Canad. Math. Bull. Vol. 28 (4), 1985

LINEAR MAPS ON HERMITIAN MATRICES: THE STABILIZER OF AN INERTIA CLASS

BY
CHARLES R. JOHNSON AND STEPHEN PIERCE

Dedicated to the memory of Robert Arnold Smith

ABSTRACT. Let T be a linear transformation acting on the space of $n \times n$ complex matrices. Let $G(k)$ be the set of all hermitian matrices with k positive and $n - k$ negative eigenvalues. Let T map some indefinite inertia class $G(k)$ onto itself. We classify all such T. The possibilities are congruence, congruence followed by transposition, and, if $n = 2k$, it is possible that $-T$ can be a congruence or a congruence followed by transposing. In other words, negation is an admissible transformation when $n = 2k$.

1. **Introduction**. Let $H(n)$ be the set of all $n \times n$ complex hermitian matrices. If $A \in H(n)$ has r positive, s negative, and t zero eigenvalues, the *inertia* of A is defined to be triple $i(A) = (r, s, t)$. If A is invertible, i.e., if $t = 0$, write $i(A) = (r, s)$. We note that $H(n)$ is not a complex vector space; in fact, the span of $H(n)$ is all $n \times n$ complex matrices $M(n, C)$.

Fix a particular inertia class (k, l, m) in $H(n)$. Let T be a linear transformation on $M(n, C)$ which maps the given inertia class into itself. It is an open problem to determine all such T. Obviously, any congruence or any congruence followed by transposition would qualify. By congruence we mean a transformation of the form $A \rightarrow X^*AX$ where $X \in M(n, C)$ is fixed and non-singular. We suspect that the following is true:

A. If $n > 2$, k and l are positive, and $k \neq l$, then T is a congruence or a congruence followed by transposition.

B. If $n > 2$, and $k = l > 0$, then T is one of the two types in part A, possibly followed by negation.

REMARK 1.1. Note that there is no initial assumption that T is non-singular; this must be proven.

REMARK 1.2. If k or l is zero, i.e., if the inertia class is semi-definite, then T could be a sum of congruences and in fact T could be singular. As an example, project each

Received by the editors May 29, 1984 and, in revised form, March 19, 1985.

The work of the first author was partially supported by National Science Foundation grant MCS 80-01611 and by Air Force Wright Aeronautical Laboratory contract F33615-81-K-3224.

The work of the second author was partially supported by NSERC Grant A-7862.

Key Words: Hermitian matrix, inertia class.

AMS Subject Classification: Primary, 15A21, 15A27.

matrix onto its diagonal part. Then T maps the inertia class $(n, 0, 0)$ into itself, T is singular, and one can easily verifiy that T is a sum of congruences.

REMARK 1.3. The assumption $n > 2$ is necessary. If $n = 2$, consider the linear map that doubles the $(1, 2)$ and $(2, 1)$ entries of each matrix. This preserves the inertia class $(1, 1)$, is nonsingular, but is not achievable by congruence.

Part of the difficulty with problems indicated by conjectures A and B may be that the hypotheses do not readily produce an algebraic set which is mapped into itself by T. Most "preserver" problems do not have this difficulty.

It is possible to obtain results similar to A and B by adding assumptions about T. We mention two of them here; one of them will be used later in this manuscript. The first is by Helton and Rodman [2] and the second by Schneider [4].

THEOREM 1.4. [2]. *Let $n > 2$. Fix an integer k, $0 < k < n$, and assume $2k \neq n$. Suppose T is a nonsingular linear transformation on $M(n, C)$ mapping the inertia class $(k, n - k)$ into itself. In addition, suppose that T is unital, i.e., $T(I) = I$, where I is the $n \times n$ identity matrix. Then T is a unitary congruence or a unitary congruence followed by transposition.*

REMARK 1.5. The assumption that T is unital allows the authors to use eigenvalue arguments in their proof. In the case that $k = 1$ or $n - 1$, the invertibility assumption on T can be removed.

THEOREM 1.6. [4]. *Let T be a linear transformation on $M(n, C)$. Suppose that T has one of the following two properties:*

(i) *T maps the positive definite hermitian matrices onto themselves;*

(ii) *T maps the set of positive semi-definite hermitian matrices onto itself.*

Then T is a congruence or a congruence followed by transposition.

REMARK 1.8. In [4], Schneider also proves the same result for a linear map T on the vector space of real symmetric matrices.

For additional information in the positive definite case, see the work of Choi [1].

2. **Statement of Results**. Briefly, our aim is to trade the unital assumption in [2] for an onto assumption, or equivalently, extend the result of [4] to other inertia classes. For convenience, we use the following notation. For each integer k between 0 and n, let $G(k)$ be the class of all matrices in $H(n)$ with inertia $(k, n - k)$. Thus $G(n)$ is the set of all positive definite matrices. Let P be the closure of $G(n)$, i.e., the positive semi-definite matrices and let N be the closure of $G(0)$. Fix an integer r, $0 < r < n$, and for the remainder of the paper, G is the particular (indefinite) inertia class $(r, n - r)$. We now state our result.

THEOREM 2.1. *Let T be a linear transformation on $M(n, C)$. If $r \neq n - r$, then T maps G onto itself if and only if T is a congruence or a congruence followed by transposition. If $r = n - r$, then T maps the inertia class G onto itself if and only if T or $-T$ is a* *congruence or a congruence followed by transposition. In other words, negation is an admissible map when* $r = n - r$.

REMARK 2.2. A similar result holds for T a linear map on the $n \times n$ real symmetric matrices which maps G onto itself.

3. **Proofs**. Our main idea is to show that if T maps G onto itself then T maps P onto itself, or, if $r = n - r$, that $T(P) = P$ or N. Then we appeal to Schneider's result.

We first observe that G contains n^2 linearly independent matrices in $M(n, C)$ and hence the "onto" assumption on T immediately implies that T is non-singular.

Let A and B be $n \times n$ hermitian matrices. Let $G(A, B)$ consist of all real numbers θ such that $\theta A + B$ is in G. Obviously $G(A, B)$ is an open set (possibly empty) in R. The following lemma is central to our argument.

LEMMA 3.1. *Let* $A \in H(n)$. *The set* $G(A, B)$ *is a single open interval (possibly empty, infinite, or semi-infinite) for every* B *in* $H(n)$, *if and only if* A *is positive or negative semi-definite.*

PROOF. Suppose A is indefinite. The problem is invariant to within congruence on A. Thus we take A to be the diagonal matrix $\operatorname{diag}(1, -1, \lambda_3, \ldots, \lambda_n)$ where the λ_i are chosen suitably small, some possibly zero. Next select $B = \operatorname{diag}(-2, 3, M_3, \ldots, M_n)$ where the M_i are chosen suitably large in absolute value and such that $B \in G$. With this choice of the λ_i and M_i, it follows that 0 and 4 are in $G(A, B)$, but 2.5 is not. Thus $G(A, B)$ is not a single open interval for any indefinite A.

Conversely, suppose that A is positive semi-definite. Let B be any hermitian matrix, and assume the eigenvalues of B are $\lambda_1 \geq \ldots \geq \lambda_n$. Let θ be a positive real number, and let $\lambda_1(\theta) \geq \ldots \geq \lambda_n(\theta)$ be the eigenvalues of $\theta A + B$. A well known inequality ([3], p. 510) states that $\lambda_i(\theta) \geq \lambda_i$, $i = 1, \ldots, n$, if $\theta \geq 0$ and $\lambda_i(\theta) \leq \lambda_i$, $i = 1, \ldots, n$ if $\theta \leq 0$. The result is now evident. □

REMARK 3.2. The converse proved above is not needed in the proof of Theorem 2.1; we add it for the reader's interest.

Returning to our transformation T in Theorem 2.1, we note that because T maps G onto itself, $\theta A + B \in G$ if and only if $T(\theta A + B)$ is in G. Thus $G(T(A), T(B)) = G(A, B)$ for any A, B in $H(n)$. It is also clear that if I is the $n \times n$ identity matrix, then $G(I, B)$ is a single open interval as specified in Lemma 3.1. Therefore, $T(I)$ must be positive or negative semi-definite. Let K be any member of $G(n)$ and let L be its positive definite square root. Set S to be the map $S(A) = T(LAL)$. Since S is the composition of a congruence and the map T, S also maps G onto itself and hence $S(I) = T(K)$ is positive or negative semi-definite. It follows that T maps every positive or negative definite matrix to a member of P or N. By continuity, T maps $P \cup N$ into itself.

We now assert that $T(P)$ is a subset of P or N. Let A and B be linearly independent members of P, and suppose that $T(A)$ and $T(B)$ are in P and N respectively. Since P is a convex cone, $rT(A) + T(B)$ is in $P \cup N$ for all non-negative r. If r is large enough,

$rT(A) \in P$. If $r = 0$, $T(B) \in N$. By the inequality in ([3], p. 510), together with the fact that P and N are closed sets, $\{r | rT(A) + T(B) \in P\}$ is a semi-infinite closed interval $[s, \infty)$ contained in the positive reals, and $\{r > 0 | rT(A) + T(B) \in N\}$ is a finite closed interval $[0, s]$. Thus, $sT(A) + T(B)$ is in $P \cap N$, and hence $sT(A) + T(B) = 0$. But T is nonsingular and this contradicts the linear independence of $T(A)$ and $T(B)$. Thus $T(P)$ is in P or N.

Assume for the moment that $T(P) \in P$. Because T maps G onto itself, the same is true of T^{-1}. Applying the previous argument to T^{-1}, we see that T^{-1} maps P into P and hence T acts bijectively on P. Then Theorem 2.1 follows from Schneider's result.

Now suppose $T(P)$ is a subset of N. Since T maps G onto itself, we have $T^{-1}(N)$ is in P. Thus, $T(P) = N$ and hence $-T(P) = P$. By Schneider's result, $-T$ is a congruence or a congruence followed by transposition. Hence, both T and $-T$ map G onto itself, and therefore G is the inertia class (r, r) where $2r = n$. Hence $r = n - r$, and our result is established. □

REFERENCES

1. M. D. Choi, *Positive linear maps*, Proc. Symp. in Pure Math., **38**(II) (1982), pp. 583–590.

2. J. W. Helton and L. Rodman, *Signature preserving linear maps on hermitian matrices*, submitted.

3. A. Marshall and I. Olkin, *Inequalities: The Theory of Majorization and its Applications*, Academic Press, 1979.

4. H. Schneider, *Positive operators and an inertia theorem*, Numer. Math., **7** (1965), pp. 11–17.