AN APPLICATION OF JACOBI TYPE
POLYNOMIALS TO IRRATIONALITY MEASURES

ARI HEIMONEN, TAPANI MATALA-AHO AND KEIJO VÄÄNÄNEN

The paper provides irrationality measures for certain values of binomial functions and definite integrals of some rational functions. The results are obtained using Jacobi type polynomials and divisibility considerations of their coefficients.

1. INTRODUCTION

In the previous work [13] we considered the irrationality measures of the values of the Gauss hypergeometric series

$$(1) \qquad F(z) = {}_2F_1\left(\begin{array}{cc} 1, & b \\ & c \end{array}\bigg| z\right) = \sum_{n=0}^{\infty} \frac{(b)_n}{(c)_n} z^n,$$

where $b, c \neq 0, -1, -2, \ldots$ are rational parameters, and $(b)_0 = 1$, $(b)_n = b(b+1) \ldots (b+n-1)$, $n = 1, 2, \ldots$. The general result of the work [13] was sharpened in the case of the logarithmic function by using the divisibility properties of the coefficients of the approximation polynomials. This idea was first realised for the binomial function by Chudnovsky [5], and then for the logarithmic function by Rukhadze [15], see also Dubitskas [7] and Hata [10]. In this paper we apply the results of [13], especially the general divisibility criterion, to the case $b = 1/k$, $c = 1 + 1/k$ $(k \geqslant 2)$ and to the binomial case $b = 1/k$, $c = 1$ $(k \geqslant 3)$ in order to get sharper irrationality measures both in the archimedean and p-adic case. For example in the first case we achieve generalisations of the theorems of Chudnovsky [4], Danilov [6], Dubitskas [7], Hata [10, 11] and Huttner [14]. In the binomial case we obtain the archimedean results of Chudnovsky [5] (see also Easton [8]), which considerably improve the work of Baker [2], and p-adic analogues improving the work of Bundschuh [3].

2. NOTATION AND RESULTS

We shall denote by $\mathbb{Q}_v$ the v-adic completion of $\mathbb{Q}$, where $v \in \{\infty, \text{primes } p\}$, in particular $\mathbb{Q}_\infty = \mathbb{R}$. For an irrational number $\theta \in \mathbb{Q}_v$, by an irrationality measure $m_v(\theta)$ of θ we mean the infimum of all m satisfying the following condition: for any $\varepsilon > 0$ there exists an $H_0 = H_0(\varepsilon)$ such that

$$\left| \theta - \frac{P}{Q} \right|_v > H^{-m-\varepsilon}$$

for all rationals P/Q satisfying $H = \max\{|P|, |Q|\} > H_0$. In the following we denote $m_\infty(\theta) = m(\theta)$. All our measures are effective in the sense that H_0 can be effectively determined.

In our first case we have $b = 1/k$, $c = 1 + 1/k$, where $k \geqslant 2$ is a natural number. For a given rational r/s, $(r,s) = 1$, we denote the denominators of $(s-r)/k$ and $(s-r) \big/ \left(k \prod_{p|k} p \right)$ by k^* and k^{**}, respectively. Further, with a given rational $\beta \geqslant 1$ we define

$$A(\beta, z) = \min_{0 < \rho < |z| + \frac{1}{2}(1 - \operatorname{sgn} z)} \left(\frac{(\rho + |z|)(\rho + |z| - \operatorname{sgn} z)^\beta}{\rho} \right) \quad (z < 0 \text{ or } z \geqslant 1),$$

$$S(\beta, z) = \max_{0 \leqslant t \leqslant 1} \frac{t(1-t)^\beta}{1 - zt} \quad (-1 \leqslant z < 1),$$

$$\omega_1(\beta) = k\mu_k \min \left\{ \left(\frac{k^*}{k} \right)^\beta, \left(\frac{k^{**}}{k\mu_k} \right)^\beta \right\} e^{\beta\lambda(k) - \sigma(\beta, k)},$$

$$Q(\beta, k) = \omega_1(\beta) |r|^\beta A\left(\beta, \frac{s}{r} \right), \quad R(\beta, k) = \omega_1(\beta) |r|^{1+\beta} s^{-1} S\left(\beta, \frac{r}{s} \right),$$

where

$$\mu_h = \prod_{p|h} p^{1/(p-1)}, \quad \lambda(h) = \frac{h}{\phi(h)} \sum_{\substack{i=1 \\ (i,h)=1}}^{h} \frac{1}{i} \quad (h \in \mathbb{Z}^+)$$

and $\sigma(\beta, k)$ is given in the formula (19). Then the following theorem holds.

THEOREM 1. Let $r/s \in (-1,1)$ be a rational number satisfying $(r,s) = 1$. Then

$$m\left({}_2F_1 \left(\begin{array}{c} 1, \frac{1}{k} \\ 1 + \frac{1}{k} \end{array} \bigg| \frac{r}{s} \right) \right) \leqslant \inf_\beta {}^* \left\{ 1 - \frac{\ln Q(\beta, k)}{\ln R(\beta, k)} \right\},$$

where $\inf_\beta^*$ means that for a given r/s the infimum is taken over all rationals $\beta \geqslant 1$ satisfying $R(\beta, k) < 1$.

We note that the choice $\beta = 1$ implies

$$m\left({}_2F_1 \left(\begin{array}{c} 1, \frac{1}{k} \\ 1 + \frac{1}{k} \end{array} \bigg| \frac{r}{s} \right) \right) \leqslant 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) + \lambda(k) + \ln\min\{k^*\mu_k, k^{**}\}}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| + \lambda(k) + \ln\min\{k^*\mu_k, k^{**}\}}$$

for all rationals $r/s \in (-1, 1)$ satisfying

$$\min\{k^*\mu_k, k^{**}\}e^{\lambda(k)}\left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1.$$

This is a sharpening of Huttner's Theorem [14]. In Hata [11] some improvements of Huttner's results were obtained under the assumption $r = 1 \equiv s\left(\mod k \prod_{p|k} p\right)$, which implies $k^{**} = 1$. Further, we note that the numbers in Theorem 1 are definite integrals of the form

$$\frac{1}{x}\int_0^x \frac{dt}{1 \pm t^k},$$

and the transcendence of this kind of numbers was considered by van der Poorten [16] using Baker's results on linear forms of logarithms.

Some numerical examples are given in the table of the following page.

Some of the numbers given explicitly in the table have been considered in Hata [10, 11], where the bounds are the same as we have (compere also with Huttner [14]). Recently Hata has achieved an improvement $m(\pi/\sqrt{3}) \leqslant 4.601579\ldots$ in the work [12], where he also reached a remarkable result $m(\pi) \leqslant 8.0161$. Further, we note that the table contains some examples where the value $\beta = 1$ does not give any measure.

Our p-adic result is

THEOREM 1p. *Let $p \nmid k$ be a prime and let us assume $|r/s|_p < 1$.*

(1) *If $r/s > 1$ is a rational number satisfying*

$$\min\{k^*\mu_k, k^{**}\}e^{\lambda(k)}r\,|r|_p^2 < 1,$$

then

$$m_p\left({}_2F_1\left(\begin{matrix} 1, \frac{1}{k} \\ 1 + \frac{1}{k} \end{matrix}\,\middle|\, \frac{r}{s}\right)\right) \leqslant \frac{2\ln|r|_p}{2\ln|r|_p + \ln r + \lambda(k) + \ln\min\{k^*\mu_k, k^{**}\}}.$$

(2) *If $r/s < 1$ is a rational number satisfying*

$$\min\{k^*\mu_k, k^{**}\}e^{\lambda(k)}\left(\sqrt{s} + \sqrt{s-r}\right)^2 |r|_p^2 < 1,$$

then

$$m_p\left({}_2F_1\left(\begin{matrix} 1, \frac{1}{k} \\ 1 + \frac{1}{k} \end{matrix}\,\middle|\, \frac{r}{s}\right)\right) \leqslant \frac{2\ln|r|_p}{2\ln|r|_p + 2\ln\left|\sqrt{s} + \sqrt{s-r}\right| + \lambda(k) + \ln\min\{k^*\mu_k, k^{**}\}}.$$

$\frac{r}{s}$	k	number	β	upper bound	$\beta = 1$
$-\frac{1}{3}$	2	$\frac{\pi}{\sqrt{3}}$	$\frac{3}{2}$	$5.0874\ldots$	$8.3099\ldots$
$-\frac{3}{13}$	2		$\frac{6}{5}$	$97.719\ldots$	$-$
$-\frac{1}{7}$	2	$\sqrt{7}\arctan\frac{1}{\sqrt{7}}$	$\frac{4}{3}$	$4.0298\ldots$	$4.8569\ldots$
$\frac{1}{45}$	2	$\sqrt{5}\log\frac{3\sqrt{5}+1}{3\sqrt{5}-1}$	$\frac{9}{8}$	$3.1346\ldots$	$3.2571\ldots$
$\frac{1}{8}$	2	$\sqrt{2}\log\frac{2\sqrt{2}+1}{2\sqrt{2}-1}$	$\frac{7}{6}$	$41.032\ldots$	$493.12\ldots$
$\frac{1}{5}$	2	$\sqrt{5}\log\frac{\sqrt{5}+1}{\sqrt{5}-1}$	$\frac{3}{2}$	$4.4937\ldots$	$6.5082\ldots$
$-\frac{2}{7}$	3		$\frac{7}{6}$	$84.960\ldots$	$-$
$-\frac{1}{5}$	3		$\frac{4}{3}$	$11.531\ldots$	$21.327\ldots$
$-\frac{1}{8}$	3	$\log 3 + \frac{\pi}{\sqrt{3}}$	$\frac{4}{3}$	$4.5586\ldots$	$5.5280\ldots$
$-\frac{2}{25}$	3		$\frac{7}{6}$	$6.7886\ldots$	$7.8411\ldots$
$\frac{1}{7}$	3		$\frac{4}{3}$	$9.2848\ldots$	$14.242\ldots$
$\frac{1}{4}$	3		$\frac{4}{3}$	$30.435\ldots$	$-$
$-\frac{1}{7}$	4		$\frac{4}{3}$	$6.6372\ldots$	$9.2718\ldots$
$-\frac{3}{29}$	4		$\frac{10}{9}$	$94.453\ldots$	$-$
$-\frac{1}{11}$	4		$\frac{8}{7}$	$12.086\ldots$	$16.347\ldots$
$-\frac{1}{64}$	4	$\log\frac{13}{5} + 2\arctan\frac{4}{7}$	$\frac{28}{27}$	$13.164\ldots$	$13.764\ldots$
$\frac{1}{81}$	4	$\log 2 + \arctan\frac{3}{4}$	$\frac{12}{11}$	$3.6073\ldots$	$3.7160\ldots$
$\frac{1}{9}$	4	$\sqrt{3}\left(\log(2+\sqrt{3})+\frac{\pi}{3}\right)$	$\frac{4}{3}$	$6.1382\ldots$	$8.2100\ldots$
$-\frac{1}{4}$	5		$\frac{5}{4}$	$33.251\ldots$	$-$
$-\frac{1}{9}$	5		$\frac{5}{4}$	$8.5809\ldots$	$11.538\ldots$
$-\frac{1}{24}$	5		$\frac{6}{5}$	$4.2604\ldots$	$4.6295\ldots$
$\frac{1}{31}$	5		$\frac{10}{9}$	$4.9971\ldots$	$5.3453\ldots$
$\frac{1}{11}$	5		$\frac{5}{4}$	$7.8909\ldots$	$10.230\ldots$
$-\frac{1}{35}$	6		$\frac{7}{6}$	$6.5200\ldots$	$7.3106\ldots$
$\frac{1}{49}$	6		$\frac{13}{12}$	$16.211\ldots$	$18.508\ldots$

Our second case considers the binomial function. Let $b = 1/k$, $c = 1$, where $k \geqslant 3$ is again a natural number, and for a given $r \in \mathbb{Z}$ let us denote the denominators of r/k and $r / \left(k \prod_{p|k} p \right)$ by k_* and k_{**}, respectively. Further, let

$$\tau(k) = \frac{2}{\phi(k)} \sum_{\substack{k/2 < q < k \\ (q,k)=1}} \left(\Psi(1) - \Psi\left(\frac{q}{k}\right) \right),$$

where Ψ is the digamma function, see [9].

THEOREM 2. *If $r/s \in (-1,1)$ satisfies*

$$\min\{k_* \mu_k, k_{**}\} e^{-\tau(k)} \left(\sqrt{s} - \sqrt{s-r}\right)^2 < 1,$$

then

$$m\left(\left(1 - \frac{r}{s}\right)^{-1/k}\right) \leqslant 1 - \frac{2\ln\left(\sqrt{s} + \sqrt{s-r}\right) - \tau(k) + \ln\min\{k_*\mu_k, k_{**}\}}{2\ln\left|\sqrt{s} - \sqrt{s-r}\right| - \tau(k) + \ln\min\{k_*\mu_k, k_{**}\}}.$$

For example we have $m\left(\sqrt[3]{2}\right) \leqslant \underline{2.4297\ldots}$ and $m\left(\sqrt[3]{6}\right) \leqslant \underline{2.3205\ldots}$, which are among the examples given by Chudnovsky [5, p.377]. In addition we obtain $m\left(\sqrt[4]{5}\right) \leqslant \underline{2.7473\ldots}$, $r/s = 1/81$, $m\left(\sqrt[7]{17}\right) \leqslant 5.0311\ldots$, $r/s = 11/3^7$, and $m\left(\sqrt[7]{127}\right) \leqslant \underline{2.70468\ldots}$, $r/s = 1/2^7$, where the underlined bounds are better than Thue's bound.

In the p-adic case we have an improvement of the work of Bundschuh [3].

THEOREM 2p. *Let $p \nmid k$ be a prime and let us assume $|r/s|_p < 1$.*

(1) *If $r/s > 1$ and*

$$\min\{k_* \mu_k, k_{**}\} e^{-\tau(k)} r \, |r|_p^2 < 1,$$

then

$$m_p\left(\left(1 - \frac{r}{s}\right)^{-1/k}\right) \leqslant \frac{2\ln |r|_p}{2\ln |r|_p + \ln r - \tau(k) + \ln\min\{k_*\mu_k, k_{**}\}}.$$

Especially we have

$$m_p\left(\left(1 - p^l\right)^{-1/k}\right) \leqslant \frac{2l \ln p}{l \ln p + \tau(k) - \ln k\mu_k}$$

for all $p^l > k\mu_k e^{-\tau(k)}$.

(2) *If $r/s < 1$ and*

$$\min\{k_* \mu_k, k_{**}\} e^{-\tau(k)} \left(\sqrt{s} + \sqrt{s-r}\right)^2 |r|_p^2 < 1,$$

then

$$m_p\left(\left(1 - \frac{r}{s}\right)^{-1/k}\right) \leqslant \frac{2\ln |r|_p}{2\ln |r|_p + 2\ln\left|\sqrt{s} + \sqrt{s-r}\right| - \tau(k) + \ln\min\{k_*\mu_k, k_{**}\}}.$$

For example

$$m_2\left(\sqrt[3]{3}\right) \leqslant \underline{2.4597\ldots}, \; \frac{r}{s} = \frac{128}{125}, \qquad m_2\left(\sqrt[3]{5}\right) \leqslant 2.7088\ldots, \; \frac{r}{s} = \frac{32}{27},$$

$$m_2\left(\sqrt[3]{41}\right) \leqslant \underline{2.4597\ldots}, \; \frac{r}{s} = \frac{3\cdot 128}{343}, \quad m_5\left(\sqrt[3]{11}\right) \leqslant 2.7845\ldots, \; \frac{r}{s} = \frac{3\cdot 25}{64},$$

$$m_3\left(\sqrt[4]{-23}\right) \leqslant 3.1125\ldots, \; \frac{r}{s} = \frac{8\cdot 3^4}{5^4}, \quad m_5\left(\sqrt[4]{6}\right) \leqslant 3.2746\ldots, \; \frac{r}{s} = \frac{2\cdot 5^3}{4^4} < 1,$$

$$m_3\left(\sqrt[4]{13}\right) \leqslant 3.1717\ldots, \; \frac{r}{s} = \frac{3^5}{2^8} < 1, \quad m_5\left(\sqrt[4]{46}\right) \leqslant \underline{2.8704\ldots}, \; \frac{r}{s} = \frac{2\cdot 5^4}{6^4} < 1,$$

$$m_2\left(\sqrt[5]{13}\right) \leqslant \underline{2.6349\ldots}, \; \frac{r}{s} = \frac{2^8}{3^5}, \qquad m_2\left(\sqrt[5]{53}\right) \leqslant \underline{3.2736\ldots}, \; \frac{r}{s} = \frac{3\cdot 4^5}{5^5} < 1.$$

3. PADÉ TYPE APPROXIMATIONS

Throughout this paper we shall assume that $c > b > 0$, $b = a/f$, $c = g/h$, where a, f, g, h are natural numbers such that $(a, f) = (g, h) = 1$. Let us denote $B = b-1 = E/F$, $C = c-b-1 = G/H$ with $E, G \in \mathbb{Z}$, $F, H \in \mathbb{N}$, $(E, F) = (G, H) = 1$. Further, let l, m and n denote positive integer parameters satisfying $l \leqslant \min\{m, n\}$.

By using the Jacobi type polynomial

$$(2) \qquad A_{l,m,n}(z) = \frac{1}{z^B(1-z)^C}\frac{1}{l!}\left(\frac{d}{dz}\right)^l\left(z^{n+B}(1-z)^{m+C}\right)$$

and the integral representation

$$(3) \qquad F(z) = \int_0^1 \frac{\omega(t)}{1-zt}\,dt, \quad \omega(t) = \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}t^B(1-t)^C \quad (|z| < 1)$$

we obtained in [13] the approximation formula

$$(4) \qquad R_{l,m,n}(z) = Q_{l,m,n}(z)F(z) - P_{l,m,n}(z),$$

where

$$Q_{l,m,n}(z) = z^{n+m-l}A_{l,m,n}\left(\frac{1}{z}\right) = (-1)^l(z-1)^{m-l}\sum_{k=0}^{l}\binom{n+B}{k}\binom{m+C}{l-k}(1-z)^k,$$

$$P_{l,m,n}(z) = z^{n+m-l-1}\int_0^1 \frac{A_{l,m,n}(1/z) - A_{l,m,n}(t)}{1/z - t}\omega(t)\,dt,$$

$$R_{l,m,n}(z) = z^{n+m-l}\int_0^1 \frac{A_{l,m,n}(t)\omega(t)}{1-zt}\,dt$$

$$= (-1)^l z^{n+m}\frac{(b)_n(c-b)_m}{(c)_{n+m}}\,{}_2F_1\left(\begin{array}{c} l+1 ,\; n+b \\ n+m+c \end{array}\bigg| z\right).$$

In the following we suppose that $l = n$, $m = [\beta n]$, where $\beta \geqslant 1$ is a rational number, and we use the notation

$$P_n(z) = P_{l,m,n}(z), \quad Q_n(z) = Q_{l,m,n}(z), \quad R_n(z) = R_{l,m,n}(z).$$

Further, let $\delta = \delta(v)$ be 1, if $v = \infty$, and 0, if v is finite. By $c_1, c_2, \ldots$ we shall denote positive constants independent of n. We now have the following lemma.

LEMMA 1.

(i) *If $|z|_v < 1$ and in the finite case $v \nmid fh$, then we have*

$$|R_n(z)|_v \leqslant c_1 n^{1-\delta} \left(|z|_v^{1+\beta} S(\beta, z)^\delta \right)^n$$

for all $n \geqslant c_2$. In the archimedean case the bound on the right-hand side of this inequality is an asymptotic for $|R_n(z)|$ $(n \to \infty)$, and the bound holds at $z = -1$, too.

(ii) *If $z \in [-1, 1)$ and $S(\beta, z) \leqslant A(\beta, 1/z)/|z|$, then*

$$\max\{|Q_n(z)|, |P_n(z)|\} \leqslant c_3 \left(|z|^\beta A\left(\beta, \frac{1}{z}\right) \right)^n.$$

(iii) *If $q = \max\{B, C\} \geqslant -1/2$, $\beta = 1$ and $z > 1$, then*

$$\max\{|Q_n(z)|, |P_n(z)|\} \leqslant c_4 n^{q+2} |z|^n.$$

(iv) *If $\beta = 1$ and $z < -1$, then*

$$\max\{|Q_n(z)|, |P_n(z)|\} \leqslant c_5 \left(|z| A\left(1, \frac{1}{z}\right) \right)^n.$$

Further, all these bounds (i)–(iv) with the value $\beta = 1$ are true, if $P_n(z)$, $Q_n(z)$ and $R_n(z)$ are replaced by

$$\widehat{P}_n(z) = P_{n,n,n+1}(z), \quad \widehat{Q}_n(z) = Q_{n,n,n+1}(z), \quad \widehat{R}_n(z) = R_{n,n,n+1}(z),$$

respectively.

PROOF: Our lemma follows immediately from Lemmas 1 and 3 of [13]. We only need to note that the the polynomial $A_{n,n,n+1}(z)$ with parameters B and C equals the polynomial $z A_{n,n,n}(z)$ with parameters $1 + B$ and C.   ☐

LEMMA 2. *If $\beta = 1$, then*

$$\begin{vmatrix} Q_n(z) & P_n(z) \\ Q_{n+1}(z) & P_{n+1}(z) \end{vmatrix} = (-1)^n \frac{(b)_n (c-b)_n}{(c)_{2n}} \binom{-n-c}{n+1} z^{2n},$$

and

$$\begin{vmatrix} \widehat{Q}_n(z) & \widehat{P}_n(z) \\ \widehat{Q}_{n+1}(z) & \widehat{P}_{n+1}(z) \end{vmatrix} = (-1)^n \frac{(b)_{n+1} (c-b)_n}{(c)_{2n+1}} \binom{-n-c-1}{n+1} z^{2n+1}.$$

The first equality is proved in [13, Lemma 6], and the other can be proved similarly.

4. ON THE ARITHMETIC PROPERTIES OF THE COEFFICIENTS OF P_n AND Q_n

From the definition of P_n and Q_n it follows that (see [13])

$$Q_n(z) = \sum_{j=0}^{n} (-1)^m a_j (1-z)^{m-n+j},$$

$$P_n(z) = -z^{m-1} \sum_{j=m-n}^{m} c_j \sum_{i=0}^{j-1} \left(\frac{z-1}{z}\right)^{j-1-i} \frac{(C+1)\ldots(C+i)}{c(c+1)\ldots(c+i-1)},$$

where

$$(5) \quad a_j = \binom{n+B}{j}\binom{m+C}{n-j}, \qquad c_j = \sum_{i=0}^{j-m+n} (-1)^{m-j} a_{n-m+j-i} \binom{m-j+i}{i}.$$

Let us consider our first case, when $B = 1/k - 1$, $C = 0$. As in the proof of Lemma 4 in [13] we have

$$(6) \qquad\qquad a_j \in \frac{1}{k^j \mu_k(j)} \mathbb{Z} \qquad (j = 0, \ldots, n),$$

$$(7) \qquad\qquad c_j \in \frac{1}{k^{j-m+n} \mu_k(j-m+n)} \mathbb{Z} \qquad (j = m-n, \ldots, m),$$

$$(8) \qquad\qquad \frac{k^i i!}{(1+k)\ldots(1+ik)} \in \frac{k^i \mu_k(i)}{d_i(k)} \mathbb{Z},$$

where we have used the notation

$$\mu_k(i) = \prod_{p|k} p^{v_p(i!)}, \qquad d_i(k) = \mathrm{lcm}\{1+k, \ldots, 1+ik\}.$$

(As usual, if $r \in \mathbb{Q}$, $r \neq 0$, then $v_p(r)$ is defined by $r = p^{v_p(r)} R/S$, where $(R,S) = (R,p) = (S,p) = 1$.) Thus the terms in the sum presenting $P_n(r/s)$ are of the form

$$c_j \left(\frac{r-s}{r}\right)^{j-i-1} \frac{k^i i!}{(1+k)\ldots(1+ik)} \in \frac{(r-s)^{j-i} r k^{m-n} \mu_k(i)}{(r-s)(rk)^{j-i} \mu_k(j-m+n) d_i(k)} \mathbb{Z},$$

where we used (8) to obtain the last expression. Since

$$v_p((m-n)!) + v_p((j-m+n)!) - v_p(i!) - v_p((j-i)!)$$
$$\leqslant \sum_{k \leqslant \left[\frac{\ln m}{\ln p}\right]} \left(\left[\frac{m-n}{p^k}\right] + \left[\frac{j-m+n}{p^k}\right] - \left[\frac{i}{p^k}\right] - \left[\frac{j-i}{p^k}\right]\right) \leqslant 2\left[\frac{\ln m}{\ln p}\right],$$

it follows that the above expression belongs to

$$\frac{k^{m-n}\mu_k(m-n)}{(r-s)r^{j-1-i}d_i(k)\mu_k(j-i)\prod_{p|k}p^{C(n)}}\left(\frac{r-s}{k}\right)^{j-i}\mathbb{Z},$$

where $C(n)=2\left[\dfrac{\ln\beta n}{\ln p}\right]$. Further, by (6), the terms in the sum giving $Q_n(r/s)$ satisfy

$$a_j\left(\frac{r-s}{s}\right)^{m-n+j}\in\frac{k^{m-n}\mu_k(m-n)(r-s)^{m-n+j}}{s^{m-n+j}k^{m-n+j}\mu_k(m-n+j)}\mathbb{Z}.$$

These considerations imply

LEMMA 3. *Suppose that* $B=1/k-1$, $C=0$. *If*

$$\Omega_{n,1}=\frac{(r-s)s^m d_m(k)\prod_{p|k}p^{C(n)}}{k^{m-n}\mu_k(m-n)}\min\{(k^*)^m\mu_k(m),(k^{**})^m\},$$

then

$$\Omega_{n,1}Q_n\left(\frac{r}{s}\right),\quad\Omega_{n,1}P_n\left(\frac{r}{s}\right)\in\mathbb{Z}.$$

Next we consider our second case, where $B=(1-k)/k$, $C=-1/k$. By the Gauss summation formula (see [9, p.104])

$$_2F_1\left(\begin{array}{c}a\;,\;-n\\c\end{array}\bigg|1\right)=\frac{(c-a)_n}{(c)_n}\quad(c\neq0,-1,\dots)$$

it follows that

$$\widehat{Q}_n(z)=\sum_{j=0}^n\binom{n+\frac{1}{k}}{j}\binom{n-\frac{1}{k}}{n-j}(1-z)^j=\sum_{j=0}^n\binom{n+\frac{1}{k}}{j}\binom{2n-j}{n-j}(-z)^j.$$

Thus

$$\widehat{Q}_n\left(\frac{r}{s}\right)\in\frac{1}{s^n k^n\mu_k(n)}\mathbb{Z}.$$

This result can be improved in many cases. Namely, $\widehat{Q}_n(r/s)$ is the sum of the terms

$$\binom{2n-j}{n-j}\frac{(1+(n-j+1)k)\dots(1+nk)}{j!}\left(\frac{-r}{ks}\right)^j$$

belonging to

$$\frac{1}{s^j\mu_k(j)}\left(\frac{r}{k}\right)^j\mathbb{Z}.$$

Since $F(z) = (1 - z)^{-1/k}$ it follows that $\widehat{P}_n(z) = \lambda Q_n^*(z)$ where λ is a constant and Q_n^* is obtained from $\widehat{Q}_n$ by replacing $1/k$ by $-1/k$. Further

$$\left[\widehat{Q}_n(z)F(z)\right]_n = \widehat{P}_n(z)$$

(if $f(z) = \sum\limits_{j=0}^{\infty} f_j z^j$, then $[f(z)]_n = \sum\limits_{j=0}^{n} f_j z^j$) and $\widehat{Q}_n(0) = Q_n^*(0)$, and therefore $\lambda = 1$. These results give us the following

LEMMA 4. *If*
$$\Omega_{n,2} = s^n \min\{k_*^n \mu_k(n), k_{**}^n\},$$

then
$$\Omega_{n,2}\widehat{Q}_n\left(\frac{r}{s}\right), \quad \Omega_{n,2}\widehat{P}_n\left(\frac{r}{s}\right) \in \mathbb{Z}.$$

In many cases the coefficients a_j of the polynomial $Q_n(z)$ have common factors. We follow Chudnovsky's [5] ideas to cancel them out. Recently his ideas were used successfully by Rukhadze [15] and Hata [10, 11, 12], and in [13] we gave a further approach to apply these ideas. Our main tool is the following lemma from [13] giving a criterion for primes $p > c_6\sqrt{n}$ dividing the numbers (5).

We use for a rational number r the notation $p \mid r$ or $r \equiv 0 \pmod{p}$, if $v_p(r) \geqslant 1$. Further, if $v_p(r) \geqslant 0$, then there exists a unique $\bar{r} \in \{0, 1, \ldots, p-1\}$ satisfying $\bar{r} \equiv r \pmod{p}$.

LEMMA 5. **(Divisibility criterion for the coefficients of Q_n)** *Let $P(l,m,n)$ denote the set of all primes satisfying $p \nmid FH$,*

$$p^2 > \max\left\{l, \max_{0\leqslant j\leqslant l}\{|E + (n-j)F|, |G + (m-j)H|\}\right\},$$
(9)
$$\overline{n+B} + \overline{m+C} + 1 \leqslant \bar{l}.$$

Then
$$\left(\prod_{p\in P(l,m,n)} p\right) \left| \binom{n+B}{i}\binom{m+C}{l-i} \right. \quad (i = 0, 1, \ldots, l).$$

Now we apply this criterion to our cases 1 and 2.

Case 1: Let $k \in \mathbb{N}$, $k \geqslant 2$, $B = 1/k - 1$, $C = 0$. The inequality (9) has now the form ($l = n$)

(10)
$$\overline{n + \frac{1-k}{k}} + \overline{m} + 1 \leqslant \bar{n}.$$

If $\overline{n + (1-k)/k} = \overline{n} + \overline{(1-k)/k} \ (< p)$, then (10) cannot hold. Thus we must have

$$\overline{n + \frac{1-k}{k}} = \overline{n} + \overline{\frac{1-k}{k}} - p.$$

Therefore (10) is true, if the inequalities

$$p - \overline{\frac{1-k}{k}} \leqslant \overline{n} < p, \quad 0 \leqslant \overline{m} < p - \overline{\frac{1-k}{k}}$$

are valid.

We consider the primes p satisfying $p \geqslant c_7 \sqrt{n}$, $p \equiv q' \pmod{k}$ with some $q' \in \{1, \ldots, k-1\}$, $(q', k) = 1$. There then exists a unique $q \in \{1, \ldots, k-1\}$, $(q, k) = 1$, satisfying $qq' \equiv -1 \pmod{k}$. When q' runs over the reduced residue classes $\pmod{k}$, so does also q. Thus

$$\overline{\frac{1-k}{k}} = \frac{1-k+qp}{k}.$$

It follows that p satisfies (10), if

$$p\left(1 - \frac{q}{k}\right) - \frac{1-k}{k} \leqslant \overline{n} < p, \quad 0 \leqslant \overline{m} < p\left(1 - \frac{q}{k}\right) - \frac{1-k}{k}$$

or

$$(11) \qquad \frac{n}{N+1} < p \leqslant \frac{kn+1-k}{kN+k-q}, \quad \frac{km+1-k}{kM+k-q} < p \leqslant \frac{m}{M},$$

where we have used the notation $\overline{n} = n - Np$, $\overline{m} = m - Mp$. These inequalities give the following conditions and the corresponding intervals, where all the primes $p \equiv q' \pmod{k}$ satisfy (10):

(i) $k \leqslant (\beta + 1)q$

    (i1) $\beta N + \beta > M \geqslant \beta N + \beta - 1 + q/k$, $(n/(N+1), m/M)$;

    (i2) $\beta N + \beta - 1 + q/k > M > \beta N + \beta(1 - q/k)$,

         $((km+1-k)/(kM+k-q), m/M)$;

    (i3) $\beta N + \beta(1 - q/k) \geqslant M > \beta N + (\beta - 1)(1 - q/k)$,

         $((km+1-k)/(kM+k-q), (kn+1-k)/(kN+k-q))$;

(ii) $k > (\beta + 1)q$

    (ii1) $\beta N + \beta > M > \beta N + \beta(1 - q/k)$, $(n/(N+1), m/M)$;

    (ii2) $\beta N + \beta(1 - q/k) \geqslant M \geqslant \beta N + \beta - 1 + q/k$,

         $(n/(N+1), (kn+1-k)/(kN+k-q))$;

    (ii3) $\beta N + \beta - 1 + q/k > M > \beta N + (\beta - 1)(1 - q/k)$,

         $((km+1-k)/(kM+k-q), (kn+1-k)/(kN+k-q))$.

After getting these intervals we use the prime number theorem in arithmetic progressions to find out an asymptotic for the part of the common factor obtained from the primes $p \equiv q' \pmod{k}$ (for effectivity we refer to Adleman-Pomerance-Rumely [1]). Multiplying these we then get an asymptotic for the total common factor.

If we choose $\beta = x/y$, $x, y \in \mathbb{N}$, $(x, y) = 1$, and $N = yK + i$, $i \in \{0, \dots, y-1\}$, then (i1) is of the form

$$xK + \beta i + \beta > M \geqslant xK + \beta i + \beta - 1 + \frac{q}{k}.$$

Thus the condition

(12) $\qquad k \leqslant (\beta + 1)q, \quad \beta i + \beta \neq [\beta i + \beta] = \begin{cases} \beta i + \beta - 1 + q/k & \text{or} \\ [\beta i + \beta + q/k] \end{cases}$

gives us an integer $M = xK + [\beta i + \beta]$ satisfying (i1). The corresponding interval is

(13) $\qquad \left(\dfrac{n}{yK + i + 1}, \dfrac{[\beta n]}{xK + [\beta i + \beta]} \right).$

For fixed i and q satisfying (12) we can calculate the asymptotic contribution $e^{n\Sigma_{i,q,1}}$ for the common factor, where

$$\Sigma_{i,q,1} = \frac{1}{y\phi(k)} \sum_{K=0}^{\infty} \left(\frac{1}{K + [\beta i + \beta]/x} - \frac{1}{K + (i+1)/y} \right)$$
$$= \frac{1}{y\phi(k)} \left(\Psi\left(\frac{i+1}{y} \right) - \Psi\left(\frac{[\beta i + \beta]}{x} \right) \right).$$

The same interval (13) and $\Sigma_{i,q,1}$ is obtained, if the pair (i, q) satisfies the condition

(14) $\qquad k > (\beta + 1)q, \quad \beta i + \beta \neq [\beta i + \beta] = \left[\beta i + \beta \left(1 - \frac{q}{k} \right) \right] + 1$

given by (ii1). If the pair (i, q) satisfies neither the condition (12) nor (14), we set $\Sigma_{i,q,1} = 0$. The inequality (i2) is of the form

$$xK + \beta i + \beta - 1 + \frac{q}{k} > M > xK + \beta i + \beta \left(1 - \frac{q}{k} \right),$$

which may give more than one integer solution M, if $(\beta + 1)q/k > 2$. More precisely, let us define

$$I = \left[\beta i + \beta - 1 + \frac{q}{k} \right] - \left[\beta i + \beta \left(1 - \frac{q}{k} \right) \right],$$

$$h = \begin{cases} 0, & \text{if } \beta i + \beta - 1 + q/k \neq [\beta i + \beta - 1 + q/k], \\ 1, & \text{if } \beta i + \beta - 1 + q/k = [\beta i + \beta - 1 + q/k]. \end{cases}$$

Thus the condition

$$(15) \qquad\qquad k \leqslant (\beta + 1)q, \quad I - h \geqslant 1$$

gives $I - h$ integer solutions $M = xK + [\beta i + \beta - 1 + q/k] - j$, $j = h, \ldots, I-1$, satisfying (i2). These all give disjoint intervals and we thus have the asymptotic $e^{n\Sigma_{i,q,2}}$, where

$$\Sigma_{i,q,2} = \frac{1}{y\phi(k)} \sum_{j=h}^{I-1} \left(\Psi\left(\frac{[\beta i + \beta + q/k] - j - q/k}{x} \right) - \Psi\left(\frac{[\beta i + \beta - 1 + q/k] - j}{x} \right) \right).$$

The condition (ii2) is transformed to

$$(16) \qquad \begin{cases} k > (\beta+1)q, & \left[\beta i + \beta\left(1 - \frac{q}{k}\right)\right] = \left[\beta i + \beta - 1 + \frac{q}{k}\right] + 1 \quad \text{or} \\ & \left[\beta i + \beta\left(1 - \frac{q}{k}\right)\right] = \beta i + \beta\left(1 - \frac{q}{k}\right) \quad \text{or} \\ & \left[\beta i + \beta - 1 + \frac{q}{k}\right] = \beta i + \beta - 1 + \frac{q}{k} \end{cases}$$

leading to the asymptotic $e^{n\Sigma_{i,q,2}}$, where

$$\Sigma_{i,q,2} = \frac{1}{y\phi(k)} \left(\Psi\left(\frac{i+1}{y} \right) - \Psi\left(\frac{i+1-q/k}{y} \right) \right).$$

Again, if (i, q) satisfies neither the condition (15) nor (16), let $\Sigma_{i,q,2} = 0$. The remaining cases (i3) and (ii3) give us the conditions
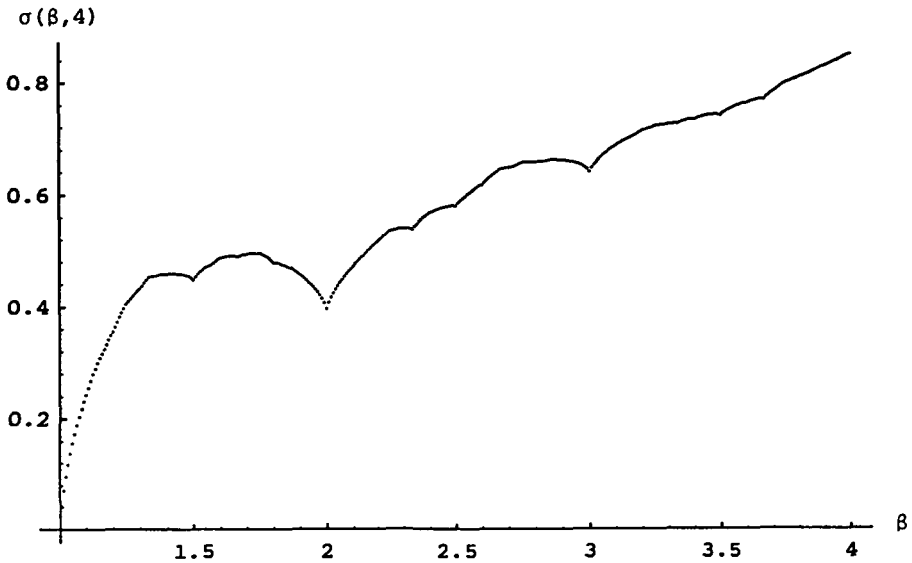
$$(17) \qquad k \leqslant (\beta+1)q, \quad \left[\beta i + \beta\left(1 - \frac{q}{k}\right)\right] = \left[\beta i + (\beta-1)\left(1 - \frac{q}{k}\right)\right] + 1,$$

$$(18) \qquad k > (\beta+1)q, \quad \beta i + \beta - 1 + \frac{q}{k} \neq \left[\beta i + \beta - 1 + \frac{q}{k}\right]$$
$$= \left[\beta i + (\beta-1)\left(1 - \frac{q}{k}\right)\right] + 1,$$

respectively, and the asymptotic in both cases is $e^{n\Sigma_{i,q,3}}$, where

$$\Sigma_{i,q,3} = \frac{1}{y\phi(k)} \left(\Psi\left(\frac{[\beta i + (\beta-1)(1 - q/k)] + 2 - q/k}{x} \right) - \Psi\left(\frac{i+1-q/k}{y} \right) \right).$$

As before, we set $\Sigma_{i,q,3} = 0$, if (i, q) satisfies neither the condition (17) nor (18). All the intervals obtained from the conditions (12), (14), (15), (16), (17) and (18) are disjoint.

Picture 1

Thus the multiplication of the contributions of all these cases gives us the asymptotic $e^{n\sigma(\beta,k)}$ of the common factor $D_{n,1}$, where

$$(19) \qquad\qquad \sigma(\beta,k) = \sum_{(i,q,t)} \Sigma_{i,q,t},$$

the summation running over the triples (i,q,t), $q = 1,\ldots,k-1$, $(q,k) = 1$, $i = 0,\ldots,y-1$, $t = 1,2,3$. Picture 1 shows the graph of $\sigma(\beta,4)$ (the interval of subsequent values of β is of length $1/120$).

    Case 2 (the binomial series): Let $k \in \mathbb{N}$, $k \geqslant 3$, $B = 1/k - 1$, $C = -1/k$. Let further $n = n' + 1$, $l = m = n'$ so that the criterion (9) has the form

$$(20) \qquad\qquad \overline{n + 1/k} + \overline{n - 1/k} + 1 \leqslant \overline{n},$$

where we have dropped the apostrophes.

    Again we consider large primes p satisfying $p \equiv q' \pmod{k}$ for some $q' \in \{1,\ldots,k-1\}$, $(q',k) = 1$, and choose $q \in \{1,\ldots,k-1\}$, $(q,k) = 1$, such that $qq' \equiv -1 \pmod{k}$. Then

$$\overline{\frac{1}{k}} = \frac{qp+1}{k}, \qquad -\overline{\frac{1}{k}} = \frac{(k-q)p-1}{k}.$$

First suppose that

$$\overline{n} + \overline{\frac{1}{k}} < p, \qquad \overline{n} + \overline{\left(-\frac{1}{k}\right)} < p.$$

In this case (20) is of the form $\overline{n} + p + 1 \leqslant 0$, which is impossible. By similar reasons the cases $\overline{n} + \overline{1/k} \geqslant p$, $\overline{n} + \overline{(-1/k)} < p$ and $\overline{n} + \overline{1/k} < p$, $\overline{n} + \overline{(-1/k)} \geqslant p$ give no contribution. Thus, we are left with the case

$$\overline{n} + \overline{\frac{1}{k}} \geqslant p, \quad \overline{n} + \overline{\left(-\frac{1}{k}\right)} \geqslant p.$$

These inequalities and (20) are satisfied if $(\overline{n} = n - Np)$

$$\max\left\{\frac{(k-q)p-1}{k}, \frac{qp+1}{k}\right\} \leqslant n - Np < p$$

or

$$\frac{n}{N+1} < p \leqslant \min\left\{\frac{kn+1}{kN+k-q}, \frac{kn-1}{kN+q}\right\}.$$

This gives us an interval

$$\left(\frac{n}{N+1}, \frac{kn+1}{kN+k-q}\right),$$

if $q < k/2$, and an interval

$$\left(\frac{n}{N+1}, \frac{kn-1}{kN+q}\right),$$

if $q > k/2$.

Therefore the above intervals imply for the common factor $D_{n,2}$ the asymptotic $e^{n\tau(k)}$, where

$$\tau(k) = \frac{2}{\phi(k)} \sum_{\substack{k/2 < q < k \\ (q,k)=1}} \sum_{N=0}^{\infty} \left(\frac{1}{N+q/k} - \frac{1}{N+1}\right) = \frac{2}{\phi(k)} \sum_{\substack{k/2 < q < k \\ (q,k)=1}} \left(\Psi(1) - \Psi\left(\frac{q}{k}\right)\right).$$

5. Proofs of the Theorems

We shall use the following well-known results (see, for example, Chudnovsky [5, Corollary 3.3] and [13, Lemma 7]). Let $x > 0$ and $y < 0$ be given. Suppose that for each $\varepsilon > 0$ there exists a constant c_8 and rational integers p_n, q_n satisfying for all $n \geqslant c_8$ the inequalities

(21) $$\frac{1}{n} \ln \max\{|q_n|, |p_n|\} < x + \varepsilon,$$

(22) $$y - \varepsilon < \frac{1}{n} \ln |r_n|_v < y + \varepsilon,$$

where $r_n = q_n \alpha - p_n$. Then the number $\alpha \in Q_v$ has an irrationality measure $m_v(\alpha)$ not greater than

$$
\begin{cases}
1 - \dfrac{x}{y}, & \text{if } v = \infty, \\[2mm]
1 - \dfrac{x}{y+x}, & \text{if } v \text{ is finite and } x + y < 0.
\end{cases}
$$

The lower bound in (22) is not needed, if $p_n q_{n+1} - q_n p_{n+1} \neq 0$ for all $n \geqslant c_9$.

In Case 1 we have an approximation formula

$$
r_n = q_n \; {}_2F_1 \left(\begin{matrix} 1 \, , \, \frac{1}{k} \\ 1 + \frac{1}{k} \end{matrix} \, \middle| \, \begin{matrix} r \\ s \end{matrix} \right) - p_n,
$$

where

$$
q_n = \frac{\Omega_{n,1} Q_n \left(r/s \right)}{D_{n,1}}, \quad p_n = \frac{\Omega_{n,1} P_n \left(r/s \right)}{D_{n,1}}
$$

are integers. Using Lemma 1 and the asymptotic bounds for $\Omega_{n,1}$ (see [13, Lemma 5]) and $D_{n,1}$ we get the following lemma.

LEMMA 6. *Let $\varepsilon > 0$ be given. If $|r/s| < 1$ and $R(\beta, k) < 1$, then*

$$
\max \{|p_n|, |q_n|\} \leqslant Q(\beta, k)^{(1+\varepsilon)n},
$$
$$
R(\beta, k)^{(1+\varepsilon)n} \leqslant |r_n| \leqslant R(\beta, k)^{(1-\varepsilon)n}
$$

for all $n \geqslant c_{10}$.

Therefore we immediately achieve the estimate

$$
m \left({}_2F_1 \left(\begin{matrix} 1 \, , \, \frac{1}{k} \\ 1 + \frac{1}{k} \end{matrix} \, \middle| \, \begin{matrix} r \\ s \end{matrix} \right) \right) \leqslant 1 - \frac{\ln Q(\beta, k)}{\ln R(\beta, k)}
$$

for any rational $\beta \geqslant 1$ satisfying $R(\beta, k) < 1$, which proves Theorem 1.

In the corresponding p-adic case we choose $\beta = 1$, $p \nmid k$ and $|r/s|_p < 1$. Thus Lemma 1 and Lemma 5 give

LEMMA 7. *Let $\varepsilon > 0$ be given and suppose $r/s > 1$. Then*

$$
\max \{|p_n|, |q_n|\} \leqslant \left(\min\{k^* \mu_k, k^{**}\} e^{\lambda(k)} |r| \right)^{(1+\varepsilon)n},
$$
$$
|r_n|_p \leqslant |r|_p^{(2-\varepsilon)n}
$$

for all $n \geqslant c_{11}$.

Now the use this lemma and the condition

$$
\min\{k^* \mu_k, k^{**}\} e^{\lambda(k)} r \, |r|_p^2 < 1,
$$

implies

$$m_p \left({}_2F_1 \left(\begin{matrix} 1 \ , \ \frac{1}{k} \\ 1 + \frac{1}{k} \end{matrix} \middle| \begin{matrix} r \\ s \end{matrix} \right) \right) \leqslant \frac{2 \ln |r|_p}{2 \ln |r|_p + \ln r + \lambda(k) + \ln \min\{k^* \mu_k, k^{**}\}}.$$

Part 2 is proved similarly using corresponding parts of Lemma 1. This completes the proof of Theorem $1p$.

In the second case we have

$$r_n = q_n \ {}_2F_1 \left(\begin{matrix} 1 \ , \ \frac{1}{k} \\ 1 \end{matrix} \middle| \begin{matrix} r \\ s \end{matrix} \right) - p_n,$$

where

$$q_n = \frac{\Omega_{n,2} \widehat{Q}_n \left(r/s \right)}{D_{n,2}}, \quad p_n = \frac{\Omega_{n,2} \widehat{P}_n \left(r/s \right)}{D_{n,2}}$$

are integers. Further, we denote

$$\omega_2 = \min \left\{ k_* \mu_k, k_{**} \right\} e^{-\tau(k)},$$
$$Q(k) = \omega_2 \left(\sqrt{s} + \sqrt{s - r} \right)^2,$$
$$R(k) = \omega_2 \left(\sqrt{s} - \sqrt{s - r} \right)^2.$$

Analogously to Lemma 6 we thus have

LEMMA 8. *Let $\varepsilon > 0$ be given and let $|r/s| < 1$, $R(k) < 1$. Then*

$$\max \left\{ |p_n|, |q_n| \right\} \leqslant Q(k)^{(1+\varepsilon)n},$$
$$R(k)^{(1+\varepsilon)n} \leqslant |r_n| \leqslant R(k)^{(1-\varepsilon)n}$$

for all $n \geqslant c_{12}$.

Clearly we get the result

$$m \left(\left(1 - \frac{r}{s} \right)^{-1/k} \right) \leqslant 1 - \frac{\ln Q(k)}{\ln R(k)}.$$

Thus Theorem 2 is proved.

By similar considerations we achieve the cases 1 and 2 in Theorem $2p$ using the corresponding polynomial bounds in Lemma 1. In our examples the p-adic values

$$(1 - z)^{1/k} = {}_2F_1 \left(\begin{matrix} 1 \ , \ -\frac{1}{k} \\ 1 \end{matrix} \middle| z \right) \quad \left(z \in \mathbb{Q}, \quad |z|_p < 1 \right)$$

are the unique solutions of the equation

$$x^k = 1 - z,$$

satisfying the property

$$x \equiv 1 \pmod{p}.$$

In these cases we denote

$$(1-z)^{1/k} = \sqrt[k]{1-z}.$$

For example

$$\sqrt[3]{1 - \frac{3 \cdot 5^2}{4^3}} = \frac{\sqrt[3]{11}}{\sqrt[3]{(-4)^3}} = -\frac{1}{4}\sqrt[3]{11} \in \mathbb{Q}_5.$$

REMARK. All the numerical computations were made using MATHEMATICA programs.

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Department of Mathematics
University of Oulu
90570 Oulu
Finland