented in section 14.4 for the $\gamma$ matrices, the solutions for the open string spectra of the models are given in table 14.2.

Table 14.2. *The gauge groups for the various orientifold models*

| Model | Sector | Gauge group | Charged hypermultiplets |
|---|---|---|---|
| $\mathbb{Z}_2^A$ | 99 | $U(16)$ | $2 \times \mathbf{120}$ |
| | 55 | $U(16)$ | $2 \times \mathbf{120}$ |
| | 59 | | $(\mathbf{16}, \mathbf{16})$ |
| $\mathbb{Z}_3^A$ | 99 | $U(8) \times SO(16)$ | $(\mathbf{28}, \mathbf{1}),\ (\mathbf{8}, \mathbf{16})$ |
| $\mathbb{Z}_4^A$ | 99 | $U(8) \times U(8)$ | $(\mathbf{28}, \mathbf{1}),\ (\mathbf{1}, \mathbf{28}),\ (\mathbf{8}, \mathbf{8})$ |
| | 55 | $U(8) \times U(8)$ | $(\mathbf{28}, \mathbf{1}),\ (\mathbf{1}, \mathbf{28}),\ (\mathbf{8}, \mathbf{8})$ |
| | 59 | | $(\mathbf{8}, \mathbf{1}; \mathbf{8}, \mathbf{1}),\ (\mathbf{1}, \mathbf{8}; \mathbf{1}, \mathbf{8})$ |
| $\mathbb{Z}_6^A$ | 99 | $U(4) \times U(4) \times U(8)$ | $(\mathbf{6}, \mathbf{1}, \mathbf{1}),\ (\mathbf{1}, \mathbf{6}, \mathbf{1})$ <br> $(\mathbf{4}, \mathbf{1}, \mathbf{8}),\ (\mathbf{1}, \mathbf{4}, \mathbf{8})$ |
| | 55 | $U(4) \times U(4) \times U(8)$ | $(\mathbf{6}, \mathbf{1}, \mathbf{1}),\ (\mathbf{1}, \mathbf{6}, \mathbf{1})$ <br> $(\mathbf{4}, \mathbf{1}, \mathbf{8}),\ (\mathbf{1}, \mathbf{4}, \mathbf{8})$ |
| | 59 | | $(\mathbf{4}, \mathbf{1}, \mathbf{1}; \mathbf{4}, \mathbf{1}, \mathbf{1})$ <br> $(\mathbf{1}, \mathbf{4}, \mathbf{1}; \mathbf{1}, \mathbf{4}, \mathbf{1})$ <br> $(\mathbf{1}, \mathbf{1}, \mathbf{8}; \mathbf{1}, \mathbf{1}, \mathbf{8})$ |
| $\mathbb{Z}_4^B$ | | — | — |
| $\mathbb{Z}_6^B$ | 55 | $U(8) \times SO(16)$ | $(\mathbf{28}, \mathbf{1}),\ (\mathbf{8}, \mathbf{16})$ |

https://doi.org/10.1017/9781009401371.015 Published online by Cambridge University Press

## 14.6 Anomalies for $\mathcal{N} = 1$ in six dimensions

As chiral models in six dimensions, these orientifold vacua that we have constructed have a chance of being afflicted by anomalies. There are anomaly polynomials which we can write for each sector, just as was done in ten dimensions in chapter 7, and in six dimensions in section 7.6.2.

Of course, we can be confident that if we have done everything properly at the string level, checking the anomaly is nothing more than a formality, but it is instructive anyway, as we have seen before. Before we proceed, we must pause to note the structure of the multiplets. In fact, they arise naturally from splitting the $\mathcal{N} = 2$ multiplets which we have seen in table 7.1. Both the vector and tensor multiplets become smaller by yielding up a hypermultiplet, whose bosonic part is four scalars, as we have already seen. Table 14.3 lists the multiplets.

So again we see that the orientifolding has thrown away many pieces of the pure K3 spectrum, and so the marvellous cancellation in equation (7.52) will not happen. As in the prototype orientifold, we have additional pieces as well, coming from the open string sectors, which may give some new interesting structures.

The first thing to check is that the irreducible parts of the anomaly cancel. For the gravitational part, we must look at the coefficient of $\mathrm{tr}R^4$. In fact, for $\mathcal{N} = 1$ models in $D = 6$ it is easy to see that the vanishing of this coefficient is equivalent to:

$$n_{\mathrm{H}} - n_{\mathrm{V}} = 244 - 29 n_{\mathrm{T}}. \tag{14.45}$$

The reader should verify this. Here, $n_{\mathrm{H}}, n_{\mathrm{V}}$ and $n_{\mathrm{T}} + 1$ are respectively the numbers of hypermultiplets, vector multiplets and tensor multiplets in the six dimensional supergravity model. This follows from direct use of the anomaly polynomials in insert 7.2, remembering to divide the polynomials for the complex fermions listed there by two to match the real fermions we

Table 14.3. *The structure of the $\mathcal{N} = 1$ multiplets in $D = 6$*

| Multiplet | Bosons | Fermions |
|---|---|---|
| vector | $\mathbf{(2,2)}$ | $2\mathbf{(1,2)}$ |
| hyper | $4\mathbf{(1,1)}$ | $2\mathbf{(2,1)}$ |
| SD tensor | $\mathbf{(1,3)}+\mathbf{(1,1)}$ | $2\mathbf{(2,1)}$ |
| ASD tensor | $\mathbf{(3,1)}+\mathbf{(1,1)}$ | $2\mathbf{(2,1)}$ |
| supergravity | $(\mathbf{3},\mathbf{3}) + (\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{1},\mathbf{1})$ | $2(\mathbf{3},\mathbf{2}) + 2(\mathbf{2},\mathbf{1})$ or $2(\mathbf{2},\mathbf{3}) + 2(\mathbf{1},\mathbf{2})$ |

https://doi.org/10.1017/9781009401371.015 Published online by Cambridge University Press

are counting with here. It is natural that the vectors and hypers contribute equal and oppositely, since they are components of the non-chiral $\mathcal{N} = 2$ vectors multiplet, as is clear from tables 7.1 and 14.3.

In fact, in the spirit of the miraculous cancellation for pure type IIB string theory in ten dimensions, and for the spectrum of type IIB on K3 to six dimensions, shown in sections 7.1.2 and 7.6.2, there is another such purely closed string example, that of the $\mathbb{Z}_4^B$ model. Indeed, equation (14.45) is satisfied, but the coefficients of the $(\mathrm{tr}R^2)^2$ terms vanish also, giving:

$$12\hat{I}_8^{(\mathbf{2},\mathbf{1})} + 8\hat{I}_8^{(\mathbf{1},\mathbf{3})} + 8\hat{I}_8^{(\mathbf{2},\mathbf{1})} + \hat{I}_8^{(\mathbf{3},\mathbf{2})} + \hat{I}_8^{(\mathbf{3},\mathbf{1})} + \hat{I}_8^{(\mathbf{1},\mathbf{3})} + \hat{I}_8^{(\mathbf{2},\mathbf{1})} = 0. \quad (14.46)$$

This is, again, another remarkable purely closed string solution of the anomaly equations, supplying an $\mathcal{N} = 1$ supersymmetric solution of orientifolded type IIB strings on K3.

Let us turn to the models which need the addition of D-branes, and hence have gauge contributions to the anomaly. For the irreducible $\mathrm{Tr}F^4$ terms, everything again cancels. Again, it is recommended that the reader who is interested should verify this. This is done with the aid of the following information, which should be set alongside that given in equations (7.39) and (7.40). For $SU(n)$ we have:

$$\begin{aligned} \mathrm{Tr}_{\mathrm{adj}}(t^2) &= 2n\mathrm{Tr}_{\mathrm{f}}(t^2), \\ \mathrm{Tr}_{\mathrm{adj}}(t^4) &= 2n\mathrm{Tr}_{\mathrm{f}}(t^4) + 6\mathrm{Tr}_{\mathrm{f}}(t^2)\mathrm{Tr}_{\mathrm{f}}(t^2), \end{aligned} \quad (14.47)$$

and for completeness, we also list the result for $Sp(n) \equiv USp(2n)$:

$$\begin{aligned} \mathrm{Tr}_{\mathrm{adj}}(t^2) &= (n+2)\mathrm{Tr}_{\mathrm{f}}(t^2), \\ \mathrm{Tr}_{\mathrm{adj}}(t^4) &= (n+8)\mathrm{Tr}_{\mathrm{f}}(t^4) + 3\mathrm{Tr}_{\mathrm{f}}(t^2)\mathrm{Tr}_{\mathrm{f}}(t^2). \end{aligned} \quad (14.48)$$

Crucially, note that for symmetric tensor representations of any of the groups mentioned, we can use the $Sp(n)$ relations just mentioned in equation (14.48), while for antisymmetric tensor representations we can use the $SO(n)$ relations given in equation (7.39).

Now let us see how the irreducible $\mathrm{Tr}F^4$ terms cancel for one example, that of $\mathbb{Z}_3^A$. Table 14.2 shows that the gauge group is $U(8) \times SO(16)$ with hypermultiplets charged as $(\mathbf{28}, \mathbf{1})$ and $(\mathbf{8}, \mathbf{16})$. We must do each separate gauge group independently. Let us first do $U(8)$, and write everything in terms of the fundamental representation. Doing so, we see that we get $16\mathrm{Tr}_{\mathrm{f}}F^4$, ignoring (and for the rest of the paragraph) the purely numerical denominator in equation (7.2), of course. The first hypermultiplet, $(\mathbf{28}, \mathbf{1})$, which in fact in the antisymmetric of $U(8)$, and so using the second line in equation (7.39), we see that its coefficient of $\mathrm{Tr}_{\mathrm{f}}F^4$ is in fact zero. This

**Insert 14.3. Another string–string duality in $D = 6$**

Very interesting is the case of the $\mathbb{Z}_2^A$ model. The $U(1)$ factors can be shown to be absent non-perturbatively[196], leaving gauge group $SU(16) \times SU(16)$. Meanwhile the remaining anomaly factorises as:

$$I_8 \sim (\mathrm{tr}R^2 - 2\mathrm{Tr}_\mathrm{f}F^2)(\mathrm{tr}R^2 - 2\mathrm{Tr}_\mathrm{f}F^2),$$

for both the D9-brane and D5-brane sectors. This similar factorisation between the two sectors is of course a reflection of the fact that there is a $\mathrm{T}_{6789}$-duality exchanging the two types of brane, but there is more. There is a signal of another duality, now between this formulation and that of a strong/weak coupling dual model also with strings in six dimensions. Looking at the anomaly two-forms in the factors (say, on the right hand side), these dual strings have a similar structure to those we associate with the two-forms on the left hand side. This is of course very special to six dimensions, where we have a chance of such a string–string duality, and the reader can probably guess what the dual string theory might be. It is in fact a K3 compactification of the *heterotic* string, of a very special sort[229, 196]. One way to make it is as follows: Recall that from equation (7.48), the field equation of $\tilde{H}^{(3)}$ is:

$$d\tilde{H}^{(3)} = -\frac{\alpha'}{4}\left[\mathrm{Tr}F^2 - \mathrm{tr}R^2\right],$$

for which, if we were to integrate this over K3, would get a contribution of 24 (up to an overall factor) which can be cancelled by precisely 24 instantons. So an interesting compactification is achieved by taking the $E_8 \times E_8$ heterotic string on K3, with 12 instantons in one $E_8$ and 12 in the other. The details are interesting to uncover, but we shall have to leave it to the reader to study the literature, since it will take us too far afield[229]. Note also that we can see the dual strings. They are perturbatively manifest on the orientifold side as solitons. One is a D1-brane which one can place in the six non-compact directions and the other is its $\mathrm{T}_{6789}$ dual, a D5-brane with four of its dimensions wrapped on the compact directions. On the heterotic side, these map over to a pair of strong/weak dual heterotic strings. One is the heterotic string itself, and the other is a K3-wrapped NS5-brane.

leaves us with the hypermultiplet $(\mathbf{8}, \mathbf{16})$, which can be treated as sixteen copies of the fundamental of the $\mathbf{8}$, and therefore contributes $-16\mathrm{Tr}_\mathrm{f}F^4$, giving a cancellation.

For the mixed anomalies, the anomalies all factorise in a way which allows for their cancellation by a generalisation[193] of the Green–Schwarz mechanism. In the models with multiple tensors, some subset of them can be given a classically anomalous gauge transformation to produce the required cancellations. In some cases, the factorisation gives a sign of interesting physics, since there is a stringy interpretation of both four-form factors, suggesting new dualities (see insert 14.3).

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