Canad. Math. Bull. Vol. **54** (3), 2011 pp. 385–395
doi:10.4153/CMB-2011-082-4

Irreducible Representations of Inner Quasidiagonal C^*-Algebras

Bruce Blackadar and Eberhard Kirchberg

Abstract. It is shown that a separable C^*-algebra is inner quasidiagonal if and only if it has a separating family of quasidiagonal irreducible representations. As a consequence, a separable C^*-algebra is a strong NF algebra if and only if it is nuclear and has a separating family of quasidiagonal irreducible representations. We also obtain some permanence properties of the class of inner quasidiagonal C^*-algebras.

1 Introduction

This article is the long-delayed third installment in the authors' study of generalized inductive limits of finite-dimensional C^*-algebras.

The basic study of such generalized inductive limits was begun in [BK97], where the classes of MF algebras, NF algebras, and strong NF algebras were defined, and a number of equivalent characterizations of each class were given. In particular, a (necessarily separable) C^*-algebra is a *strong NF algebra* if it can be written as a generalized inductive limit of a sequential inductive system of finite-dimensional C^*-algebras in which the connecting maps are complete order embeddings (and asymptotically multiplicative in the sense of [BK97]). An *NF algebra* is a C^*-algebra that can be written as the generalized inductive limit of such a system, where the connecting maps are only required to be completely positive contractions. An NF algebra is automatically nuclear (and separable). It was shown that a separable C^*-algebra is an NF algebra if and only if it is nuclear and quasidiagonal.

It was not shown in [BK97] that the classes of NF algebras and strong NF algebras are distinct. Our second paper [BK01] used the notion of inner quasidiagonality to distinguish them. We gave the following definition, a slight variation of Voiculescu's characterization of quasidiagonal C^*-algebras [Voi91].

Definition 1.1 A C^*-algebra A is *inner quasidiagonal* if, for every $x_1, \dots, x_n \in A$ and $\epsilon > 0$, there is a representation π of A on a Hilbert space $\mathcal{H}$ and a finite-rank projection $P \in \pi(A)''$ such that $\|P\pi(x_j) - \pi(x_j)P\| < \epsilon$ $\|P\pi(x_j)P\| > \|x_j\| - \epsilon$ for $1 \leq j \leq n$.

Voiculescu's characterization of quasidiagonality is the same, except that the requirement that $P \in \pi(A)''$ is deleted. We then proved that a separable C^*-algebra is a strong NF algebra if and only if it is nuclear and inner quasidiagonal.

Received by the editors November 29, 2007.
Published electronically April 27, 2011.
This work was completed while both authors were visitors at the Fields Institute, Toronto, Canada.
AMS subject classification: **46L05**.

The principal shortcoming of this result is that it is often difficult to determine directly from the definition whether a C^*-algebra is inner quasidiagonal, although we were able to give examples of separable nuclear C^*-algebras that are quasidiagonal but not inner quasidiagonal, hence of NF algebras that are not strong NF. It is immediate from the definition that a C^*-algebra with a separating family of quasidiagonal irreducible representations is inner quasidiagonal, and in [BK01] we established some very special cases of the converse that were sufficient to yield the examples.

The main result of the present article is the full converse in the separable case.

Theorem 1.2 *A separable C^*-algebra is inner quasidiagonal if and only if it has a separating family of quasidiagonal irreducible representations.*

We thus obtain a characterization of strong NF algebras that is usually much easier to check than the characterization of [BK01].

Corollary 1.3 *A separable C^*-algebra is a strong NF algebra if and only if it is nuclear and has a separating family of quasidiagonal irreducible representations.*

Although our initial interest in inner quasidiagonality was through its connection to generalized inductive limits, it now appears that inner quasidiagonality is of some interest in its own right. In the final section, we list some permanence properties of the class of inner quasidiagonal C^*-algebras that follow easily from our characterizations:

(i) An arbitrary inductive limit (with injective connecting maps) of inner quasidiagonal C^*-algebras is inner quasidiagonal.
(ii) A (minimal) tensor product of inner quasidiagonal C^*-algebras is inner quasidiagonal.
(iii) The algebra of sections of a continuous field of separable inner quasidiagonal C^*-algebras is inner quasidiagonal.
(iv) Inner quasidiagonality is an (SI) property in the sense of [Bla06, II.8.5].

As a result of (i) and (iv), the study of inner quasidiagonal C^*-algebras can be effectively reduced to studying separable inner quasidiagonal C^*-algebras, to which Theorem 1.2 applies.

2 Outline of the Proof

To prove Theorem 1.2, fix a separable inner quasidiagonal C^*-algebra A and a nonzero $x_0 \in A$ (we may assume $\|x_0\| = 1$ to simplify notation). We will manufacture a quasidiagonal irreducible representation π of A with $\pi(x_0) \neq 0$.

We will construct a sequence $(\mathcal{H}_n)$ of finite-dimensional Hilbert spaces, embeddings $I_n \colon \mathcal{H}_n \to \mathcal{H}_{n+1}$, an increasing sequence (X_n) of finite self-adjoint subsets of the unit ball of A containing x_0, with union X dense in the unit ball of A, and completely positive contractions $V_n \colon A \to \mathcal{L}(\mathcal{H}_n)$ mapping the closed unit ball of A onto the closed unit ball of $\mathcal{L}(\mathcal{H}_n)$, such that, for all n:

(i) $\|I_n^* V_{n+1}(x) I_n - V_n(x)\| < 2^{-n-2}$ for all $x \in X_{n+1}$;
(ii) $\|V_n(xy) - V_n(x)V_n(y)\| < 2^{-n-2}$ for all $x, y \in X_n$;

https://doi.org/10.4153/CMB-2011-082-4 Published online by Cambridge University Press

(iii) $V_n(X_{n+1})$ is 2^{-n-2}-dense in the unit ball of $\mathcal{L}(\mathcal{H}_n)$, *i.e.*, for all z in the unit ball of $\mathcal{L}(\mathcal{H}_n)$ there is an $x \in X_{n+1}$ with $\|V_n(x) - z\| < 2^{-n-2}$;

(iv) $\|V_1(x_0)\| > 3/4$.

Once this tower is constructed, we proceed as follows. Let $\mathcal{H} = \lim_{\rightarrow}(\mathcal{H}_n, I_n)$ be the inductive limit Hilbert space, which may be thought of as the "union" of the $\mathcal{H}_n$. Let J_n be the natural inclusion of $\mathcal{H}_n$ into $\mathcal{H}$. If $x \in X_m$, then for $n \geq m$ and $\xi, \eta \in J_n\mathcal{H}_n$, we have

$$|\langle (J_{n+1}V_{n+1}(x)J_{n+1}^* - J_nV_n(x)J_n^*)\xi, \eta\rangle| = \\ |\langle (J_nI_n^*V_{n+1}(x)I_nJ_n^* - J_nV_n(x)J_n^*)\xi, \eta\rangle| < 2^{-n-2},$$

by (i). So the sequence $(J_nV_n(x)J_n^*)$ converges weakly in $\mathcal{L}(\mathcal{H})$ to an operator we call $\pi(x)$. For $\xi \in J_m\mathcal{H}_m$, for $n \geq m$ we have

$$\|J_{n+1}V_{n+1}(x)J_{n+1}^*\xi\| \geq \|J_nJ_n^*J_{n+1}V_{n+1}(x)J_{n+1}^*\xi\| = \\ \|J_{n+1}I_n^*V_{n+1}(x)I_nJ_n^*\xi\| \geq \|J_nV_n(x)J_n^*\xi\| - 2^{n-2}$$

and thus $\|\pi(x)\xi\| \geq \limsup \|J_nV_n(x)J_n^*\xi\|$. So $J_nV_n(x)J_n^* \to \pi(x)$ strongly (cf. [Bla06, I.1.3.3]). If $x, y \in X$, it follows from (ii) and joint strong continuity of multiplication on bounded sets that $\pi(xy) = \pi(x)\pi(y)$. Since X is dense in the unit ball of A and each V_n is a contraction, $(J_nV_n(x)J_n^*)$ converges strongly for each $x \in A$ to an operator on $\mathcal{H}$ we call $\pi(x)$, and π is linear, completely positive, contractive, and multiplicative, hence a *-representation of A on $\mathcal{H}$. For each $m \in \mathbb{N}$ and $x \in X_m$, we have $\|\pi(x)J_n - J_nV_n(x)\| < 2^{-n}$ for all $n \geq m$.

To show that π is irreducible, suppose $\xi, \eta, \zeta \in \mathcal{H}$ are unit vectors and $\epsilon > 0$. Choose m with $2^{-m} < \epsilon/4$, and for some $n \geq m$ choose unit vectors $\tilde{\xi}, \tilde{\eta}, \tilde{\zeta} \in \mathcal{H}_n$ with $\|\xi - J_n\tilde{\xi}\|, \|\eta - J_n\tilde{\eta}\|, \|\zeta - J_n\tilde{\zeta}\| < \epsilon/4$. There is a unitary $u \in \mathcal{L}(\mathcal{H}_n)$ with $u\tilde{\xi} = \tilde{\eta}$. By (iii), there is an $x \in X_{n+1}$ with $\|V_n(x) - u\| < 2^{-n-2}$. Since $\|I_n^*V_{n+1}(x)I_n - V_n(x)\| < 2^{-n-2}$, we have $\|I_n^*V_{n+1}(x)I_n - u\| < 2^{-n-1}$. By iteration, $\|J_n^*\pi(x)J_n - u\| < 2^{-n}$, and hence $\|J_n^*\pi(x)J_n\tilde{\xi} - \tilde{\eta}\| < 2^{-n}$. Then

$$\begin{aligned}|\langle \pi(x)\xi - \eta, \zeta\rangle| &\leq \|\zeta - J_n\tilde{\zeta}\| + |\langle \pi(x)\xi - \eta, J_n\tilde{\zeta}\rangle| \\ &= \|\zeta - J_n\tilde{\zeta}\| + |\langle J_n^*(\pi(x)\xi - \eta), \tilde{\zeta}\rangle| \\ &\leq \|\zeta - J_n\tilde{\zeta}\| + \|\xi - J_n\tilde{\xi}\| + \|\eta - J_n\tilde{\eta}\| + |\langle J_n^*\pi(x)J_n\tilde{\xi} - \tilde{\eta}, \tilde{\zeta}\rangle| < \epsilon,\end{aligned}$$

and so (fixing ξ and η and letting ζ vary) η is in the weak closure of $\pi(A)\xi$.

To show that π is quasidiagonal, let $P_n = J_nJ_n^*$ be the projection of $\mathcal{H}$ onto $J_n\mathcal{H}_n$. Then P_n has finite rank, $P_n \to 1$ strongly, and, for $x \in X_m$ and $n \geq m$,

$$\begin{aligned}\|P_n\pi(x) - \pi(x)P_n\| &= \max(\|(1 - P_n)\pi(x)P_n\|, \|P_n\pi(x)(1 - P_n)\|) \\ &= \max(\|(1 - P_n)\pi(x)P_n\|, \|(1 - P_n)\pi(x^*)P_n\|), \\ \|(1 - P_n)\pi(x)P_n\| &\leq \|\pi(x)P_n - J_nV_n(x)J_n^*\| + \|P_nJ_nV_n(x)J_n^* - P_n\pi(x)P_n\| \\ &\leq \|\pi(x)J_n - J_nV_n(x)\| + \|J_nV_n(x) - \pi(x)J_n\| < 2^{-n+1}\end{aligned}$$

since $P_n J_n = J_n$. Similarly, $\|(1-P_n)\pi(x^*)P_n\| < 2^{-n+1}$ since $x^* \in X_m$.

Finally, note that $\|J_1 V_1(x_0) J_1^*\| = \|V_1(x_0)\| > 3/4$ and $\|J_1^* \pi(x_0) J_1 - V_1(x_0)\| < 1/2$, so $\|\pi(x_0)\| > 1/4$.

3 Pure Matricial States

In this section, A will be a general C^*-algebra, not the specific C^*-algebra of Section 2.

Definition 3.1 A *pure matricial n-state* on a C^*-algebra A is a completely positive contraction $V\colon A \to \mathbb{M}_n = M_n(\mathbb{C})$ such that there is an irreducible representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$ and an isometry $I\colon \mathbb{C}^n \to \mathcal{H}$ such that $V(a) = I^*\pi(a)I$ for $a \in A$. (Note that π and I are uniquely determined up to unitary equivalence (of the $\mathcal{H}$) via the Stinespring dilation.)

Remarks 3.2 (i) We have chosen a definition that is easily applicable to our needs. There are other characterizations, which are elementary functional analysis exercises, *e.g.,* $V\colon A \to \mathbb{M}_n$ maps the *open* unit ball onto the *open* unit ball of $\mathbb{M}_n$ and V is an extreme point of the convex set $CPC(A, \mathbb{M}_n)$ of completely positive contractions $T\colon A \to \mathbb{M}_n$. But, even for $A = \mathbb{M}_n$, the extreme points of the unital maps in of $CPC(A, \mathbb{M}_n)$ are in general not pure n-states, *i.e.,* are not automorphisms of $\mathbb{M}_n$ (for example, if ϕ is a pure state of A, then the map $x \mapsto \phi(x)1$ from A to $\mathbb{M}_n$ is an extreme point of $CPC(A, \mathbb{M}_n)$, but not a pure matricial state of A).

(ii) If one applies the Kadison Transitivity Theorem (see *e.g.,* [Bla06, II.6.1.12]) to the irreducible representation π and the image of I, then one can see that the restriction of V to the multiplicative domain

$$A^V := \{a \in A : V(ab) = V(a)V(b) \text{and} V(ba) = V(b)V(a), b \in A\}$$

is an epimorphism from A^V onto $\mathbb{M}_n$, see [BK01, 3.4]. In particular, V maps the closed unit ball of A onto the closed unit ball of $\mathbb{M}_n$, and hence maps the open unit ball of A onto the open unit ball of $\mathbb{M}_n$.

(iii) Up to unitary equivalence (up to an automorphism of $\mathbb{M}_n$) a pure matricial n-state is defined by a projection p in the socle of the second conjugate A^{**} of A with $pAp \cong \mathbb{M}_n$. Consider the support projection z of the normal extension $\overline{\pi}\colon A^{**} \to \mathcal{L}(\mathcal{H})$ of π in the center of A^{**}. The restriction φ of $\overline{\pi}$ to $A^{**}z$ defines an isomorphism from $A^{**}z$ onto $\mathcal{L}(\mathcal{H})$. Let $p_V := \varphi^{-1}(II^*)$, then $b \mapsto I^*\varphi(b)I$ is an isomorphism λ_V from $p_V A p_V = p_V A^{**} p_V$ onto $\mathbb{M}_n$, such that $V(b) = \lambda_V(p_V b p_V)$ for $b \in A$.

Note that p_V is just the support projection of the normal extension $\overline{V}$ of V to A^{**}. Projections corresponding to disjoint pure matricial states have orthogonal central supports.

(iv) A pure matricial n-state $V\colon A \to \mathbb{M}_n$ always extends to a unital pure matricial n-state on the unitization $\tilde{A}$ of A (see (ii)). In particular, V is unital if A is unital.

(v) Pure matricial n-states are in obvious one to one correspondence with those pure states η on $A \otimes \mathbb{M}_n$ that have the additional property that their (unique) extension $\tilde{\eta}$ to $\tilde{A} \otimes \mathbb{M}_n$ satisfies $\tilde{\eta}(1 \otimes b) = \tau(b)$, where τ denotes the tracial state on $\mathbb{M}_n$. The

https://doi.org/10.4153/CMB-2011-082-4 Published online by Cambridge University Press

correspondence is given by $\eta_V(a \otimes b) := (1/n)\mathrm{Tr}(V(a)b^\top)$, where $b^\top$ denotes the *transposed* matrix of b. Not every pure state of $A \otimes \mathbb{M}_n$ has this property (see (i)).

[To see that such an η_V extends to a pure state on $\tilde{A} \otimes \mathbb{M}_n$, note that $a \otimes b \mapsto (1/n)\mathrm{Tr}(b^\top a)$ defines a pure state η on $\mathbb{M}_n \otimes \mathbb{M}_n$. Conversely, every pure state η on $A \otimes \mathbb{M}_n$ is given by an irreducible representation $\pi\colon A \to \mathcal{L}(\mathcal{H})$ and a map $I\colon \mathbb{C}^n \to \mathcal{H}$ such that $\eta(a \otimes b) = (1/n)\mathrm{Tr}(V(a)b^\top)$ for $a \in A$ and $b \in \mathbb{M}_n$, where $V(a) := I^*\pi(a)I$. The condition $\eta(1 \otimes b) = (1/n)\mathrm{Tr}(b)$ implies that I is an isometry.]

Here is an alternate way of viewing the situation. A pure state on $A \otimes \mathbb{M}_n$ is a vector state from an irreducible representation of $A \otimes \mathbb{M}_n$. Up to unitary equivalence, every irreducible representation of $A \otimes \mathbb{M}_n$ is of the form $\rho \otimes \sigma$, where ρ is an irreducible representation of A on a Hilbert space $\mathcal{H}$, and σ is the standard representation of $\mathbb{M}_n$ on $\mathbb{C}^n$. Let $\{\zeta_1, \dots, \zeta_n\}$ be the standard basis for $\mathbb{C}^n$. Then every unit vector in $\mathcal{H} \otimes \mathbb{C}^n$ can be written in the form $\sum_{j=1}^n \alpha_j(\xi_j \otimes \zeta_j)$, where the ξ_j are unit vectors in $\mathcal{H}$, $\alpha_j \geq 0$, and $\sum \alpha_j^2 = 1$; the representation is unique if all $\alpha_j > 0$. Then the vector state from this vector corresponds to a pure matricial n-state on A if and only if the ξ_j are mutually orthogonal and all α_j are equal to $n^{-1/2}$.

(vi) Every pure matricial n-state $V\colon A \to \mathbb{M}_n$ on $A \subset B$ extends to a pure matricial n-state $V_e\colon B \to \mathbb{M}_n$. Simply extend the pure state $\tilde{\eta}$ on $\tilde{A} \otimes \mathbb{M}_n$ to a pure state on $\tilde{B} \otimes \mathbb{M}_n$.

If $T\colon A \to B$ is completely isometric *and* completely positive, then there is a pure matrical state $W\colon B \to \mathbb{M}_n$ with $W \circ T = V$.

[Indeed, T extends to a unital completely isometric map $T_1\colon A_1 \to B_1$ of the *outer* unitizations A_1 and B_1. An extremal extension of the extremal state on $T_1(A_1) \otimes \mathbb{M}_n \subset B_1 \otimes \mathbb{M}_n$, related to V, defines the desired extension of $V \circ T^{-1}$ to all of B_1 by (iv).]

(vii) Since up to unitary equivalence the standard representation of $\mathbb{M}_n$ on $\mathbb{C}^n$ is the only irreducible representation of $\mathbb{M}_n$, every pure matricial state $V\colon \mathbb{M}_n \to \mathbb{M}_m$ (where necessarily $m \leq n$) is the compression of the identity representation to an m-dimensional subspace of $\mathbb{C}^n$, *i.e.*, there is a unique isometry $I\colon \mathbb{C}^m \to \mathbb{C}^n$ with $V(x) = I^*xI$ for all $x \in \mathbb{M}_n$, where $\mathbb{M}_n$ and $\mathbb{M}_m$ are identified with $\mathcal{L}(\mathbb{C}^n)$ and $\mathcal{L}(\mathbb{C}^m)$ in the standard way.

The next result is the crucial technical tool needed for construction of the tower.

Lemma 3.3 *Suppose that $\{\psi_\mu\colon A \to B_\mu; \mu \in \Gamma\}$ is a separating family of C^*-algebra homomorphisms. Then for every pure matricial n-state $V\colon A \to \mathbb{M}_n$, every $\delta > 0$ and every finite subset $F \subset A$ there is a $\nu \in \Gamma$ and a pure matricial n-state $W\colon B_\nu \to \mathbb{M}_n$ such that*

$$\|W\psi_\nu(x) - V(x)\| < \delta \quad \forall x \in F.$$

Proof If $n = 1$ (the case of pure states), the result is well known (see [Dix69, 3.4.2(ii)]). For the general case, we may assume A and the B_μ are unital. Replace A and B_μ by $A \otimes \mathbb{M}_n$ and $B_\mu \otimes \mathbb{M}_n$, and ψ_μ by $\psi_\mu \otimes id$. Let $F \otimes E = \{x \otimes e_{ij} : x \in F, 1 \leq i, j \leq n\}$, where the e_{ij} are the standard matrix units in $\mathbb{M}_n$. The pure state η_V on $A \otimes \mathbb{M}_n$ corresponding to the pure matricial state V on A can be approximated arbitrarily closely (within $\delta/6n^4$ will do) on $F \otimes E$ by a pure state θ on $B_\nu \otimes \mathbb{M}_n$ for

https://doi.org/10.4153/CMB-2011-082-4 Published online by Cambridge University Press

some ν. The restriction of θ to $1 \otimes \mathbb{M}_n$ is not (obviously) exactly τ, the tracial state on $\mathbb{M}_n$, but is at least approximately τ. We must perturb θ to make the restriction exactly τ.

When θ is represented as a vector state with vector $\sum_{j=1}^{n} \alpha_j(\xi_j \otimes \zeta_j)$ as in Remark 3.2(v), the ξ_j are almost mutually orthogonal and α_j satisfy $|\alpha_j - n^{-1/2}| < \delta/6n^4$ for all j. Let φ be the (pure) state of $B_\nu \otimes \mathbb{M}_n$ corresponding to the vector $\sum_{j=1}^{n} n^{-1/2}(\tilde{\xi}_j \otimes \zeta_j)$, where the $\tilde{\xi}_j$ are obtained from the ξ_j by the Gram–Schmidt process. We have $\|\tilde{\xi}_j - \xi_j\| < \delta/3n^3$, so $\|\varphi - \theta\| < \delta/2n^2$ and $\|\varphi(x) - \eta_V(x)\| < \delta/n^2$ for $x \in F \otimes E$. Then $\varphi = \eta_W$ for some pure matricial n-state W on A factoring through B_ν, and $\|W(x) - V(x)\| < \delta$ for all $x \in F$. ■

4 Constructing the Tower

We now construct the tower used in the proof in Section 2, using the following two lemmas.

Lemma 4.1 *Let B be an inner quasidiagonal C^*-algebra, F a finite subset of the unit ball of B, $b \in F$, and $\epsilon > 0$. Then there is a pure matricial state $V\colon B \to \mathbb{M}_n$ for some n, such that $\|V(xy) - V(x)V(y)\| < \epsilon$ for all $x, y \in F$ and $\|V(b)\| > \|b\| - \epsilon$.*

Proof In the separable case, this is just (i) ⇒ (ii) of [BK01, 3.7] (note that there is a misprint in the published statement of [BK01, 3.16(ii)]). We give the simple argument, which was omitted in [BK01], and does not require separability.

By the definition of inner quasidiagonality, there is a representation π of B on a Hilbert space $\mathcal{H}$ and a finite-rank projection $P \in \pi(B)'' \subseteq \mathcal{L}(\mathcal{H})$ such that $\|P\pi(x) - \pi(x)P\| < \epsilon$ for all $x \in F$ and $\|P\pi(b)P\| > \|b\| - \epsilon$. The central support Q of P in $\pi(B)''$ is Type I and is a sum of finitely many minimal central projections $Q_1, \dots, Q_m$. If $R_1, \dots, R_m$ are minimal projections in $\pi(B)'$ with Q_j the central support of R_j, and $P_j = PR_j$, then $\|P\pi(x)P\| = \max_j \|P_j\pi(x)P_j\|$ for all $x \in B$. Fix j with $\|P_j\pi(b)P_j\| > \|b\| - \epsilon$. Then $\rho = \pi|_{R_j\mathcal{H}}$ is irreducible. For $x \in B$, let $V(x) = P_j\rho(x)P_j \in \mathcal{L}(P_j\mathcal{H}) \cong \mathbb{M}_n$, where $n = dim(P_j\mathcal{H})$; then V is the desired pure matricial state of B: $\|P_j\pi(x) - \pi(x)P_j\| < \epsilon$ for all $x \in F$, so

$$\begin{aligned}\|V(xy) - V(x)V(y)\| &= \|P_j\pi(xy)P_j - P_j\pi(x)P_j\pi(y)P_j\| \\ &= \|P_j(P_j\pi(x) - \pi(x)P_j)\pi(y)P_j\| < \epsilon\end{aligned}$$

for all $x, y \in F$. ■

Lemma 4.2 *Let B be a separable inner quasidiagonal C^*-algebra, F a finite subset of B, $V\colon B \to \mathbb{M}_n \cong \mathcal{L}(\mathbb{C}^n)$ a pure matricial state of B, and $\epsilon > 0$. Then there is a pure matricial state $W\colon B \to \mathbb{M}_m \cong \mathcal{L}(\mathbb{C}^m)$ for some m and an isometry $I\colon \mathbb{C}^n \to \mathbb{C}^m$ such that*

(i) *$\|W(xy) - W(x)W(y)\| < \epsilon$ for all $x, y \in F$.*
(ii) *$\|I^*W(x)I - V(x)\| < \epsilon$ for all $x \in F$.*

Proof Let $X = \{x_1, x_2, \dots\}$ be a countable dense subset of the unit ball of B, and $X_k = \{x_1, \dots, x_k\}$. For each k, apply Lemma 4.1 to obtain a pure matricial state

https://doi.org/10.4153/CMB-2011-082-4 Published online by Cambridge University Press

$W_k\colon B \to \mathbb{M}_{m_k}$ such that $\|W_k(xy) - W_k(x)W_k(y)\| < 2^{-k}$ for all $x, y \in X_k$, and $\|W_k(x_k)\| > \|x_k\| - 2^{-k}$. Set $M_k = \mathbb{M}_{m_k}$, $M = \prod_k M_k$, $J = \oplus_k M_k$. Let $\varphi_k\colon M \to M_k$ be the k-th coordinate map. Then $\{\varphi_k\}$ is a separating family of *-homomorphisms on M (since for each k_0, $\{x_k : k \geq k_0\}$ is dense in the unit ball of B).

The map $\Psi\colon b \mapsto (W_1(b), W_2(b), \dots)$ from B to M drops to an injective *-homomorphism ψ from B to M/J. By Remark 3.2(vi), the pure matricial state $V\colon B \to \mathbb{M}_n$ extends to a pure matricial state, also called V, from M/J to $\mathbb{M}_n$; V may be regarded as a pure matricial state on M by composing with the quotient map from M to M/J. By Lemma 3.3, for some k there is a pure matricial state U on M_k with $\|U(\varphi_k(x)) - V(x)\| < \epsilon$ for all $x \in F$ (where B is identified with $\Psi(B)$). By Remark 3.2(vii), there is an isometry $I\colon \mathbb{C}^n \to \mathbb{C}^{m_k}$ such that $U(y) = I^* y I$ for $y \in M_k$. Set $m = m_k$, $W = W_k$ (note that $W_k(x) = \varphi_k(x)$ for $x \in B \subseteq M$). ∎

We now construct the tower. Let A and x_0 be as in Section 2, and let X be a self-adjoint, countable, dense subset of the unit ball of A containing x_0, closed under multiplication. Enumerate X as

$$X = \{x_0, x_0^*, x_1, x_1^*, \dots, x_n, x_n^*, \dots\}$$

Set $X_1 = \{x_0, x_0^*\}$, and by Lemma 4.1 choose a pure matricial m_1-state $V_1\colon A \to \mathbb{M}_{m_1} \cong \mathcal{L}(\mathbb{C}^{m_1})$ of $B = A$ with $F = X_1$, $b = x_0$, and $\epsilon = 2^{-3}$, and set $\mathcal{H}_1 = \mathbb{C}^{m_1}$.

Suppose $X_j, \mathcal{H}_j = \mathbb{C}^{m_j}, V_j, I_j$ have been defined for $1 \leq j \leq n$. Since V_n maps the closed unit ball of A onto the closed unit ball of $\mathbb{M}_{m_n}$, there is a k_n such that $X_{n+1} := \{x_0, x_0^*, \dots, x_{k_n}, x_{k_n}^*\}$ contains X_n and X_n^2 and such that $V_n(X_{n+1})$ is 2^{-n-2}-dense in the unit ball of $\mathbb{M}_{m_n}$. By Lemma 4.2 with $B = A$, $F = X_{n+1}$, $V = V_n$, and $\epsilon = 2^{-n-2}$ choose a pure matricial m_{n+1}-state V_{n+1} of A and an isometry $I_n\colon \mathbb{C}^{m_n} \to \mathbb{C}^{m_{n+1}}$.

The $X_n, \mathcal{H}_n, V_n, I_n$ satisfy (i), (ii), (iii), and (iv) by construction. The tower thus has all the properties required in Section 2, completing the proof of Theorem 1.2.

Remark 4.3 What about the nonseparable case? Many parts of the argument have obvious generalizations to the nonseparable case. The separability hypothesis in Lemma 4.2 should be removable at the cost of some technical complication. However, it is doubtful that the argument in the proof of Theorem 1.2 can be adapted to the nonseparable case, because a slight variation of this argument can be used to give a new proof of the well-known fact that a separable prime C^*-algebra is primitive. This is known to be false in general for nonseparable C^*-algebras [Wea03].

5 Inductive Limits

We do not know whether a general inner quasidiagonal C^*-algebra has a separating family of quasidiagonal irreducible representations. However, it follows immediately from [BK01, 3.6] that every inner quasidiagonal C^*-algebra is an inductive limit (with injective connecting maps) of separable inner quasidiagonal C^*-algebras, for which Theorem 1.2 holds. Thus the theory of inner quasidiagonal C^*-algebras can be largely reduced to the separable case. But to complete this reduction, we must know that an arbitrary inductive limit of inner quasidiagonal C^*-algebras is inner quasidiagonal.

In [BK01, 2.11], it was stated that "it is obvious from the definition" that an inductive limit of an inductive system of inner quasidiagonal C^*-algebras (with injective connecting maps) is inner quasidiagonal. In fact, this is not so obvious from the definition, but it is obvious from the equivalence of (i) and (ii) of [BK01, 3.7] in the separable case (see Corollary 5.3). So to prove that general inductive limits of inner quasidiagonal C^*-algebras are inner quasidiagonal, it suffices to remove the separability hypothesis in this equivalence.

Proposition 5.1 *Let A be a C^*-algebra. Then A is inner quasidiagonal if and only if the following condition is satisfied.*

For every $a_1, \dots, a_n, b \in A$ and $\epsilon > 0$, there is a pure matricial state V of A such that $\|V(a_i)V(a_j) - V(a_ia_j)\| < \epsilon$ for $1 \le i, j \le n$ and $\|V(b)\| > \|b\| - \epsilon$.

Proof The "only if" direction follows immediately from Lemma 4.1. To prove the converse, we will show by induction on m that if the statement in Proposition 5.1 holds, the following condition $P(m)$ holds for every m. Then, given $a_1, \dots, a_n \in A$ and $\epsilon > 0$, applying $P(n)$ with $b_j = a_j$ shows that A is inner quasidiagonal.

$P(m)$: For every $a_1, \dots, a_n, b_1, \dots, b_m \in A$ and $\epsilon > 0$, there are finitely many pure matricial states $V_1, \dots, V_r$ of A such that

(i) $\|V_k(a_i)V_k(a_j) - V_k(a_ia_j)\| < \epsilon$ for $1 \le i, j \le n$, $1 \le k \le r$;
(ii) $\max_k \|V_k(b_j)\| > \|b_j\| - \epsilon$ for $1 \le j \le m$;
(iii) the V_k are pairwise disjoint (the corresponding irreducible representations are pairwise inequivalent).

Note that $P(1)$ is exactly the condition in the statement of Proposition 5.1.

Assume $P(m)$ holds (and thus $P(1)$ also holds), and let $a_1, \dots, a_n, b_1, \dots, b_{m+1} \in A$ and $\epsilon > 0$. Fix $\delta > 0$ such that $\delta < \frac{1}{4}$ and

$$2\delta + 12\delta(\max_{i,j}\{\|a_i\|, \|b_j\|\}) < \epsilon.$$

Let π_k $(1 \le k \le r)$ be pairwise inequivalent irreducible representations of A on $\mathcal{H}_k$ with finite-rank projections $p_k \in \mathcal{L}(\mathcal{H}_k)$ such that the pure matricial states $U_k(\,\cdot\,) = p_k\pi_k(\,\cdot\,)p_k$ satisfy

(i) $\|U_k(x)U_k(y) - U_k(xy)\| < \delta$ for $x, y \in \{a_1, \dots, a_n, b_1, \dots, b_{m+1}\}$, $1 \le k \le r$.
(ii) $\max_k \|V_k(b_j)\| > \|b_j\| - \delta$ for $1 \le j \le m$.

If $\pi_k(A)$ contains $\mathcal{K}(\mathcal{H}_k)$ (*i.e.*, if $\pi_k(A) \cap \mathcal{K}(\mathcal{H}_k) \neq \{0\}$), then there is a $c_k \in A_+$, $\|c_k\| = 1$, with $\pi_k(c_k) = p_k$. If $\pi_k(A) \cap \mathcal{K}(\mathcal{H}_k) = \{0\}$, set $c_k = 0$.

By $P(1)$ there is an irreducible representation π_{r+1} of A on $\mathcal{H}_{r+1}$ and a finite-rank projection $p_{r+1} \in \mathcal{L}(\mathcal{H}_{r+1})$ such that the pure matricial state $W(\,\cdot\,) = p_{r+1}\pi_{r+1}(\,\cdot\,)p_{r+1}$ satisfies the following:

(i) $\|W(x)W(y) - W(xy)\| < \delta/2$ for $x, y \in \{a_1, \dots, a_n, b_1, \dots, b_{m+1}, c_1, \dots, c_r\}$.
(ii) $\|W(b_{m+1})\| > \|b_{m+1}\| - \delta$.

If π_{r+1} is not equivalent to any π_k, $k \le r$, then we can set $V_k = U_k$ for $1 \le k \le r$ and $V_{r+1} = W$, and we are done (since $\delta < \epsilon$). The difficulty comes when π_{r+1} is equivalent to some π_k, say π_r without loss of generality. In this case, there is an isometry I from $p_{r+1}\mathcal{H}_{r+1}$ into $\mathcal{H}_r$ such that $W(\,\cdot\,) = I^*\pi_r(\,\cdot\,)I$.

If $\pi_r(A) \cap \mathcal{K}(\mathcal{H}_r) = \{0\}$, then (see [Arv77]) there is a sequence of isometries I_t from $p_{r+1}\mathcal{H}_{r+1}$ to $(1-p_r)\mathcal{H}_r$ such that

$$W(x) = \lim_{t\to\infty} I_t^* \pi_r(x) I_t \text{ for } x \in \{a_1, \dots, a_n, b_1, \dots, b_{m+1}\}.$$

For sufficiently large t, we may take $V_r(\,\cdot\,) = (p_r + I_t I_t^*)\pi_r(\,\cdot\,)(p_r + I_t I_t^*)$ and $V_k = U_k$ for $1 \le k \le r-1$.

The most difficult case is where π_{r+1} is equivalent to π_r and $\pi_r(A)$ contains $\mathcal{K}(\mathcal{H}_r)$. If $q = II^*$, then we have

$$\|qp_r - p_r q\| = \|q\pi_r(c_r) - \pi(c_r)q\| < \delta.$$

By the following lemma, let $\tilde{q}$ be a projection in $\mathcal{L}(\mathcal{H}_r)$ with $\tilde{q}p_k = p_k\tilde{q}$ and $\|\tilde{q} - q\| < 3\delta$. Set $\tilde{p}_r = p_r + \tilde{q}(1-p_r)$ and $V_r(\,\cdot\,) = \tilde{p}_r\pi(\,\cdot\,)\tilde{p}_r$, and $V_k = U_k$ for $1 \le k \le r-1$. These V_k have the desired properties, completing the inductive step and thus the proof of 5.1. ∎

Lemma 5.2 *Let A be a C^*-algebra, and p and q projections in A. If $\|qp - pq\| < \epsilon < \frac{1}{4}$, then there is a projection $\tilde{q} \in A$ with $\|\tilde{q} - q\| < 3\epsilon$ and $\tilde{q}p = p\tilde{q}$. If $r = p + \tilde{q}(1-p)$, then for every $x \in A$ we have*

$$\|rx - xr\| \le 2\|xp - px\| + 2\|xq - qx\| + 12\epsilon\|x\|.$$

Proof We may assume A is unital. We have

$$\|q - [pqp + (1-p)q(1-p)]\| = \|(1-p)qp\| = \|(qp - pq)p\| \le \|qp - pq\| < \epsilon.$$

Also,

$$\|pqp - (pqp)^2\| = \|pqp - pqpqp\| = \|pq(qp - pq)p\| < \epsilon$$

and so $\sigma(pqp) \subseteq [1, \gamma] \cup [\gamma, 1]$, where $\gamma = \frac{1-\sqrt{1-4\epsilon}}{2} < 2\epsilon$ since $\epsilon < \frac{1}{4}$. Thus by functional calculus, there is a projection $r \in pAp$ with $\|r - pqp\| < \gamma < 2\epsilon$. Similarly, there is a projection $s \in (1-p)A(1-p)$ with $\|s - (1-p)q(1-p)\| < 2\epsilon$. If $\tilde{q} = r + s$, then

$$\|\tilde{q} - q\| \le \|\tilde{q} - [pqp + (1-p)q(1-p)]\| + \|[pqp + (1-p)q(1-p)] - q\| < 3\epsilon.$$

If $x \in A$, then

$$\begin{aligned}\|rx - xr\| &\le \|xp - px\| + \|x\tilde{q} - \tilde{q}x\| + \|xp\tilde{q} - p\tilde{q}x\| \\ &\le 2(\|xp - px\| + \|x\tilde{q} - \tilde{q}x\|) \\ &\le 2\|xp - px\| + 2(2\|x\|\|\tilde{q} - q\| + \|xq - qx\|).\end{aligned}$$ ∎

Corollary 5.3 *An arbitrary inductive limit (with injective connecting maps) of inner quasidiagonal C^*-algebras is inner quasidiagonal.*

Proof Let $A = \lim_{\to}(A_i, \phi_{ij})$, with each A_i inner quasidiagonal. Regard each A_i as a C^*-subalgebra of A. If $a_1, \dots, a_n, b \in A$ and $\epsilon > 0$, fix an A_i and elements $\tilde{a}_1, \dots, \tilde{a}_n, \tilde{b} \in A_i$ with $\|a_j - \tilde{a}_j\| < \delta$ for each j and $\|b - \tilde{b}\| < \delta$, where $\delta = \epsilon/3\max(1, \|a_1\|, \dots, \|a_n\|, \|b\|)$. Let V be a pure matricial state of A_i such that $\|V(\tilde{a}_j)V(\tilde{a}_k) - V(\tilde{a}_j\tilde{a}_k)\| < \delta$ for all j, k, and $\|V(\tilde{b})\| > \|\tilde{b}\| - \delta$. Extend V to a pure matricial state W on A. Then $\|W(a_j)W(a_k) - W(a_ja_k)\| < \epsilon$ for all j, k, and $\|W(b)\| > \|b\| - \epsilon$. Thus A is inner quasidiagonal by Proposition 5.1. ∎

6 Permanence Properties

We finish by recording some other permanence properties of the class of inner quasidiagonal C^*-algebras. The first one is an easy consequence of the definition of inner quasidiagonality and could have been noted in [BK01].

Proposition 6.1 *The minimal tensor product of inner quasidiagonal C^*-algebras is inner quasidiagonal.*

Proof If A and B are inner quasidiagonal and $Z = \{z_1, \dots, z_n\} \subseteq A \otimes_{\min} B$, approximate z_k by an element $\sum_{j=1}^{n_k} x_{jk} \otimes y_{jk}$ of the algebraic tensor product $A \odot B$. Let $E = \{x_{jk}\}$ and $F = \{y_{jk}\}$. If (π, P) and (ρ, Q) are representations of A and B with projections as in the definition for E and F with sufficiently small ϵ, then $(\pi \otimes \rho, P \otimes Q)$ will be the desired representation for $A \otimes_{\min} B$ and Z. ■

It is doubtful that the result holds for maximal tensor products. Note that no separability hypothesis is necessary in Proposition 6.1.

The next property is an immediate consequence of Theorem 1.2 and essentially generalizes [BK01, 3.10].

Proposition 6.2 *The algebra of sections of a continuous field of separable inner quasidiagonal C^*-algebras is inner quasidiagonal.*

Proof Let $\langle A(t) \rangle$ be a continuous field of separable continuous trace C^*-algebras over a locally compact Hausdorff space X, and A the C^*-algebra of continuous sections vanishing at infinity. Each fiber $A(t)$ has a separating family of quasidiagonal irreducible representations by 1.2, so by composition with the fiber maps from A to the $A(t)$, A also has a separating family of quasidiagonal irreducible representations, hence is inner quasidiagonal. ■

Any C^*-subalgebra of a quasidiagonal C^*-algebra is quasidiagonal. This is false for inner quasidiagonality: if A is an NF algebra that is not strong NF (see [BK01, 5.6]), let π be a faithful quasidiagonal representation of A on a Hilbert space $\mathcal{H}$. Then $\pi(A) + \mathcal{K}(\mathcal{H})$ is inner quasidiagonal (by Theorem1.2 or [BK01, 5.8]), in fact strong NF, but the C^*-subalgebra $\pi(A)$ is not inner quasidiagonal. But we have the following.

Proposition 6.3 *Inner quasidiagonality is an (SI) property in the sense of* [Bla06, II.8.5].

Proof This is just a combination of Corollary 5.3 and [BK01, 3.6]. ■

References

[Arv77] W. Arveson, *Notes on extensions of C^*-algebras.* Duke Math. J. **44**(1977), no. 2, 329–355. doi:10.1215/S0012-7094-77-04414-3

[BK97] B. Blackadar and E. Kirchberg, *Generalized inductive limits of finite-dimensional C^*-algebras.* Math. Ann. **307**(1997), no. 3, 343–380. doi:10.1007/s002080050039

[BK01] ———, *Inner quasidiagonality and strong NF algebras.* Pacific J. Math. **198**(2001), no. 2, 307–329. doi:10.2140/pjm.2001.198.307

[Bla06] B. Blackadar, *Operator algebras. Theory of C^*-algebras and von Neumann algebras.* In: Encyclopaedia of Mathematical Sciences, 122, Operator Algebras and Non-commutative Geometry, III, Springer-Verlag, Berlin, 2006.

[Dix69] J. Dixmier, *Les C^*-algèbres et leurs représentations.* Cahiers Scientifiques, 29, Gauthier-Villars & Cie., Paris, 1964.

[Voi91] D. Voiculescu, *A note on quasi-diagonal C^*-algebras and homotopy.* Duke Math. J. **62**(1991), no. 2, 267–271. doi:10.1215/S0012-7094-91-06211-3

[Wea03] N. Weaver, *A prime C^*-algebra that is not primitive.* J. Funct. Anal. **203**(2003), no. 2, 356–361. doi:10.1016/S0022-1236(03)00196-4

Department of Mathematics, University of Nevada, Reno, Reno, NV, U.S.A.
e-mail: bruceb@unr.edu

Institut für Mathematik, Humboldt Universität zu Berlin, Berlin, Germany
e-mail: kirchbrg@mathematik.hu-berlin.de