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GROUPS IN WHICH ALL SUBGROUPS OF INFINITE RANK ARE SUBNORMAL

LEONID A. KURDACHENKO

Algebra Department, Dnepropetrovsk University, Vul. Naukova 13, Dnepropetrovsk 50, Ukraine 49000

and HOWARD SMITH

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.
e-mail: howsmith@bucknell.edu

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Abstract. Let G be a locally soluble-by-finite group in which every non-subnormal subgroup has finite rank. It is proved that either G has finite rank or G is soluble and locally nilpotent (and even a Baer group). On the other hand, a group G is constructed that has infinite rank and satisfies the given hypothesis, but does not have every subgroup subnormal.

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1. Introduction. The following result is established in [**5**].

Theorem 1 [**5**, Theorem 1.22]. *Let G be a group whose non-subnormal subgroups have finite rank. If G is soluble then either G is a Baer group or G has finite rank.*

We recall here that a Baer group is one in which every cyclic (and hence every finitely generated) subgroup is subnormal, and that a group G has finite rank r, say, if every finitely generated subgroup of G is r-generated. (Thus G has *finite special rank* r.) In [**3**] the authors show that (in particular) a locally soluble-by-finite group G in which every subgroup of infinite rank is subnormal of defect at most d, where d is some fixed positive integer, is either nilpotent of bounded class or of finite rank. This result generalizes to some extent the well-known theorem of Roseblade [**9**] that a group in which every subgroup is subnormal of bounded defect is nilpotent (of bounded class). In the present article we return to the theme of [**5**] and establish the following improvement on Theorem 1 above.

Theorem 2. *Let G be a locally soluble-by-finite group in which every subgroup of infinite rank is subnormal. If G has infinite rank then G is soluble, and hence a Baer group.*

As noted in [**5**], the solubility of G is a reasonable hypothesis in the statement of Theorem 1, since a group with *all* subgroups subnormal is in any case soluble, a remarkable result due to Möhres [**6**]. On the other hand, such a group need not be nilpotent, as the famous Heineken-Mohamed examples [**4**] indicate – these groups have trivial centre and all subgroups subnormal. Whether the local hypothesis on G presented in Theorem 2 can be weakened to any significant extent is not clear – the conclusion may hold for locally graded groups G i.e. groups in which all finitely generated nontrivial subgroups have nontrivial finite images. However, as it is not even known whether a finitely generated p-group (p a prime) that is residually finite, and has all subgroups either finite or of finite index (and hence subnormal), need itself be finite, we are nowhere near being able to answer the above question at present.

Papers [**10**] and [**11**] address the topic of groups in which every subgroup is either subnormal or nilpotent, and we shall have occasion here to use the main results of these articles. Indeed, it is shown in [**10**] that a torsion-free, locally soluble-by-finite group in which every non-nilpotent subgroup is subnormal is itself nilpotent, and since a locally nilpotent group of finite rank that is torsion-free is nilpotent [**7**, Theorem 6.36], we deduce immediately that a torsion-free locally nilpotent group that satisfies the hypotheses of Theorem 2 is nilpotent. We therefore obtain the following consequence of Theorem 2.

THEOREM 3. *Let G be a torsion-free locally soluble-by-finite group in which every subgroup of infinite rank is subnormal. If G has infinite rank then G is nilpotent.*

It is also reasonable to ask whether the conclusion of Theorem 2 can be replaced by the stronger one that such a group G, if of infinite rank, must have all subgroups subnormal. The next result shows that this is not the case.

THEOREM 4. *There exists a metabelian, locally nilpotent group G of infinite rank such that the torsion subgroup T of G has finite rank and contains a non-subnormal subgroup of G, but every subgroup that is not contained in T is subnormal in G.*

Our final result shows that, in one sense, the above example is optimal.

THEOREM 5. *Let G be a group that satisfies the hypotheses of Theorem 2, and suppose that the torsion subgroup T of G has infinite rank. Then every subgroup of G is subnormal.*

2. The proof of Theorem 2. Let G be a locally soluble-by-finite group in which all subgroups of infinite rank are subnormal, and suppose that G has infinite rank. We wish to show that G is soluble, for then G is a Baer group, by Theorem 1. If H is an arbitrary subgroup of G of infinite rank then H is subnormal in G, as is every subgroup that contains H, and by considering a (finite) subnormal series from H to G we see that every factor of this series is soluble [**6**], so that some term of the derived series of G is contained in H. Now, since G has infinite rank it has a locally soluble subgroup H of infinite rank [**2**], and if H is soluble then so is G. Thus we may assume that G is locally soluble. If G is not soluble then it has finitely generated subgroups of arbitrarily high derived length and rank, and so we may also suppose that G is countable.

We now proceed to establish some properties that will allow us to pass to successive sections of G as our proof proceeds. If H is an infinite rank subgroup of any section K of G then $K/Core_K(H)$ is soluble, as above. In particular, if H is soluble then so is K – in other words, each insoluble section K of G has no soluble subgroups of infinite rank. Now a locally nilpotent group of infinite rank has an infinite rank abelian subgroup [**7**, Corollary 2 to Theorem 6.36], and the same holds for hyperabelian groups [**1**] and locally finite groups [**12**]. Again let K be an infinite rank section of G. If M is an infinite rank subgroup of K generated by K-invariant subgroups M_λ of finite rank then, since each M_λ has a characteristic and hence K-invariant abelian series [**7**, Lemma 10.39], M is hyperabelian and K is therefore soluble (since M has an abelian subgroup of infinite rank). On the other hand, if M is a normal subgroup of K of finite rank such that K/M is soluble then K is hyperabelian and therefore soluble. What all this means in the present context is that, in order to establish that G is soluble, we may restrict our attention to the following case.

(1) *G has no nontrivial normal subgroups of finite rank, and both the locally nilpotent and locally finite radicals of G are trivial. G has no infinite rank soluble subgroups.*

Note that hypotheses (1) imply that G is insoluble.

LEMMA 1. *G is locally of finite rank, and every proper image of G is soluble and locally nilpotent.*

Proof. Let F be a finitely generated subgroup of G; then F is soluble and therefore, by (1), of finite rank. If N is an arbitrary nontrivial normal subgroup of G then, again by (1), N has infinite rank and so all subgroups of G/N are subnormal. Thus G/N is locally nilpotent and, by [**6**], soluble. □

Next, let G_0 denote the intersection of all normal subgroups N of G such that G/N is torsion-free and locally nilpotent. Certainly G/G_0 is torsion-free, and if $G_0 \neq 1$ then G/G_0 is soluble, by Lemma 1. If M is a proper nontrivial normal subgroup of G_0 then M has infinite rank, for otherwise M has a characteristic abelian series (as before) and hence contains a nontrivial abelian subnormal subgroup of G, contradicting the fact that the locally nilpotent radical of G is trivial. But M of infinite rank implies that G_0/M is soluble, so there is a G-invariant subgroup L of M with G/L soluble and hence locally nilpotent. If U/L denotes the torsion subgroup of G_0/L then we have G/U torsion-free and locally nilpotent, which implies that $U = G_0$. Thus all proper images of G_0 are periodic (in the case $G_0 \neq 1$). On the other hand, if $G_0 = 1$ then G is residually torsion-free nilpotent.

LEMMA 2. *Let X be a locally soluble group that is residually torsion-free nilpotent. If X is locally of finite rank then X is locally nilpotent.*

Proof. Let F be a finitely generated subgroup of X; then F too is residually torsion-free nilpotent. Since F has finite Hirsch length there is no infinite descending chain of normal subgroups of F with each factor being nontrivial and torsion-free. It follows that F is nilpotent, as required. □

From Lemma 2 we see that if $G_0 = 1$ then G is locally nilpotent, contradicting (1). Thus G_0 is also a counterexample to Theorem 2, and we may assume that

(2) *every proper image of G is periodic, soluble and locally nilpotent.*

The intersection of all nontrivial normal subgroups of G cannot be nontrivial, as a minimal normal subgroup of a locally soluble group is abelian. It follows that G is residually periodic. Since G is countable it is the ascending union of finitely generated subgroups $F_1 \leq F_2 \leq \ldots$, and each F_i is soluble of finite rank, by Lemma 1, hence minimax, by [**7**, Theorem 10.38], and therefore nilpotent-by-abelian-by-finite [**7**, Proof of Theorem 10.38]. So, for each i, there are normal subgroups U_i, V_i of F_i with $V_i \leq U_i$, V_i nilpotent, U_i/V_i abelian and F_i/U_i finite. V_i may be chosen to be the Fitting radical of F_i (in this case, the unique maximal normal nilpotent subgroup of F_i), and U_i/V_i may be assumed torsion-free for each i. Since G is residually periodic, each F_i is residually finite. Let $V = \langle V_i : i = 1, 2, 3, \ldots \rangle$.

LEMMA 3. *If U_1/V_1 is nontrivial then V has finite rank.*

Proof. Suppose for a contradiction that V has infinite rank. Then V is subnormal in G, as is every subgroup that contains V, and so some term D of the derived series

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of G is contained in V, while G/D is periodic, by (2). Choose $u \in U_1 \backslash V_1$. Then there exists a positive integer n such that $u^n \in V \cap F_1$, and so $u^n \in (V_1 \ldots V_r) \cap F_1$ for some integer r. Now, for all $j > 1$, $(V_1 \ldots V_j) \cap F_1 = (V_1 \ldots V_j) \cap F_{j-1} \cap F_1 = (V_1 \ldots V_{j-1})(V_j \cap F_{j-1}) \cap F_1 = (V_1 \ldots V_{j-1}) \cap F_1$, since $V_j \cap F_{j-1}$ is a normal nilpotent subgroup of F_{j-1} and hence contained in V_{j-1}. It follows that $(V_1 \ldots V_r) \cap F_1 = V_1$, and we obtain the contradiction that U_1/V_1 is periodic. The lemma is proved. □

Thus, if U_1/V_1 is nontrivial then V has finite rank r, say, and so each V_i has rank (at most) r. We proceed to dispense with this possibility. Firstly, suppose that G is not torsion-free. Then there is an element x of prime order p in G, and $\langle x \rangle^G$ has infinite rank and is insoluble, by (1). We may assume in this case that G is generated by elements of order p, and so every proper image of G is a locally nilpotent p-group, by (2). In particular, G is residually a p-group and so every finitely generated subgroup is residually finite-p. Thus, in general, we may suppose that every periodic subgroup is a p-group, where p is some fixed prime.

For each i, let T_i denote the torsion subgroup of V_i, and let S be the subgroup of V generated by all T_i. Clearly S is a locally finite p-group and, since it has finite rank, S is Černikov. But S is also residually of finite exponent, so it is finite of order m, say, and each T_i therefore has order at most m. For each i, let D_i be the centralizer of T_i in F_i, so that F_i/D_i has bounded order and therefore bounded derived length.

Now V_i/T_i is a torsion-free nilpotent group of rank at most r and hence of nilpotency class c that is also at most r, as may be seen by noting that the rank of V_i/T_i is the sum of the ranks of the upper central factors of V_i/T_i. Let $\bar{Z}_{i,j}$ denote one such upper central factor, and let $C_{i,j}$ be the centralizer of $\bar{Z}_{i,j}$ in F_i; then $F_i/C_{i,j}$ embeds in $GL(r, \mathbb{Q})$ and therefore has r-bounded derived length, by the well-known theorem of Zassenhaus [**7**, Theorem 3.2.3]. Denoting by C_i the intersection of the $C_{i,j}$, $j = 1, \ldots, c$, we deduce that F_i/C_i has bounded derived length, as therefore has F_i/E_i, where $E_i = C_i \cap D_i$. If E_iV_i/V_i is non-trivial then, since F_i/V_i is soluble, there is a nontrivial F_i-invariant abelian subgroup B_iV_i/V_i of E_iV_i/V_i. But B_i acts nilpotently on V_i and so B_iV_i is nilpotent, and since V_i is the Fitting radical of F_i we have a contradiction. It follows that E_i is contained in V_i and hence that F_i/V_i has bounded derived length, so that F_i also has bounded derived length. But this gives the contradiction that G is soluble.

The above argument establishes that V cannot have finite rank, and Lemma 3 now tells us that U_1/V_1 is trivial. But if $U_i > V_i$ for any i then a suitable relabelling allows us to suppose that $U_1 > V_1$, and we are thus forced to conclude that $U_i = V_i$ for all i, so each F_i is finitely generated nilpotent-by-finite and hence polycyclic. We know from (1) that every abelian subgroup of G has finite rank; if the ranks of the torsion-free abelian subgroups of G were bounded then we would obtain from [**2**, Corollary 3.7] that G has finite rank. By this contradiction we see that there is no bound for the torsion-free ranks of the F_i.

LEMMA 4. *Let X be a polycyclic group, Y a subgroup of X, and let V be a normal subgroup of finite index in Y. Suppose that the derived length of X is exactly d. Then there is a normal subgroup U of finite index in X such that X/U has derived length exactly d and $U \cap Y \leq V$.*

Proof. Since X is residually finite there is certainly a normal subgroup U_0 of finite index in X with the derived length of X/U_0 being exactly d. Also, using a theorem of Mal'cev [**8**, 5.4.16], there is a subgroup U_1 of finite index in X such that $U_1 \cap Y = V$. Let U_2 be the core of U_1 in X and set $U = U_0 \cap U_2$. □

Conclusion of the proof of Theorem 2. For an arbitrary group H, we denote by $H^{(n)}$ the nth term of the derived series of H, and if H is soluble we denote by $d(H)$ the derived length of H. Suppose that $d(F_1) = d_1$ and choose N_1 normal in F_1 with F_1/N_1 finite and $d(F_1/N_1) = d_1$. By relabelling if necessary we may assume that $d(F_2) = d_2$ is greater that $2d_1$, and by Lemma 4 we may choose $N_2 \lhd F_2$ with F_2/N_2 finite, $d(F_2/N_2) = d_2$ and $N_2 \cap F_1 \leq N_1$. Since $d(N_1N_2/N_2) \leq d(N_1) \leq d_1$, we see that $F_2^{(d_1)} \not\leq N_1N_2$. Note that $N_1N_2 \cap F_1 = N_1(N_2 \cap F_1) = N_1$. Next, we may assume that $d(F_3) = d_3 > 2d_2 + d_1$, and so there exists a normal subgroup N_3 of F_3 with F_3/N_3 finite, $d(F_3/N_3) = d_3$ and $N_3 \cap F_2 \leq N_2$. We have $d(N_1N_2N_3/N_3) \leq d(N_1N_2) \leq d_1 + d_2$, so that $F_3^{(d_2)} \not\leq N_1N_2N_3$, also $N_1N_2N_3 \cap F_2 \leq N_1N_2$. We continue in this manner, and with the obvious notation. Write $J_i = N_1 \dots N_i$ for each i, and set $J = \langle J_i : i = 1, 2, \dots \rangle$. Since the F_i have unbounded torsion-free rank, so do the N_i, and it follows that J has infinite rank and hence that every subgroup containing J is subnormal in G. We deduce that $G^{(d)} \leq J$ for some integer d, and hence that $F_{r+1}^{(dr)} \leq J$ for some r (chosen so that $d_r \geq d$). Certainly therefore $F_{r+1}^{(d_r)} \leq J_i$ for some $i > r + 1$. But $J_i \cap F_{r+1} = (N_1 \dots N_i) \cap F_{r+1} \leq (N_1 \dots N_i) \cap F_{i-1} = (N_1 \dots N_{i-1})(N_i \cap F_{i-1}) = J_{i-1}$, and continuing (if necessary) we see that $F_{r+1}^{(d_r)}$ is contained in J_{r+1}, contradicting the choice of N_{r+1}. The theorem is therefore proved. □

3. The proof of Theorem 5. Let G be as stated, so that G is both soluble and a Baer group, by Theorem 2. Note that G/T is nilpotent, by the remarks preceding the statement of Theorem 3. Assuming for a contradiction that not every subgroup of G is subnormal, there is a subgroup H of finite rank that is not subnormal in G. If every p-component of T has finite rank then, since T has infinite rank, we may write $T = T_1 \times T_2$ for some infinite rank subgroups T_1, T_2, where the sets of primes involved in T_1 and T_2 are disjoint. Since G/T is nilpotent and each of HT_1, HT_2 is subnormal in G, there is a positive integer n such that $[G,_n H] \leq T \cap HT_1 \cap HT_2 = T_1(T \cap H) \cap T_2(T \cap H) = T_1(T_2 \cap H) \cap T_2(T_1 \cap H) = (T_2 \cap H)(T_1 \cap H) \leq H$, giving the contradiction that H is subnormal. Hence there is a p-component P of T that has infinite rank, and since HP is subnormal in G we may suppose that $G = HP$. There is a term R of the derived series of P that has infinite rank while R' has finite rank, and so there exist G-invariant subgroups A and B of P, with B of finite rank and contained in A and A/B infinite abelian and of exponent p. Since every subgroup of G/A is subnormal, we may further suppose that $G = AH$.

Let D denote the divisible component of B; then D is central in the Baer p-group A [**7**, Lemma 3.13] and hence, as DH is a Baer group of finite rank r, say, we have $D \leq Z_r(G)$, the rth term of the upper central series of G [**7**, Vol. 2, p. 38]. But B/D is finite, so $B \leq Z_s(G)$ for some integer s, and it follows that H is subnormal in HB, so we may assume that B is contained in H. But now $H \cap A$ is normal in G, and we may again factor and hence suppose that A is abelian and of exponent p. By induction on the derived length of H we may assume that H' is subnormal in G, so that $[A,_k H'] \leq H' \cap A = 1$, for some positive integer k. Writing $A_i = [A,_i H']$ for each $i \geq 0$ (where $A_0 = A$), we see that HA_{i+1} is not subnormal in HA_i for some integer i; certainly A_i/A_{i+1} must be infinite, and so factoring once more allows us to suppose that $[A, H'] = 1$. This in turn means that H' is normal in G, so we set $H' = 1$ and assume that H is abelian.

Since H has finite rank, it has a finitely generated subgroup F such that H/F is periodic. Because G is a Baer group there is a positive integer m such that $[A,_m F] = 1$, and repeating the above argument with F in place of H' we reduce to the case where H is itself periodic. But the p'-component of H may clearly be assumed trivial, so that now G is a p-group. Let S be an arbitrary subgroup of G. If S has finite rank then, since it is also a Baer p-group, S is nilpotent [**7**, Vol. 2, p. 38], and it follows that every subgroup of the p-group G is either nilpotent or subnormal. By [**11**, Theorem 3] we deduce that every subgroup of G is subnormal, our final contradiction. □

4. The proof of Theorem 4. Let p_1 be an arbitrary prime and define inductively a set of primes $p_1, p_2, \ldots$ such that $p_{n+1} > n(p_1 + \ldots + p_n)$ for each $n \geq 1$. For each n, let $H_n = \langle c_n \rangle \rtimes \langle b_n \rangle$, where $|c_n| = p_n^{n+1}$, $|b_n| = p_n^n$ and $[c_n, b_n] = c_n^{p_n}$, so that H_n is a finite p-group of nilpotency class n. Let K denote the cartesian product of the H_n, and let T be the torsion subgroup of K, namely their direct product. Write $d_1 = (1, c_2^{p_1}, c_3^{p_2}, c_4^{p_3}, \ldots)$, $d_2 = (1, 1, c_3^{p_1}, c_4^{p_2}, \ldots), \ldots$. (so that d_n has 1 for each of its first n entries, followed by $c_{n+1}^{p_1}, c_{n+2}^{p_2}, \ldots$). The direct product D of the $\langle d_n \rangle$ is a torsion-free abelian subgroup of K that has infinite rank. Our group G is then the subgroup $T \rtimes D$ of K. Setting $C = Dr\langle c_n \rangle$, $B = Dr\langle b_n \rangle$, we have $T = C \rtimes B$, $G' = T'[T, D] \leq C$, and so G is metabelian. Clearly $[G, D, D] = 1$, and so D is subnormal in G, and G is a Baer group. However, B is not subnormal in G, since $[\langle c_n \rangle,_{n-1} \langle b_n \rangle] \neq 1$, for each n. Finally, T is (non-nilpotent) of rank 2, and G has infinite rank.

Let U be a subgroup of G not contained in T, and choose an element u of $U \backslash T$. Then $u = d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_k^{\alpha_k} t$, for some $t \in T$, some positive integer k and some $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{Z}$, where not every α_i is zero. Certainly $t \in Dr_{i=1}^{l} H_i$ for some l. Choose an integer $N > \max(|\alpha_1| + k, \ldots, |\alpha_k| + k, l)$, and let $n \geq N$. We have $[G, U] \leq C$, and so it suffices to prove that U is subnormal in UC. If $V = Dr_{i=1}^{l} C_i$ then we see that $V \leq Z_{N+1}(G)$, and so U is subnormal in UV. We claim that now UV is normal in UC, and to show this we may as well suppose that V is contained in U. It remains to show that $[\langle c_n \rangle, U] \leq U$ for all $n \geq N$.

Let n be an arbitrary integer greater than or equal to N. Certainly DC is normal in G (since $G' \leq C$) and DC centralizes $\langle c_n \rangle$. If U projects trivially onto $\langle b_n \rangle$ via the restriction of the natural epimorphism from G onto G/DC then we have $[\langle c_n \rangle, U] = 1$, and there is nothing to prove. Otherwise, there is an element v of U of the form $v = b_n^{p_n^{\sigma n}} w$, where w is contained in the (G-invariant) subgroup generated by DC and $Dr_{i \neq n}\langle b_i \rangle$, and where $\sigma(n)$, the least such non-negative integer, is less than n. Observe that $[\langle c_n \rangle, U]$ is precisely $[\langle c_n \rangle, \langle b_n^{p_n^{\sigma n}} \rangle]$. It therefore suffices to prove that $g_n := [c_n, b_n^{p_n^{\sigma n}}]$ is contained in U. (Note that this does indeed suffice, since every subgroup of C is normal in G, while g_n generates $[\langle c_n \rangle, U]$ as a G-group.) Now U contains $[u, v] = [d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_k^{\alpha_k} t, b_n^{p_n^{\sigma n}} w]$, and since $G' \leq C$, which is the direct product of its primary components, it follows without difficulty that U contains $[d_1^{\alpha_1} d_2^{\alpha_2} \ldots d_k^{\alpha_k}, b_n^{p_n^{\sigma n}}]$ (note that $[t, G]$ and $[D, w]$ both intersect H_n trivially). So U contains $[c_n^{\lambda(n)}, b_n^{p_n^{\sigma n}}]$, where $\lambda(n) = \alpha_1 p_{n-1} + \alpha_2 p_{n-2} + \ldots + \alpha_k p_{n-k}$. It now suffices to prove that $0 < |\lambda(n)| < p_n$. We have $|\lambda(n)| \leq (\max(|\alpha_i|)(p_{n-1} + p_{n-2} + \ldots + p_{n-k}) < n(p_{n-1} + p_{n-2} + \ldots + p_{n-k}) \leq n(p_{n-1} + p_{n-2} + \ldots + p_1) < p_n$. Also, if i is least such that $\alpha_i \neq 0$, then $\lambda(n) = \alpha_i p_{n-i} + \alpha_{i+1} p_{n-i-1} + \ldots + \alpha_k p_{n-k}$, and $|\alpha_i p_{n-i}| \geq p_{n-i} > (n-i)(p_{n-i-1} + \ldots + p_1) \geq (n-i)(p_{n-i-1} + \ldots + p_{n-k}) \geq (n-k)(p_{n-i-1} + \ldots + p_{n-k}) > (\max|\alpha_j|)(p_{n-i-1} + \ldots + p_{n-k}) \geq |\alpha_{i+1} p_{n-i-1} + \ldots + \alpha_k p_{n-k}|$, so $\lambda(n) \neq 0$. Hence $0 < |\lambda(n)| < p_n$, and the result follows. □

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