CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

The Local Langlands Conjecture for G_2

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Abstract
We prove the local Langlands conjecture for the exceptional group $G_2(F)$ where F is a non-archimedean local field of characteristic zero.

Contents

1. Introduction

Let F be a non-archimedean local field of characteristic 0 and residue characteristic p. Let W_F be the Weil group of F and let $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil-Deligne group. For a connected reductive group G over F, which we assume to be split for simplicity, Langlands conjectured that there is a surjective finite-to-one map from the set $\mathrm{Irr}(G)$ of (equivalence classes of) irreducible smooth representations of $G(F)$ to the set $\Phi(G)$ of (equivalence classes of) L-parameters

$$WD_F \longrightarrow G^\vee$$

where $G^\vee$ is the Langlands dual group of G and the homomorphisms are taken up to $G^\vee$-conjugacy. This leads to a partition of the set of equivalence classes of irreducible representations of $G(F)$ into a disjoint union of finite subsets, which are the fibers of the map and are called L-packets. Moreover, one would like to characterize the map

$$\mathcal{L}_G : \mathrm{Irr}(G) \longrightarrow \Phi(G)$$

by requiring that it satisfies a number of natural conditions and to have a refined parametrization of its fibers.

    This local Langland conjecture (LLC) has now been proved for GL_n by Harris-Taylor [HT] and Henniart [He] where the map $\mathcal{L}$ is a bijection. Building upon this, the LLC has now been shown for the quasi-split classical groups (i.e. symplectic, special orthogonal and unitary groups) by the work of Arthur [A], Moeglin and Mok [M], as a consequence of the theory of twisted endoscopy using the stable twisted trace formula of GL_n, and extended to pure inner forms by various authors by various means (see for example [KMSW, MR, CZ1, CZ2]). It has also been shown for the group GSp(4) and its inner forms in [GT] and [GTW] using theta correspondence as the main tool. For general G, the recent work of Fargues-Scholze [FS] gives a general geometric construction of a semisimplified LLC (which is a priori weaker than the full LLC). In another direction, for general tamely ramified groups, the work of Kaletha [K1, K2] constructs an LLC for supercuspidal L-packets.

1.1. *The main result*

The purpose of this paper is to establish the local Langlands conjecture for the split exceptional group $G = G_2$. More precisely, we prove:

Main Theorem
There is a natural surjective map

$$\mathcal{L} : \mathrm{Irr}(G_2) \longrightarrow \Phi(G_2)$$

with finite fibers satisfying the following properties:

(i) $\pi \in \mathrm{Irr}(G_2)$ is square-integrable (or a discrete series representation) if and only if $\mathcal{L}(\pi)$ does not factor through a proper Levi subgroup of $G_2^\vee = G_2(\mathbb{C})$.

(ii) $\pi \in \mathrm{Irr}(G_2)$ is tempered if and only if $\mathcal{L}(\pi)(W_F)$ is bounded in $G_2^\vee = G_2(\mathbb{C})$. More precisely, if π is contained in an induced representation $\mathrm{Ind}_P^{G_2}\tau$, with $P = MN$ a parabolic subgroup and τ a unitary discrete series representation of its Levi factor M, then $\mathcal{L}(\pi)$ is the composite of the L-parameter of τ with the natural inclusion $M^\vee \hookrightarrow G_2^\vee = G_2(\mathbb{C})$, Here, note that the Levi subgroup M is either GL_2 or GL_1^2, for which the LLC is known.

(iii) If $\pi \in \mathrm{Irr}(G_2)$ is nontempered, so that π is the unique Langlands quotient of a standard module $\mathrm{Ind}_P^{G_2}\tau$ induced from a proper parabolic subgroup $P = MN$, then $\mathcal{L}(\pi)$ is the composite of the L-parameter of τ with the natural inclusion $M^\vee \hookrightarrow G_2^\vee = G_2(\mathbb{C})$.

(iv) The map $\mathcal{L}$ features in the following two commutative diagrams:

(a)

$$
\begin{array}{ccc}
\mathrm{Irr}^\spadesuit(G_2) & \xrightarrow{\ \mathcal{L}\ } & \Phi(G_2) \\
{\scriptstyle\theta}\downarrow & & \uparrow{\scriptstyle\iota'_*} \\
\mathrm{Irr}(PD^\times) & \xrightarrow{\ L\ } & \Phi(\mathrm{PGL}_3)
\end{array}
$$

(b)

$$
\begin{array}{ccc}
\mathrm{Irr}^\heartsuit(G_2) & \xrightarrow{\ \mathcal{L}\ } & \Phi(G_2) \\
{\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\iota_*} \\
\mathrm{Irr}(\mathrm{PGSp}_6) & & \Phi(\mathrm{PGSp}_6) \\
{\scriptstyle\mathrm{rest}}\downarrow & & \downarrow{\scriptstyle\mathrm{std}_*} \\
\mathrm{Irr}(\mathrm{Sp}_6)_{/\mathrm{PGSp}_6} & \xrightarrow{\ L\ } & \Phi(\mathrm{Sp}_6).
\end{array}
$$

In these diagrams, the symbol L refers to the known LLC for $PD^\times$ and Sp_6 and the maps on the right hand columns of the diagrams are induced by the natural maps of dual groups

$$
\begin{array}{ccc}
 & & \mathrm{SO}_7(\mathbb{C}) \\
 & \overset{std}{\nearrow} & \\
\mathrm{SL}_3(\mathbb{C}) \xrightarrow{\ \iota'\ } G_2(\mathbb{C}) \xrightarrow{\ \iota\ } \mathrm{Spin}_7(\mathbb{C}) & & \\
 & \underset{spin}{\searrow} & \\
 & & \mathrm{GL}_8(\mathbb{C}).
\end{array}
$$

For the left hand columns in the two diagrams, the symbol θ refers to an appropriate theta correspondence (to be explained below) and rest refers to the restriction of representations,

with the notation $\mathrm{Irr}(\mathrm{Sp}_6)_{/\mathrm{PGSp}_6}$ denoting the set of $\mathrm{PGSp}_6(F)$-orbits on $\mathrm{Irr}(\mathrm{Sp}_6)$. Moreover, $\mathrm{Irr}(G_2) = \mathrm{Irr}^\heartsuit(G_2) \sqcup \mathrm{Irr}^\spadesuit(G_2)$ is a decomposition of $\mathrm{Irr}(G_2)$ into the disjoint union of two subsets (see Theorem 1.1 below).

(v) The map $\mathcal{L}$ is uniquely characterized by the properties (i), (ii), (iii) and (iv).

(vi) The map $\mathcal{L}$ also features in the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Irr}_{gen,ds}(G_2) & \xrightarrow{\ \mathcal{L}\ } & \Phi(G_2) \\
{\scriptstyle \theta}\downarrow & & \downarrow{\scriptstyle \iota_*} \\
\mathrm{Irr}_{gen,ds}(\mathrm{PGSp}_6) & & \Phi(\mathrm{PGSp}_6) \\
{\scriptstyle \mathrm{spin}_*}\downarrow & & \downarrow{\scriptstyle \mathrm{spin}_*} \\
\mathrm{Irr}(\mathrm{GL}_8) & \xrightarrow{\ L\ } & \Phi(\mathrm{GL}_8).
\end{array}
$$

Here, $\mathrm{Irr}_{gen,ds}(\bullet)$ refers to the subset of generic discrete series representations in $\mathrm{Irr}(\bullet)$, the left hand column of the diagram is induced by the relevant maps of dual groups mentioned in (iv) above and spin_* refers to a certain spin lifting whose construction will be explained in §4.

(vii) For each $\phi \in \Phi(G_2)$, the fiber of $\mathcal{L}$ over ϕ is in natural bijection with $\mathrm{Irr}S_\phi$, where

$$
S_\phi = \pi_0(Z_{G_2(\mathbb{C})}(\phi)),
$$

when $p \neq 3$. When $p = 3$, this is still true unless perhaps for $\phi \in \iota'_*(\Phi_{ds}(\mathrm{PGL}_3))$. Moreover, for tempered ϕ, the trivial character of S_ϕ corresponds to the unique generic element in $\mathcal{L}^{-1}(\phi)$.

(viii) The LLC for G_2 satisfies the following global-local compatibility. Suppose that Π is a globally generic regular algebraic cuspidal automorphic representation of G_2 over a totally real number field k with a Steinberg local component. Suppose that

$$
\rho_\Pi : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{GL}_7(\overline{\mathbb{Q}}_l)
$$

is a Galois representation associated to Π, in the sense that for almost all places v, $\rho_\Pi(\mathrm{Frob}_v)_{ss}$ is conjugate to the Satake parameter $s_{\Pi_v} \in G_2(\overline{\mathbb{Q}}_l)$. Then ρ_Π factors through $G_2(\overline{\mathbb{Q}}_l)$ (uniquely up to $G_2(\overline{\mathbb{Q}}_l)$-conjugation) and for each finite place v, the restriction of ρ_Π to the local Galois group $\mathrm{Gal}(\overline{k}_v/k_v)$ corresponds to the local L-parameter $\mathcal{L}(\Pi_v)$.

(ix) Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. In [GS24], we have developed a theory of γ-factors $\gamma(s, \pi \times \tau, \psi)$ for every $\pi \in \mathrm{Irr}(G_2)$ and $\tau \in \mathrm{Irr}(\mathrm{GL}_r)$. Then

$$
\gamma(s, \pi \times \tau, \psi) = \gamma(s, \mathrm{std} \circ \mathcal{L}(\pi)) \otimes \phi_\tau, \psi)
$$

where $\phi_\tau : WD_F \longrightarrow \mathrm{GL}_r(\mathbb{C})$ is the L-parameters of τ and $\mathrm{std} : G_2(\mathbb{C}) \hookrightarrow \mathrm{GL}_7(\mathbb{C})$.

(x) The map $\mathcal{L}$ is also uniquely characterized by the properties (i), (ii), (iii) and (ix).

1.2. Construction of $\mathcal{L}$

Let us make a few remarks regarding the undefined notation in (iv) and (vi) of the Main Theorem above and give a brief sketch of the main ideas used in the construction of $\mathcal{L}$. The starting point of our work is the local theta correspondences furnished by the following dual pairs:

$$
\begin{cases}
PD^\times \times G_2 \subset E_6^D \\
G_2 \times \mathrm{PGSp}_6 \subset E_7
\end{cases}
$$

where the exceptional groups of type E are of adjoint type and D denotes a cubic division F-algebra, so that $PD^\times$ is the unique inner form of PGL_3. One can thus consider the restriction of the minimal representation of E to the relevant dual pair and obtain a local theta correspondence. In particular, for a

representation π of one member of a dual pair, one has a big theta lift $\Theta(\pi)$ on the other member of the dual pair, and its maximal semisimple quotient $\theta(\pi)$. In [GS23], the following theorem was shown:

Theorem 1.1. (i) *(Howe duality) The Howe duality theorem holds for the above dual pairs, i.e.* $\Theta(\pi)$ *has finite length and its maximal semisimple quotient* $\theta(\pi)$ *is irreducible or zero.*

(ii) *(Theta dichotomy) Let* $\pi \in \mathrm{Irr}(G_2)$. *Then* π *has nonzero theta lift to exactly one of* $\mathrm{PD}^\times$ *or* PGSp_6.

In view of the above dichotomy theorem, one has a decomposition

$$\mathrm{Irr}(G_2) = \mathrm{Irr}^\heartsuit(G_2) \sqcup \mathrm{Irr}^\spadesuit(G_2)$$

where $\mathrm{Irr}^\heartsuit(G_2)$ consists of those irreducible representations which participate in theta correspondence with PGSp_6 and $\mathrm{Irr}^\spadesuit(G_2)$ consists of those which participate in theta correspondence with $\mathrm{PD}^\times$. This explains the left hand sides of the two commutative diagrams in (iv) of the Main Theorem.

We can now sketch how one can define the map $\mathcal{L} : \mathrm{Irr}(G_2) \longrightarrow \Phi(G_2)$. Suppose that $\pi \in \mathrm{Irr}(G_2)$, then we define its L-parameter $\phi_\pi := \mathcal{L}(\pi)$ as follows:

- if $\pi \in \mathrm{Irr}^\spadesuit(G_2)$, say $\theta_D(\pi) = \tau_D \in \mathrm{Irr}(\mathrm{PD}^\times)$, consider the Jacquet-Langlands lift τ of τ_D to PGL_3 with L-parameter ϕ_τ. Then we set

$$\phi_\pi = \iota' \circ \phi_\tau : WD_F \longrightarrow \mathrm{SL}_3(\mathbb{C}) \subset G_2(\mathbb{C}).$$

  where $\iota' : \mathrm{SL}_3(\mathbb{C}) \hookrightarrow G_2(\mathbb{C})$ is the natural inclusion.
- if $\pi \in \mathrm{Irr}^\heartsuit(G_2)$, say $\theta(\pi) = \sigma \in \mathrm{Irr}(\mathrm{PGSp}_6)$, then

$$\mathrm{rest}(\sigma) \in \mathrm{Irr}(\mathrm{Sp}_6(F))_{/\mathrm{PGSp}_6(F)},$$

gives rise to an L-parameter (à la Arthur)

$$\phi_{\mathrm{rest}(\sigma)} : WD_F \longrightarrow \mathrm{SO}_7(\mathbb{C}).$$

At this point, one needs to show that $\phi_{\mathrm{rest}(\sigma)}$ factors through $G_2(\mathbb{C})$, uniquely up to $G_2(\mathbb{C})$-conjugacy. We shall achieve this by a global argument, using various globalization results which are given in Appendices A and B (i.e. §9 and §11), the construction of Galois representations associated to cohomological cuspidal representations of GL_7 and the group theoretic results of Chenevier [C] and Greiss [Gr95]. After this, we set $\phi_\pi := \phi_{\mathrm{rest}(\theta(\pi))}$ as a map valued in $G_2(\mathbb{C})$, well-defined up to $G_2(\mathbb{C})$-conjugacy.

In other words, using the theta correspondence and theta dichotomy, we deduce the LLC for G_2 from the known LLC for $\mathrm{PD}^\times$ and Sp_6. Moreover, by construction, one has the commutative diagrams in (iv), and the various results alluded to above give the characterization of $\mathcal{L}$ in (v).

1.3. Fibers of $\mathcal{L}$

Parametrizing the fibers of $\mathcal{L}$ (i.e. showing (vii) of the Main Theorem) is the most delicate part of this paper which requires key new ideas. In constructing the map $\mathcal{L}$, we had only needed to appeal to the LLC for Sp_6. To explicate the fibers of $\mathcal{L}$ (for example, to show that $\mathcal{L}$ is surjective), it would help greatly if the LLC for PGSp_6 is known. Of course, the requirement of compatibility of restriction with the LLC of Sp_{2n} places severe constraints on the LLC for PGSp_{2n}, but there is an inherent "quadratic ambiguity" on both the representation theoretic and Galois theoretic sides that the LLC for Sp_{2n} cannot resolve. We circumvent the lack of the full LLC for PGSp_6 by making crucial use of the following inputs:

- In his thesis and subsequent work [Xu1, Xu2], Bin Xu has resolved the quadratic ambiguities on the representation theoretic side, defining a partition of $\mathrm{Irr}(\mathrm{PGSp}_{2n})$ into candidate local L-packets and establishing Arthur's multiplicity formula in the global setting based on these local partitions. We will summarize his results that we need in §7;

- We make use of results of Kret and Shin [KS] who gave a construction of Galois representations valued in $\mathrm{GSpin}_{2n+1}(\mathbb{C})$ associated to certain cuspidal automorphic representations of GSp_{2n}. This is summarized in §5;
- By a combination of the similitude classical theta lifting from PGSp_6 to PGSO_8 and the theory of triality on PGSO_8, we construct on the representation theoretic side a Spin lifting

$$\mathrm{spin}_* : \mathrm{Irr}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{SO}_8) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8)$$

which should be associated to the map of dual groups

$$\mathrm{spin} : \mathrm{Spin}_7(\mathbb{C}) \longrightarrow \mathrm{SO}_8(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C})$$

given by the spin representation of $\mathrm{Spin}_7(\mathbb{C})$; this is given in §4.

The combination of these various inputs to understand the fibers of $\mathcal{L}$ requires an execution too delicate to explain in the introduction. An outcome of this is the commutative diagram in (vi) of the Main Theorem, which plays a key role in the proof of (vii).

This exploitation of triality to produce the Spin lifting is certainly one of the main innovations of this paper; it will be discussed in greater detail in §4. The idea can be pushed further to yield a weak LLC for PGSp_6; this is carried out in Appendix C of this paper. Such applications of triality in the global setting were first obtained in the paper [CG2] of G. Chenevier and the first author.

1.4. Prior work, related literature and further remarks

We conclude this introduction by mentioning some prior work towards the LLC for G_2 and their relevance to the current paper:

- In [SW], the second author and M. Weissman showed that an irreducible generic supercuspidal representation of G_2 has nonzero theta lift to a generic supercuspidal representation of exactly one of PGL_3 or PGSp_6; this is a precursor of the dichotomy theorem of [GS23] mentioned above.
- In [HKT], Harris-Khare-Thorne constructed a bijection

$$\mathrm{Irr}_{sc,gen}(G_2(F)) \longleftrightarrow \Phi_{sc}(G_2)$$

where the LHS refers to the set of irreducible generic supercuspidal representations, whereas the RHS refers to supercuspidal L-parameters, i.e. discrete L-parameters $W_F \longrightarrow G_2(\mathbb{C})$ trivial on the Deligne SL_2. In doing this, they made use of the results of Savin-Weissman [SW], Chenevier [C] and Hundley-Liu [HL] (on an automorphic descent from GL_7 to G_2). In addition, they made crucial use of potential modularity results from the theory of Galois representations. For this paper, we shall not make use of the results of [HKT]. Rather, we shall reprove their result by different means in the course of the proof of the Main Theorem. More precisely, we do not appeal to automorphic descent [HL] or potentially modularity results, but exploit the theory of triality and [SW].
- In [AMS], Aubert-Moussaoui-Solleveld defined the notion of cuspidal support for enhanced L-parameters and formulated the conjecture [AMS, Conj. 7.8] that the LLC map should be compatible with the cuspidal support maps on both sides. In a recent preprint [AX], Aubert-Xu explored the implications of [AMS, Conj. 7.8] (which is formulated as [AX, Property 2.3.14]) for the LLC of G_2. For $p \neq 2$ or 3, they took Kaletha's construction of the LLC for supercuspidal L-packets as a starting point and tried to use the usual desiderata of the LLC and [AMS, Conj. 7.8] as guiding principles to extend it to non-supercuspidal representations, as well as the remaining (singular) supercuspidal ones.

  More precisely, for non-supercuspidal representations of G_2, they used essentially the same process as [GS23, §3.5] to define the LLC and then verify that this is consistent with [AMS, Conj. 7.8]. An interesting aspect of their work is the use of [AMS, Conj. 7.8] to limit the possible options for the enhanced L-parameters of singular supercuspidal representations. This places nontrivial constraints

on the possible enhanced L-parameters, but is ultimately not sufficient for them to obtain a definitive LLC map for G_2.

From our point of view, the association of (enhanced) L-parameters to non-supercuspidal representations of G_2 is completely dictated by the usual desiderata of the LLC, as we explained in [GS23, §3.5]. Hence, for us, the main point in constructing the LLC for G_2 is to take care of the supercuspidal representations, especially the non-generic ones. In particular, we do not use the results of Kaletha [K1, K2] in this paper. It will of course be interesting to see if the LLC supplied by our Main Theorem agrees with the supercuspidal L-packets constructed by Kaletha [K1, K2] in the case of G_2.

- Likewise, it will be interesting to verify the compatibility of our LLC map with the semisimple one defined by Fargues-Scholze [FS]. Currently, besides the case of GL_n and related groups, such compatibility results are known for GSp_4 and odd unitary groups. The proofs proceed by global means, using Shimura varieties and Arthur multiplicity formula. Both these ingredients are not available in the setting of G_2.

- We have given two independent characterizations of $\mathcal{L}$ in (v) and (x) of the Main Theorem. The characterization of $\mathcal{L}$ in (v) via the commutative diagrams in (iv) is of an extrinsic nature, involving transfers of representations to other groups. One may complain that this is not-so-satisfactory. On the other hand, the construction and characterization of the LLC for classical groups G by twisted endoscopic transfer to GL_n in the work of Arthur [A] is in the same spirit as our characterization in (v). Namely, the LLC map $\mathcal{L}_G$ for a classical G is the unique map for which one has a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Irr}(G) & \xrightarrow{\mathcal{L}_G} & \Phi(G) \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle \mathrm{std}_*} \\
\mathrm{Irr}(GL_n) & \xrightarrow{\mathcal{L}_{GL_n}} & \Phi(GL_n),
\end{array}
$$

where t stands for the twisted endoscopic transfer and $\mathrm{std} : G^\vee \to GL_n(\mathbb{C})$ is the standard representation.

The characterization by gamma factors of pairs in (x) is more intrinsic and closer in spirit to the characterization of the LLC for GL_n. Such a characterization is perhaps more useful for applications to number theory. since it is more closely connected with L-functions.

- We do not address the issues of stability and endoscopic character identities for our L-packets in this paper. This is obviously a natural problem to consider. One should be able to show these using a combination of theta correspondence and the stable trace formula, along the lines of [CG] for the LLC of GSp_4.

- In Appendix C (i.e. §12), we show how the exploitation of the principle of triality and the results of Kret-Shin allow one to construct a weak LLC map for $PGSp_6$. We also explain how this refines the results of Bin Xu [Xu2] for similitude classical groups in the limited context of $PGSp_6$.

2. Theta Dichotomy

We begin by recalling in greater detail the main results of [GS23] that were briefly alluded to in the introduction.

2.1. *Theta correspondences*

In [GS23], we studied the local theta correspondence for the following dual pairs:

$$
\begin{array}{ccc}
 & & PGSp_6 \\
 & G_2 \diagup & \\
 & \diagdown & \\
PD^\times \diagup & & PGL_3 \rtimes \mathbb{Z}/2\mathbb{Z},
\end{array}
$$
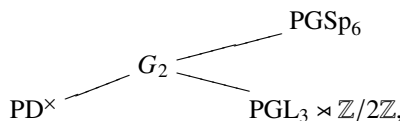
where D denotes a cubic division F-algebra. More precisely, one has the dual pairs

$$\begin{cases} (\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2 \subset E_6 \rtimes \mathbb{Z}/2\mathbb{Z} \\ \mathrm{PD}^\times \times G_2 \subset E_6^D \\ G_2 \times \mathrm{PGSp}_6 \subset E_7 \end{cases}$$

where the exceptional groups of type E are of adjoint type. One can thus consider the restriction of the minimal representation of E to the relevant dual pair and obtain a local theta correspondence. In particular, for a representation π of one member of a dual pair, one has a big theta lift $\Theta(\pi)$ on the other member of the dual pair, and its maximal semisimple quotient $\theta(\pi)$. In [GS23], the following theorem was shown:

Theorem 2.1. (i) *(Howe duality) The Howe duality theorem holds for the above dual pairs, i.e. $\Theta(\pi)$ has finite length and its maximal semisimple quotient $\theta(\pi)$ is irreducible or zero.*

(ii) *(Theta dichotomy) Let $\pi \in \mathrm{Irr}(G_2)$. Then π has nonzero theta lift to exactly one of $\mathrm{PD}^\times$ or PGSp_6. In particular, one has a decomposition:*

$$\mathrm{Irr}(G_2) = \mathrm{Irr}^\heartsuit(G_2) \sqcup \mathrm{Irr}^\spadesuit(G_2)$$

where $\mathrm{Irr}^\heartsuit(G_2)$ consists of those irreducible representations which participate in theta correspondence with PGSp_6 and $\mathrm{Irr}^\spadesuit(G_2)$ consists of those which participate in theta correspondence with $\mathrm{PD}^\times$.

(iii) *More precisely, one has:*

(a) *the theta correspondence for $\mathrm{PD}^\times \times G_2$ defines an injective map*

$$\theta_D : \mathrm{Irr}^\spadesuit(G_2) \hookrightarrow \mathrm{Irr}(\mathrm{PD}^\times),$$

which is bijective if $p \neq 3$. Moreover, $\mathrm{Irr}^\spadesuit(G_2)$ is contained in the subset $\mathrm{Irr}_{ds}(G_2)$ of discrete series representations.

(b) *the theta correspondence for $G_2 \times \mathrm{PGSp}_6$ defines an injection*

$$\theta : \mathrm{Irr}^\heartsuit(G_2) \hookrightarrow \mathrm{Irr}(\mathrm{PGSp}_6).$$

The map θ carries tempered representations to tempered representations.

(c) *the theta correspondence for $(\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times G_2$ defines an injective map*

$$\theta_{M_3} : \mathrm{Irr}^\clubsuit(G_2) \hookrightarrow \mathrm{Irr}(\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}),$$

where $\mathrm{Irr}^\clubsuit(G_2) \subset \mathrm{Irr}^\heartsuit(G_2)$ is the subset of representations which participates in theta correspondence with $\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$. The map θ_{M_3} respects tempered (resp. discrete series) representations. Moreover, its image has been completely determined.

By Theorem 2.1(iii), as a refinement of the decomposition of $\mathrm{Irr}(G_2)$ in Theorem 2.1(i), one has a further decomposition

$$\mathrm{Irr}^\heartsuit(G_2) = \mathrm{Irr}^\diamond(G_2) \sqcup \mathrm{Irr}^\clubsuit(G_2), \tag{2.2}$$

where $\mathrm{Irr}^\clubsuit(G_2)$ consists of those representations which participate in theta correspondence with $\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Irr}^\diamond(G_2)$ consists of those which participate exclusively in the theta correspondence with PGSp_6. Further, we have a trichotomy result for discrete series representations:

Proposition 2.3. *Each irreducible discrete series representation of G_2 has a nonzero discrete series theta lift to exactly one of $\mathrm{PD}^\times$, $\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}$ or PGSp_6. Setting $\mathrm{Irr}^\bullet_{ds}(G_2) = \mathrm{Irr}_{ds}(G_2) \cap \mathrm{Irr}^\bullet(G_2)$, this trichotomy is given by the disjoint union*

$$\mathrm{Irr}_{ds}(G_2) = \mathrm{Irr}^\spadesuit_{ds}(G_2) \sqcup \mathrm{Irr}^\clubsuit_{ds}(G_2) \sqcup \mathrm{Irr}^\diamond_{ds}(G_2).$$

In fact, the results of [GS23] give a precise determination of the maps θ_{M_3} and θ on nonsupercuspidal (in particular nontempered) representations. These precise results are too intricate to recall here; we refer the reader to [GS23].

2.2. Maps of L-parameters

The above dichotomy results on the representation theory side is reflected to some extent by analogous results on the side of L-parameters. Recall the natural morphisms of Langlands dual groups

$$\mathrm{SL}_3(\mathbb{C}) \xrightarrow{\ \iota'\ } G_2(\mathbb{C}) \xrightarrow{\ \iota\ } \mathrm{Spin}_7(\mathbb{C}) \xrightarrow{\ \mathrm{std}\ } \mathrm{SO}_7(\mathbb{C})$$

which induce natural maps

$$\Phi(\mathrm{PGL}_3) \xrightarrow{\ \iota'_*\ } \Phi(G_2) \xrightarrow{\ \iota_*\ } \Phi(\mathrm{PGSp}_6) \xrightarrow{\ \mathrm{std}_*\ } \Phi(\mathrm{Sp}_6).$$

Moreover, the outer automorphism of $\mathrm{SL}_3(\mathbb{C})$ induces an action of $\mathbb{Z}/2\mathbb{Z}$ on $\Phi(\mathrm{PGL}_3)$. We note:

Lemma 2.4. (i) *The map ι'_* gives an injection*

$$\Phi(\mathrm{PGL}_3)/_{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow \Phi(G_2),$$

whose image we denote by $\Phi^{\clubsuit}(G_2)$. It restricts to an injection

$$\Phi_{ds}(\mathrm{PGL}_3)/_{\mathbb{Z}/2\mathbb{Z}} \hookrightarrow \Phi_{ds}(G_2),$$

and the image of $\Phi_{ds}(\mathrm{PGL}_3)$ is characterized as those $\phi \in \Phi_{ds}(G_2)$ such that the local L-factor $L(s, \mathrm{std} \circ \iota \circ \phi)$ has a pole at $s = 0$.

   Moreover, for $\phi \in \Phi_{ds}(\mathrm{PGL}_3)$ with component group $S_\phi = \mu_3$, one has

$$S_{\iota'_*(\phi)} = \begin{cases} \mu_3 & \text{if } \phi \text{ is not self-dual;} \\ S_3 & \text{if } \phi \text{ is self-dual.} \end{cases}$$

   (ii) *The map ι_* gives an injection*

$$\iota_* : \Phi(G_2) \longrightarrow \Phi(\mathrm{PGSp}_6)$$

which restricts to an injection

$$\Phi^{\diamond}_{ds}(G_2) := \Phi_{ds}(G_2) \setminus \Phi^{\clubsuit}(G_2) \longrightarrow \Phi_{ds}(\mathrm{PGSp}_6).$$

The image of $\Phi^{\diamond}_{ds}(G_2)$ is characterized as those $\phi' \in \Phi_{ds}(\mathrm{PGSp}_6)$ such that the local L-factor $L(s, \mathrm{spin} \circ \phi')$ has a pole at $s = 0$. Moreover, for $\phi \in \Phi^{\diamond}_{ds}(G_2)$ with component group S_ϕ, one has

$$\iota_* : S_\phi \cong S_{\iota_*(\phi)}/Z(\mathrm{Spin}_7),$$

where $Z(\mathrm{Spin}_7) \cong \mu_2$ is the center of $\mathrm{Spin}_7(\mathbb{C})$.
   (iii) *The map*

$$\mathrm{std}_* \circ \iota_* : \Phi(G_2) \longrightarrow \Phi(\mathrm{Sp}_6)$$

is injective and restricts to an injection

$$\Phi^{\diamond}_{ds}(G_2) \hookrightarrow \Phi_{ds}(\mathrm{Sp}_6).$$

Proof. (i) If $\phi_1, \phi_2 \in \Phi(\mathrm{PGL}_3)$ are conjugate in $G_2(\mathbb{C})$, then the following semi-simple representations are isomorphic:

$$\phi_1 + \phi_1^\vee + 1 \cong \phi_2 + \phi_2^\vee + 1.$$

We need to show that this isomorphism implies $\phi_1 \cong \phi_2$ or $\phi_1 \cong \phi_2^\vee$. This is obvious if ϕ_1 is irreducible. If ϕ_1 contains an irreducible two dimensional summand, then that summand must be contained in ϕ_2 or $\phi_2^\vee$. Assume that it is contained in ϕ_2. The one-dimensional summand of ϕ_1 is the determinant inverse of the two-dimensional summand, hence it is also contained in ϕ_2 and thus $\phi_1 \cong \phi_2$. We leave the case of three summands as an exercise. Assume $\phi \in \Phi_{ds}(\mathrm{PGL}_3)$. Then the centralizer of ϕ in $\mathrm{SL}_3(\mathbb{C})$ is μ_3. The stabilizer in $G_2(\mathbb{C})$ of $1 \subset 1 + \phi + \phi^\vee$ is $\mathrm{SL}_3(\mathbb{C}) \rtimes \mathbb{Z}/2\mathbb{Z}$. It follows that the centralizer of ϕ in $G_2(\mathbb{C})$ is either μ_3 or S_3, depending on whether $\phi \not\cong \phi^\vee$ or $\phi \cong \phi^\vee$ respectively.

For (ii) and (iii), we refer the reader to [GrS2, Chapter 4, Section 1].   □

3. Definition of $\mathcal{L}$

In this section, we will define the LLC map

$$\mathcal{L} : \mathrm{Irr}(G_2) \longrightarrow \Phi(G_2)$$

using the results of the previous section and establish some initial properties of $\mathcal{L}$.

3.1. First definition of $\mathcal{L}$

We shall give two slightly different constructions of the map $\mathcal{L}$.

For the first construction (which was sketched in the introduction), we make use of the decomposition

$$\mathrm{Irr}(G_2) = \mathrm{Irr}^\spadesuit(G_2) \sqcup \mathrm{Irr}^\heartsuit(G_2)$$

introduced in Theorem 2.1(i). First consider $\pi \in \mathrm{Irr}^\spadesuit(G_2) \subset \mathrm{Irr}_{ds}(G_2)$, and set $\tau = \theta_D(\pi)$ for a unique $\tau \in \mathrm{Irr}(\mathrm{PD}^\times)$. By the Jacquet-Langlands correspondence and the LLC for PGL_3, one has a discrete L-parameter

$$\phi_\tau : WD_F \longrightarrow \mathrm{SL}_3(\mathbb{C}).$$

Composing this with the natural inclusion

$$\iota' : \mathrm{SL}_3(\mathbb{C}) \longrightarrow G_2(\mathbb{C}),$$

we set

$$\mathcal{L}(\pi) = \iota' \circ \phi_\tau : WD_F \longrightarrow G_2(\mathbb{C}),$$

which is an element of $\Phi_{ds}(G_2)$ by Lemma 2.4(i).

For $\pi \in \mathrm{Irr}^\heartsuit(G_2)$, let $\sigma = \theta(\pi) \in \mathrm{Irr}(\mathrm{PGSp}_6)$ and consider its restriction

$$\mathrm{rest}(\sigma) \in \mathrm{Irr}(\mathrm{Sp}_6)/_{\mathrm{PGSp}_6}.$$

By the LLC for Sp_6, $\mathrm{rest}(\sigma)$ gives rise to an L-parameter

$$\phi_{\mathrm{rest}(\sigma)} \in \Phi(\mathrm{Sp}_6).$$

In view of Lemma 2.4, it remains to verify:

Proposition 3.1. *For* $\pi \in \mathrm{Irr}^{\heartsuit}(G_2)$*, with* $\sigma = \theta(\pi) \in \mathrm{Irr}(\mathrm{PGSp}_6)$*,*

$$\phi_{\mathrm{rest}(\sigma)} \in \mathrm{Im}(\mathrm{std}_* \circ \iota_*),$$

so that it gives rise to a uniquely determined element $\mathcal{L}(\pi) \in \Phi(G_2)$ *by Lemma* 2.4*(iii).*

We shall verify this in §3.3 by global means.

3.2. Second definition of $\mathcal{L}$

Our second construction is closely related to the first and is based on the more refined decomposition

$$\mathrm{Irr}(G_2) = \mathrm{Irr}^{\spadesuit}(G_2) \sqcup \mathrm{Irr}^{\clubsuit}(G_2) \sqcup \mathrm{Irr}^{\diamondsuit}(G_2)$$

in (2.2).

For $\pi \in \mathrm{Irr}^{\spadesuit}(G_2)$ or $\mathrm{Irr}^{\diamondsuit}(G_2) \subset \mathrm{Irr}^{\heartsuit}(G_2)$, the definition of $\mathcal{L}(\pi)$ is as in the first construction above. Thus, the difference only arises for those $\pi \in \mathrm{Irr}^{\clubsuit}(G_2) \subset \mathrm{Irr}^{\heartsuit}(G_2)$. Such a π has nonzero theta lift to a unique $\tilde{\tau} \in \mathrm{Irr}(\mathrm{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z})$. Restricting $\tilde{\tau}$ to PGL_3 gives a well-defined element

$$\{\tau, \tau^{\vee}\} \in \mathrm{Irr}(\mathrm{PGL}_3)/_{\mathbb{Z}/2\mathbb{Z}},$$

and thus an L-parameter

$$\phi_{\tilde{\tau}} : WD_F \longrightarrow \mathrm{SL}_3(\mathbb{C})$$

which is well-defined up to the outer automorphism action. By Lemma 2.4, we may set

$$\mathcal{L}(\pi) = \iota' \circ \phi_{\tilde{\tau}} \in \Phi(G_2).$$

There is of course a need to reconcile the two definitions for $\pi \in \mathrm{Irr}^{\spadesuit}(G_2)$. This follows readily from the explicit theta correspondences computed in [GS23, Thm. 8.2, Thm. 8.5, Thm. 15.1 and Thm. 15.2].

3.3. Proof of Proposition 3.1

To complete the construction of $\mathcal{L}$, it remains to verify Proposition 3.1. For a non-supercuspidal representation $\pi \in \mathrm{Irr}^{\heartsuit}(G_2)$, we have determined in [GS23, Thm. 15.1, Thm 15.2 and Thm. 15.3] the representation $\theta(\pi) \in \mathrm{Irr}(\mathrm{PGSp}_6)$ explicitly. From this, Proposition 3.1 follows readily. Indeed, the results of [GS23] shows that $\mathcal{L}(\pi) \in \Phi(G_2)$ is given precisely by the discussion in [GS23, §3.5].

It remains to treat supercuspidal $\pi \in \mathrm{Irr}^{\heartsuit}(G_2)$. We shall use a global argument.

Let k be a totally real number field with a place w such that $k_w \cong F$. Let $\mathbb{O}$ be a totally definite octonion algebra over F with automorphism group $G = \mathrm{Aut}(\mathbb{O})$. Then G_v is anisotropic at all archimedean places v and G_v is the split G_2 at all finite places v. By Proposition 10.5 and Corollary 10.6 in Appendix A, we can find a cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ of G so that

- $\Pi_w \cong \pi$;
- Π_{∞} has sufficiently regular infinitesimal character at each archimedean place ∞;
- $\Pi_u \cong$ the Steinberg representation St at some finite place $u \neq w$.
- Π has nonzero cuspidal global theta lift $\Sigma := \theta(\Pi)$ to the split group PGSp_6 over k.

Here, in the last bullet point, we are considering the dual pair $G \times \mathrm{PGSp}_6$ in the k-rank 3 form of E_7 (commonly denoted by $E_{7,3}$); this global theta correspondence was studied in [GrS2].

Now the nonzero cuspidal representation $\theta(\Pi) =: \Sigma = \otimes_v \Sigma_v$ of PGSp_6 satisfies:

- Σ_w is the representation $\theta(\pi)$ of $\mathrm{PGSp}_6(k_w)$.
- Σ_{∞} is a holomorphic discrete series with sufficiently regular infinitesimal character for every archimedean place ∞.
- Σ_u is the Steinberg representation of $\mathrm{PGSp}_6(k_u)$.

By Arthur [A], the restriction of Σ to Sp_6 has a cuspidal transfer $\mathcal{A}(\mathrm{rest}(\Sigma))$ to GL_7 (because of the Steinberg local component at u) which is regular algebraic at all real places. Now at each finite place v, it follows by local-global compatibility of the transfer from classical groups to GL_N that the local L-parameter of $\mathcal{A}(\mathrm{rest}(\Sigma))_v$ is the local L-parameter of $\mathrm{rest}(\Sigma_v)$. Moreover, by a result of Chenevier [C, Thm. E], at all finite places v, the local L-parameter of $\mathcal{A}(\mathrm{rest}(\Sigma))_v$ factors through $G_2(\mathbb{C})$, uniquely up to G_2-conjugacy. Hence, the local L-parameter of $\mathrm{rest}(\Sigma_w) = \mathrm{rest}(\theta(\pi))$ takes value in $G_2(\mathbb{C}) \subset \mathrm{SO}_7(\mathbb{C})$, as desired.

We have thus completed the proof of Proposition 3.1.

3.4. Initial properties

The definition of $\mathcal{L}$ given above and the results of [GS23] allow us to read off some initial properties of $\mathcal{L}$ readily. We highlight some of these here:

- (Nontempered representations) As mentioned above, the results of [GS23, Thm. 15.1] show that our definition of $\mathcal{L}$ gives the Langlands parametrization of nontempered representations discussed in [GS23, §3.5] (based on the consideration of usual desiderata). In particular, $\mathcal{L}$ gives a bijection

$$\mathcal{L} : \mathrm{Irr}_{nt}(G_2) \longleftrightarrow \Phi_{nt}(G_2)$$

where the subscript "nt" stands for "nontempered". In particular, nontempered L-packets of G_2 are singletons.

- (Tempered representations) The results of [GS23, Thm 15.2, 15.3] imply that one has a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Irr}_t(G_2) & \xrightarrow{\ \mathcal{L}\ } & \Phi_t(G_2) \\
\uparrow & & \uparrow \\
\mathrm{Irr}_{ds}(G_2) & \xrightarrow{\ \mathcal{L}\ } & \Phi_{ds}(G_2).
\end{array}
$$

- (Tempered but non-discrete series) One has a surjection

$$\mathcal{L} : \mathrm{Irr}_t(G_2) \setminus \mathrm{Irr}_{ds}(G_2) \twoheadrightarrow \Phi_t(G_2) \setminus \Phi_{ds}(G_2)$$

such that the fiber over ϕ has size 1 or 2, according to whether $|S_\phi| = 1$ or 2. The fiber $\mathcal{L}^{-1}(\phi)$ has a unique generic representation, and we attach it to the trivial character of S_ϕ.

- (Summary) In summary, we have established items (i), (ii) and (iii) of the Main Theorem. Moreover, the commutativity of the two diagrams in (iv) of the Main Theorem follows by the construction of $\mathcal{L}$, as does the characterization of $\mathcal{L}$ given in (v) of the Main Theorem. Furthermore, we have also demonstrated (vii) of the Main Theorem for non-discrete-series parameters ϕ (for all p).

- (Compatibility with global Langlands)
  We can also show (viii) of the Main Theorem. In the context of (viii) in the Main Theorem, the global theta lift Σ of Π to PGSp_6 is globally generic, regular algebraic and cuspidal with a Steinberg local component. Moreover, the transfer of $\mathrm{rest}(\Sigma)$ to GL_7 is a regular algebraic cuspidal automorphic representation $\mathcal{A}(\mathrm{rest}(\Sigma))$ (with cuspidality a consequence of the Steinberg local component). Hence, the Galois representation ρ_Π is associated with $\mathcal{A}(\mathrm{rest}(\Sigma))$. By [C, Thm. 6.4], ρ_Π takes value in $G_2(\overline{\mathbb{Q}}_l)$ (after conjugation). By local-global compatibility [TY, Ca], the local Galois representation $\rho_{\Pi,v}$ at each finite place v corresponds to the local L-parameter of $\mathrm{rest}(\Sigma_v)$. But by the construction of $\mathcal{L}$, the local L-parameter of $\mathrm{rest}(\Sigma_v)$ is $\mathrm{std} \circ \iota \circ \mathcal{L}(\Pi_v)$. This establishes (viii).

- (Preservation of γ-factors) We now show (ix) of the Main Theorem. Let us first briefly recall the definition of the relevant γ-factors given in [GS24]. The results of [GS23] can be summarized as the construction of a map

$$\text{Lif} : \text{Irr}(G_2) \longrightarrow \text{Irr}(\text{GL}_7)$$

which is given in the following commutative diagram:

$$\text{Irr}(G_2)$$

$$\text{Irr}(PD^\times) \xleftarrow{\ \theta\ } \text{Irr}^\spadesuit(G_2) \cup \text{Irr}^\heartsuit(G_2) \xrightarrow{\ \theta\ } \text{Irr}(\text{PGSp}_6)$$

$$\downarrow{\scriptstyle \text{JL}} \qquad\qquad\qquad \downarrow{\scriptstyle \text{Lif}} \qquad\qquad\qquad \downarrow{\scriptstyle \text{rest}}$$

$$\text{Irr}_{ds}(\text{PGL}_3) \xrightarrow{\ \boxplus\ } \text{Irr}(\text{GL}_7) \xleftarrow{\ \text{Art}\ } \text{Irr}(\text{Sp}_6)/_{\text{PGSp}_6}.$$

Here,
- the two θ's refer to the local theta correspondence;
- JL refers to the Jacquet-Langlands transfer;
- rest refers to the restriction of representations;
- Art refers to the functorial transfer of irreducible representations from Sp_6 to GL_7 established by Arthur;
- $\boxplus : \text{Irr}_{ds}(\text{PGL}_3) \longrightarrow \text{Irr}(\text{GL}_7)$ is the map

$$\tau \mapsto \tau \times 1 \times \tau^\vee.$$

With the above preparation, we can now define the local γ-factors. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$. For $\pi \in \text{Irr}(G_2)$ and $\tau \in \text{Irr}(\text{GL}_r)$, set

$$\gamma(s, \pi \times \tau, \psi) := \gamma(s, \text{Lif}(\pi) \times \tau, \psi).$$

where the γ-factor on the RHS is the Rankin-Selberg γ-factor defined by Jacquet-Piatetski-Shapiro-Shalika and Shahidi [GeSh].

Let $\phi_\pi : WD_F \longrightarrow G_2(\mathbb{C})$ and $\phi_\tau : WD_F \longrightarrow \text{GL}_r(\mathbb{C})$ be the L-parameters of π and τ respectively and recall that $\text{std} : G_2(\mathbb{C}) \longrightarrow \text{GL}_7(\mathbb{C})$ is the standard degree 7 irreducible representation of $G_2(\mathbb{C})$. Then it follows from the construction of the LLC for G_2 and the definition of the relevant γ-factors that one has:

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, (\text{std} \circ \phi_\pi) \otimes \phi_\tau, \psi),$$

using the fact that the LLC for GL_r respects the Rankin-Selberg γ-factors. In other words, (ix) follows. The reader will no doubt complain that our definition of the γ-factor makes the statement (ix) almost a formality, which would not be inaccurate description. However, the main result of [GS24] states that the family of invariants $\gamma(s, \pi \times \tau, \psi)$ can be characterized by a (usual) list of properties which is independent of the LLC, including multiplicativity and the global functional equations (for certain cuspidal representations). This characterization of the γ-factors shows that the definition given in [GS24] (and recalled above) is the only possible candidate, and it is this which gives non-trivial content to the statement (ix).

Finally, the characterization of $\mathcal{L}$ by (ix) asserted in (x) follows from the standard argument that the preservation of γ-factors of pairs allows one to determine the 7-dimensional representation $\text{std} \circ \phi_\pi$ uniquely (for tempered π). Since $\text{GL}_7(\mathbb{C})$-conjugacy of $G_2(\mathbb{C})$-valued L-parameters implies $G_2(\mathbb{C})$-conjugacy (as we have exploited in the construction of $\mathcal{L}$), it follows that ϕ_π is uniquely characterized up to $G_2(\mathbb{C})$-conjugacy,

3.5. The packets for $\Phi_{ds}^{\spadesuit\clubsuit}(G_2)$

By the above discussion, it remains to understand the map

$$\mathcal{L} : \mathrm{Irr}_{ds}(G_2) \longrightarrow \Phi_{ds}(G_2)$$

on discrete series representations. In fact, by construction, $\mathcal{L}$ restricts to give:

$$\mathcal{L}^{\spadesuit\clubsuit} : \mathrm{Irr}_{ds}^{\spadesuit}(G_2) \sqcup \mathrm{Irr}_{ds}^{\clubsuit}(G_2) \longrightarrow \Phi_{ds}^{\spadesuit\clubsuit}(G_2).$$

and

$$\mathcal{L}^{\diamond} : \mathrm{Irr}_{ds}^{\diamond}(G_2) \longrightarrow \Phi_{ds}^{\diamond}(G_2).$$

The results of [GS23, §7 and §8] allow us to understand the fibers of the map $\mathcal{L}^{\spadesuit\clubsuit}$ quite precisely.

Proposition 3.2. *The map $\mathcal{L}^{\spadesuit\clubsuit}$ is surjective. Moreover, for $\phi \in \Phi_{ds}^{\spadesuit\clubsuit}(G_2)$, the fiber $\mathcal{L}^{-1}(\phi)$ contains a unique generic element (belonging to $\mathrm{Irr}_{ds}^{\spadesuit}(G_2)$). Further, for $p \neq 3$, one has a natural bijection*

$$\mathcal{L}^{-1}(\phi) \longleftrightarrow \mathrm{Irr}(S_\phi)$$

such that the unique generic element corresponds to the trivial character of S_ϕ. For $p = 3$, we only know that there is an injection $\mathcal{L}^{-1}(\phi) \hookrightarrow \mathrm{Irr}(S_\phi)$ in general, though this injection is bijective when $\phi|_{\mathrm{SL}_2}$ is the subregular SL_2 in $G_2(\mathbb{C})$.

Proof. As almost all statements of the proposition follow from [GS23, §7 and §8], we will focus on the main new information here: the natural bijection

$$\mathcal{L}^{-1}(\phi) \longleftrightarrow \mathrm{Irr}(S_\phi)$$

when $p \neq 3$. To obtain this, we need to set things up rather carefully.

Let M_3 be the F-algebra of 3×3-matrices, with associated Jordan algebra M_3^+. The automorphism group $\mathrm{Aut}(M_3^+)$ of M_3^+ is a disconnected group whose identity component is $H = \mathrm{PGL}_3$, with Langlands dual group $H^\vee = \mathrm{SL}_3(\mathbb{C})$. Note that $H^1(F, H) = H^1(F, \mathrm{PGL}_3)$ classifies the isomorphism classes of central simple F-algebras of degree 3, with the distinguished point corresponding to M_3. Moreover, the invariant map gives a bijection:

$$\mathrm{inv} : H^1(F, H) = H^1(F, \mathrm{PGL}_3) \longrightarrow \mathbb{Z}/3\mathbb{Z}.$$

For each nontrivial $j \in \mathbb{Z}/3\mathbb{Z}$, the associated central simple algebra D_j is a cubic division algebra. Moreover, D_1 and D_2 are opposite of each other, so that the inverse map $x \mapsto x^{-1}$ gives a canonical isomorphism of multiplicative groups $i : D_1^\times \cong D_2^\times$.

Suppose that τ is a discrete series representation of PGL_3. Then for $j \in \mathbb{Z}/3\mathbb{Z}$, one has a Jacquet-Langlands transfer $JL_{D_j}(\tau)$ of τ to $D_j^\times$. Via the isomorphism $i : D_1^\times \cong D_2^\times$, we may compare $JL_{D_1}(\tau)$ and $JL_{D_2}(\tau)$. One has

$$JL_{D_1}(\tau)^\vee \cong i^*(JL_{D_2}(\tau)).$$

Consider now $[\phi] \in \Phi_{ds}^{\spadesuit\clubsuit}(G_2)$, which is a G_2-conjugacy class of homomorphisms $\phi : WD_F \longrightarrow G_2(\mathbb{C})$. One has a projective system (relative to G_2-conjugacy) of centralizer subgroups $Z_{G_2}(\phi)$ which are finite of order 3, and thus identified with the component groups S_ϕ. We denote this projective system of local component groups by $S_{[\phi]}$ and set $C_{[\phi]} = Z_{G_2}(S_{[\phi]})$: this is a projective system of subgroups of $G_2(\mathbb{C})$ which are simply-connected of type A_2. We fix a projective system (relative to G_2-conjugacy)

of isomorphisms

$$i_{[\phi]} : H^\vee = \mathrm{SL}_3(\mathbb{C}) \longrightarrow C_{[\phi]}.$$

Note that there are basically two inner automorphism classes of such projective systems of isomorphisms, exchanged by an outer automorphism of $\mathrm{SL}_3(\mathbb{C})$.

By virtue of $i_{[\phi]}$, one has:

(a) $[\phi]$ gives rise to a well-defined $\mathrm{SL}_3(\mathbb{C})$-conjugacy classes of maps

$$[\rho] : WD_F \longrightarrow \mathrm{SL}_3(\mathbb{C}),$$

i.e. a discrete series L-parameter of $H = \mathrm{PGL}_3$, so that

$$[\phi] = i_{[\phi]} \circ [\rho].$$

Let τ_ρ be the discrete series representation of PGL_3 with L-parameter ρ.

(b) One also has a projective system of isomorphism

$$i_{[\phi]} : Z(H^\vee) = Z(\mathrm{SL}_3(\mathbb{C})) \longrightarrow S_{[\phi]} = Z(C_{[\phi]})$$

inducing a projection system of bijections:

$$j : \mathrm{Irr}(S_{[\phi]}) \xrightarrow{i^*_{[\phi]}} \mathrm{Irr}(Z(H^\vee)) \xrightarrow{\text{Kottwitz}} H^1(F, \mathrm{PGL}_3) \xrightarrow{\text{inv}} \mathbb{Z}/3\mathbb{Z}.$$

Hence, an element $\eta \in \mathrm{Irr}(S_{[\phi]})$ gives rise to a central simple algebra $D_{j(\eta)}$ of degree 3, determined by its invariant $j(\eta) \in \mathbb{Z}/3\mathbb{Z}$.

(c) By combining (a) and (b), one sees that $\eta \in \mathrm{Irr}(S_{[\phi]})$ gives rise to a representation

$$\tau_{\phi,\eta} = JL_{D_{j(\eta)}}(\tau_\rho) \in \mathrm{Irr}(PD^\times_{j(\eta)}).$$

Now the central simple algebra $D_{j(\eta)}$, or rather its associated Jordan algebra $D^+_{j(\eta)}$ gives rise to a local theta correspondence for the dual pair

$$\mathrm{Aut}(D^+_{j(\eta)}) \times G_2 \subset E_6^{D^+_{j(\eta)}}.$$

This is the local theta correspondence used in the definition of θ_D and θ_{M_3} in the Main Theorem. Observe that $D^+_{j(\eta)} \cong D^+_{-j(\eta)}$ and we have canonical (up to inner automorphisms) isomorphisms

$$PD^\times_{j(\eta)} = \mathrm{Aut}(D^+_{j(\eta)})^0 \cong \mathrm{Aut}(D^+_{-j(\eta)})^0 = PD^\times_{-j(\eta)},$$

where the composite $i : PD^\times_{j(\eta)} \to PD^\times_{-j(\eta)}$ is given by $i(x) = x^{-1}$ (up to inner automorphisms). Hence one can define a representation

$$\pi_\eta := \theta(\tau_{\phi,\eta}) = \theta(JL_{D_{j(\eta)}}(\tau_\rho)) \cong \theta(JL_{D_{-j(\eta)}}(\tau_\rho^\vee)),$$

where for the last isomorphism we used the fact that $i^*(JL_{D_{j(\eta)}}(\tau_\rho)) = JL_{D_{-j(\eta)}}(\tau_\rho^\vee)$ as observed earlier. When $p \neq 3$, we have seen in [GS23] that π_η is nonzero irreducible. When $p = 3$, we only know this nonvanishing when η is trivial or if ϕ gives the subregular SL_2 when restricted to the Deligne SL_2.

It is clear from the construction of $\mathcal{L}$ that one has

$$\mathcal{L}^{-1}([\phi]) = \{\pi_\eta : \eta \in \mathrm{Irr}(S_{[\phi]})\} \quad \text{for each } [\phi] \in \Phi^{\clubsuit\clubsuit}_{ds}(G_2).$$

This gives the natural parametrization of the fibers of $\mathcal{L}^{\clubsuit\clubsuit}$ in the proposition.

Note that this parametrization does not depend on the choice of the projective system of isomorphism $i_{[\phi]}$. Indeed, there are 2 such choices as noted above, but changing the choice replaces ρ by $\rho^\vee$ in (a) above (and hence τ_ρ by $\tau_\rho^\vee$) and j by $-j$ in (b). Hence, under this new regime, $\eta \in \mathrm{Irr}(S_{[\phi]})$ is associated by the same process to

$$\theta(JL_{D_{-j(\eta)}}(\tau_\rho^\vee)) \cong \theta(JL_{D_{j(\eta)}}(\tau_\rho)) = \pi_\eta,$$

as asserted.   □

After the above proposition, we see that to prove the Main Theorem, it remains to analyze the map

$$\mathcal{L}^\diamond : \mathrm{Irr}_{ds}^\diamond(G_2) \longrightarrow \Phi_{ds}^\diamond(G_2).$$

For example, a first key question to address is whether this map is surjective. Before addressing such questions, let us describe in the following two sections some ingredients we shall use.

4. Triality and Spin Lifting

In this section, we explain how the principle of triality, together with the theory of theta correspondence, can be used to construct a candidate Spin functorial lifting

$$\mathrm{spin}_* : \mathrm{Irr}_{gen}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}_{gen}(\mathrm{PGSO}_8) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8).$$

This also explains the diagram in (vi) of the Main Theorem.

4.1. Triality and Spin representations.

We begin with a brief discussion of the phenomenon of triality. Recall that there are 3 different maps over F:

$$f_1, f_2, f_3 : \mathrm{SO}_8 \longrightarrow \mathrm{PGSO}_8$$

where the groups SO_8 and PGSO_8 are split. These three morphisms are non-conjugate under PGSO_8 but are cyclically permuted by an order 3 outer automorphism of PGSO_8 (the triality automorphism). Moreover, they induce corresponding morphisms

$$f_1^\vee, f_2^\vee, f_3^\vee : \mathrm{Spin}_8(\mathbb{C}) \longrightarrow \mathrm{SO}_8(\mathbb{C})$$

on the dual side. Without loss of generality, let us fix

$$f_1 : \mathrm{SO}_8 \hookrightarrow \mathrm{GSO}_8 \longrightarrow \mathrm{PGSO}_8$$

and call it the "standard SO_8". Then $f_1^\vee = \mathrm{std} : \mathrm{Spin}_8(\mathbb{C}) \longrightarrow \mathrm{SO}_8(\mathbb{C})$ will be called the standard representation of $\mathrm{Spin}_8(\mathbb{C})$. On the other hand, $f_2^\vee$ and $f_3^\vee$ will be called the half-spin representations of $\mathrm{Spin}_8(\mathbb{C})$.

Consider now the standard embedding

$$
\begin{array}{ccc}
\mathrm{Spin}_7(\mathbb{C}) & \xrightarrow{\;\;i\;\;} & \mathrm{Spin}_8(\mathbb{C}) \\
\big\downarrow & & \big\downarrow{\scriptstyle f_1^\vee} \\
\mathrm{SO}_7(\mathbb{C}) & \longrightarrow & \mathrm{SO}_8(\mathbb{C}).
\end{array}
$$

Then as representations of $\mathrm{Spin}_7(\mathbb{C})$, one has:

$$f_1^\vee \circ i = \mathrm{std} \oplus 1 \quad \text{and} \quad f_2^\vee \circ i = f_3^\vee \circ i = \mathrm{spin}.$$

Thus, one has a very useful and convenient description of the Spin representation of $\mathrm{Spin}_7(\mathbb{C})$.

4.2. Implication.

We observe some implications in representation theory.

For an irreducible generic representation σ of PGSp_6, and $\sigma^\flat \in \mathrm{Irr}(\mathrm{Sp}_6)$ contained in the restriction $\mathrm{rest}(\sigma)$, one may consider the usual (isometry) theta lift of $\sigma^\flat$ from Sp_6 to SO_8, as well as the similitude theta lift of σ to PGSO_8. Both these theta lifts are nonzero and one has the following compatibility:

$$\theta(\sigma^\flat) \subset \theta(\sigma)|_{\mathrm{SO}_8} = f_1^*(\theta(\sigma)).$$

If the L-parameter of $\sigma^\flat$ is $\phi^\flat$, then the L-parameter of any irreducible constituent of $f_1^*(\theta(\sigma))$ is $\phi^\flat \oplus 1$.

On the other hand, if one considers $f_2^*(\theta(\sigma))$ instead, one potentially gets a very different representation of SO_8 with a very different L-parameter. Indeed, consider the special case when σ is unramified with Satake parameter $s \in \mathrm{Spin}_7(\mathbb{C})$. Then one can show that $\theta(\sigma)$ is unramified with Satake parameter $i(s) \in \mathrm{Spin}_8(\mathbb{C})$. Thus, the Satake parameter for $f_1^*(\theta(\sigma))$ is $f_1^\vee(i(s)) = 1 \oplus s \in \mathrm{SO}_8(\mathbb{C})$ (as we noted above), whereas that of $f_2^*(\theta(\sigma))$ is

$$f_2^\vee(i(s)) = \mathrm{spin}(s).$$

This suggests that the map on generic representations of PGSp_6 given by

$$\sigma \mapsto \theta(\sigma) \mapsto f_2^*(\theta(\sigma)) \in \mathrm{Irr}(\mathrm{SO}_8)/_{\mathrm{PGSO}_8}$$

is nothing but the Langlands functorial lifting corresponding to the spin representation

$$\mathrm{spin} : \mathrm{Spin}_7(\mathbb{C}) \longrightarrow \mathrm{SO}_8(\mathbb{C}) \longrightarrow \mathrm{GL}_8(\mathbb{C}),$$

after one composes the above with the functorial lifting

$$\mathcal{A} : \mathrm{Irr}(\mathrm{SO}_8) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8)$$

provided by Cogdell-Kim-Piatetski-Shapiro-Shahidi [CKPSS]) or Arthur [A].

4.3. Spin lifting.

Motivated by the above, we can now define the Spin lifting

$$\mathrm{spin}_* : \mathrm{Irr}_{gen}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8)$$

by

$$\mathrm{spin}_*(\sigma) = \mathcal{A}\left(f_2^*(\theta(\sigma))\right) \in \mathrm{Irr}(\mathrm{GL}_8).$$

Observe that the above construction can be carried out globally as well. Namely, if Σ is a globally generic cuspidal automorphic representation of PGSp_6 over a number field k, then the global theta lift $\theta(\Sigma)$ on PGSO_8 is globally generic and thus nonzero. Suppose further that $\theta(\Sigma)$ is cuspidal. Then $f_2^*(\theta(\Sigma))$ gives rise to a submodule of globally generic cusp forms on SO_8, all of whose summands are weak spin lifting of Σ. Its transfer $\mathcal{A}\left(f_2^*(\theta(\Sigma))\right)$ to GL_8 (à la Cogdell-Kim-Piatetski-Shapiro-Shahidi

[CKPSS]) is then an isobaric automorphic representation of GL_8 (of orthogonal type): this is the global analog of the local construction above. Moreover, for each place v, one has:

$$\mathrm{spin}_*(\Sigma)_v \cong \mathrm{spin}_*(\Sigma_v) \in \mathrm{Irr}(\mathrm{GL}_8(k_v)).$$

4.4. A key property

We shall now show a key property of our local spin lifting spin_*.

Proposition 4.1. *Let σ be an irreducible generic discrete series representation of $\mathrm{PGSp}_6(F)$ and set $\tau = \mathrm{spin}_*(\sigma) \in \mathrm{Irr}(\mathrm{GL}_8)$. Then the following holds:*

(i) *For the γ-factors provided by the Langlands-Shahidi theory,*

$$\gamma(s, \sigma, \mathrm{spin}, \psi) = \gamma(s, \mathrm{spin}_*(\sigma), \mathrm{std}, \psi),$$

for any non-trivial character ψ of F.

(ii) *Let χ be a quadratic character of $F^\times$. Then*

$$\mathrm{spin}_*(\sigma \otimes \chi) = \mathrm{spin}_*(\sigma) \otimes \chi.$$

Proof. We shall prove this via a classical global to local argument. Choose a totally complex number field k with some finite set $T \sqcup \{u, w\}$ of finite places (with T nonempty) such that $k_u \cong k_w \cong F \cong k_v$ for all $v \in T$. By the globalization result in Proposition 10.7 of Appendix A, there exists a globally generic cuspidal automorphic representation Σ such that

- $\Sigma_v \cong \sigma$ for all $v \in T$;
- Σ_u is the Steinberg representation St;
- Σ_w is a generic supercuspidal representation δ;
- Σ_v is a spherical representation at all other finite places v.

Fix a global additive character Ψ of $k \backslash \mathbb{A}_k$ with $\Psi_u = \Psi_w = \psi = \Psi_v$ for all $v \in T$. For a sufficiently large set S of places, containing $T \sqcup \{u, w\}$ and all archimedean places, we have a global functional equation

$$L^S(1 - s, \Sigma^\vee, \mathrm{spin}) = \prod_{v \in S} \gamma(s, \Sigma_v, \mathrm{spin}, \Psi_v) \cdot L^S(s, \Sigma, \mathrm{spin})$$

where L^S denotes the partial L-function. Since Σ is generic, its theta lift $\theta(\Sigma)$ to PGSO_8 is non-zero and generic. Moreover, since Σ_u is the Steinberg representation, its theta lift to PGSO_6 (the smaller group in the Rallis-Witt tower) is 0. Hence $\theta(\Sigma)$ is cuspidal.

By the global analog of the Spin lifting discussed in the previous subsection, we have an isobaric generic automorphic representation

$$\mathrm{spin}_*(\Sigma) = \mathcal{A}\left(f_2^*(\theta(\Sigma))\right)$$

on GL_8. Likewise, we have the global functional equation

$$L^S(1 - s, \mathrm{spin}_*(\Sigma)^\vee, \mathrm{std}) = \prod_{v \in S} \gamma(s, \mathrm{spin}_*(\Sigma)_v, \mathrm{std}, \Psi_v) \cdot L^S(s, \mathrm{spin}_*(\Sigma), \mathrm{std}).$$

Next, by the theta correspondence of unramified representations, observe that

$$L^S(s, \Sigma, \mathrm{spin}) = L^S(s, \mathrm{spin}_*(\Sigma), \mathrm{std}).$$

Moreover, one can readily check that

$$\gamma(s, \Sigma_v, \mathrm{spin}, \Psi_v) = \gamma(s, \mathrm{spin}_*(\Sigma)_v, \mathrm{std}, \Psi_v) \quad \text{for all } v \notin T \sqcup \{u, w\}.$$

Indeed, the local components at finite places outside of $T \sqcup \{u, w\}$ are generic principal series representations and hence their spin lifts and γ-factors are easily computable. For complex groups, the theta correspondence is explicitly known, [AB] (observe here that $\mathrm{PGSp}_6(\mathbb{C}) = \mathrm{Sp}_6(\mathbb{C})/\mu_2$ so the usual theta correspondence is already a correspondence for similitude groups), so that one can check the identities at complex places. Hence, we deduce that

$$\gamma(s, \Sigma_u, \mathrm{spin}, \Psi_u) \cdot \gamma(s, \Sigma_w, \mathrm{spin}, \Psi_w) \cdot \prod_{v \in T} \gamma(s, \Sigma_v, \mathrm{spin}, \Psi_v)$$

$$= \gamma(s, \mathrm{spin}_*(\Sigma)_u, \mathrm{std}, \Psi_u) \cdot \gamma(s, \mathrm{spin}_*(\Sigma)_w, \mathrm{std}, \Psi_w) \cdot \prod_{v \in T} \gamma(s, \mathrm{spin}_*(\Sigma)_v, \mathrm{std}, \Psi_v).$$

This gives the identity for representations of $G_2(F)$:

$$\gamma(s, \mathrm{St}, \mathrm{spin}, \psi) \cdot \gamma(s, \delta, \mathrm{spin}, \psi) \cdot \gamma(s, \sigma, \mathrm{spin}, \psi)^{|T|}$$

$$= \gamma(s, \mathrm{spin}_*(\mathrm{St}), \mathrm{std}, \psi) \cdot \gamma(s, \mathrm{spin}_*(\delta), \mathrm{std}, \psi) \cdot \gamma(s, \mathrm{spin}_*(\sigma), \mathrm{std}, \psi)^{|T|}.$$

Since we can vary the size of T, by considering the quotient of two such identities (say with $|T| = 1$ and 2 respectively), we deduce that

$$\gamma(s, \sigma, \mathrm{spin}, \psi) = \gamma(s, \mathrm{spin}_*(\sigma), \mathrm{std}, \psi)$$

as desired. This proves (i).

For (ii), globalize χ to a quadratic Hecke character χ of $\mathbb{A}_k^\times$. By examining places outside S, one checks readily that $\mathrm{spin}_*(\Sigma \otimes \chi)$ and $\mathrm{spin}_*(\Sigma) \otimes \chi$ are nearly equivalent. By the strong multiplicity one theorem for GL_8, it follows that $\mathrm{spin}_*(\sigma \otimes \chi) \cong \mathrm{spin}_*(\sigma) \otimes \chi$, as desired. $\qquad\square$

5. Results of Kret-Shin

In this section, we briefly recall a special case of some recent results of Kret-Shin relevant for our applications. By combining the Spin lifting of the previous section and the results of Kret-Shin recalled here, we shall prove (vi) of the Main Theorem.

5.1. Results of Kret-Shin

Assume that Σ is a cuspidal automorphic representation of PGSp_6 over a totally real number field k such that

- Σ_∞ is a sufficiently regular generic discrete series representation at each archimedean place ∞;
- Σ_u is the Steinberg representation for some finite place u of k.

Then Kret-Shin [KS, Thm. A] constructed a global Galois representation

$$\rho_\Sigma : \mathrm{Gal}(\overline{k}/k) \longrightarrow \mathrm{Spin}_7(\overline{\mathbb{Q}}_l)$$

satisfying:

- for almost all places v, $\rho_\Sigma(\mathrm{Frob}_v)_{ss}$ is conjugate to the Satake parameter s_{Σ_v} in $\mathrm{Spin}_7(\overline{\mathbb{Q}}_l)$;
- $\mathrm{std} \circ \rho_\Sigma$ is the Galois representation associated to the restriction $\mathrm{rest}(\Sigma)$ of Σ to Sp_6.

5.2. *Spin lifting and Kret-Shin parameters*

We shall apply the results of Kret-Shin to the following problem. Suppose that σ is an irreducible generic discrete series representation of PGSp_6. Then σ determines a pair

$$(\phi^\flat, \phi^\#) \in \Phi(\mathrm{Sp}_6) \times \Phi(\mathrm{GL}_8)$$

where $\phi^\flat$ is the L-parameter of $\mathrm{rest}(\sigma)$ and $\phi^\#$ is the L-parameter of the Spin lifting $\mathrm{spin}_*(\sigma)$. On the other hand, one has the natural map

$$\mathrm{std}_* \times \mathrm{spin}_* : \Phi(\mathrm{PGSp}_6) \longrightarrow \Phi(\mathrm{Sp}_6) \times \Phi(\mathrm{GL}_8)$$

and it is natural to ask if there exists $\phi \in \Phi(\mathrm{PGSp}_6)$ such that

$$\mathrm{std}_*(\phi) = \phi^\flat \quad \text{and} \quad \mathrm{spin}_*(\phi) = \phi^\#.$$

Such a ϕ would be a natural candidate for an L-parameter of σ. The results of Kret-Shin allow us to show the existence of such a ϕ.

Proposition 5.1. *Let $\sigma \in \mathrm{Irr}_{gen,ds}(\mathrm{PGSp}_6)$ be an irreducible generic discrete series representation. Then there exists $\phi \in \Phi_{ds}(\mathrm{PGSp}_6)$ such that*

- $\mathrm{std}_*(\phi)$ *is the L-parameter of* $\mathrm{rest}(\sigma)$ *and*
- $\mathrm{spin}_*(\phi)$ *is the L-parameter of* $\mathrm{spin}_*(\sigma)$.

Proof. Let k be a totally real number field with a place w such that $k_w \cong F$ and consider the group PGSp_6 over k. By the globalization result in Proposition 10.7, we can find a globally generic cuspidal automorphic representation Σ such that

- $\Sigma_w \cong \sigma$;
- Σ_u is the Steinberg representation for some finite place $u \neq w$;
- Σ_v is a generic supercuspidal representation at some other finite place $v \neq w, u$;
- Σ_∞ is a sufficiently regular generic discrete series representation for all archimedean places ∞.

Such a Σ is L-algebraic in the sense of [BG].

Now on one hand, one may consider the global Spin lifting $\mathrm{spin}_*(\Sigma)$: this is an isobaric automorphic representation of GL_8 which is L-algebraic in the sense of [BG]. On the other hand, by Kret-Shin [KS, Thm. A], one can associate to Σ a $\mathrm{Spin}_7(\overline{\mathbb{Q}}_l)$-valued Galois representation ρ_Σ. Then the 8-dimensional representation $\mathrm{spin} \circ \rho_\Sigma$ is a Galois representation associated to $\mathrm{spin}_*(\Sigma)$ (by considering unramified places). Let ϕ be the local L-parameter corresponding to the local Galois representation deduced from ρ_Σ at the place w. Then by local-global compatibility [TY, Ca], one sees readily that ϕ satisfies the requisite conditions of the proposition. □

For a given $\sigma \in \mathrm{Irr}_{gen,ds}(\mathrm{PGSp}_6)$, we will call a ϕ which satisfies the conditions of Proposition 5.1 a Kret-Shin parameter of σ. Unfortunately, it is not true that σ necessarily has a unique Kret-Shin parameter, because $\mathrm{Spin}_7(\mathbb{C})$ is not acceptable [CG1]. However, any two Kret-Shin parameters are related to each other via twisting by a quadratic character. The following says that in some cases, the Kret-Shin parameter is unique.

Proposition 5.2. *Assume that $\sigma \in \mathrm{Irr}_{gen,ds}(\mathrm{PGSp}_6)$ is the theta lift of a generic discrete series π of G_2. Then the Kret-Shin parameter of σ is unique (as an element of $\Phi(\mathrm{PGSp}_6)$) and is given by $\iota \circ \mathcal{L}(\pi)$.*

Proof. Let k be a number field as in the proof of Proposition 5.1 and consider the split group G_2 over k. By Proposition 10.7 and Lemma 10.8 in Appendix A, we can find a globally generic cuspidal representation Π such that $\Pi_w = \pi$ and Π_u is the Steinberg representation for some finite place $u \neq w$. The global theta lift of Π to PGSp_6 is then a nonzero globally generic cuspidal representation Σ of

PGSp_6. As in the proof of Proposition 5.1, by considering the global Spin lifting of Σ to GL_8 and looking at unramified places, we deduce by the strong multiplicity one theorem that

$$\mathrm{spin}_*(\Sigma) = 1 \boxplus \mathrm{std}_*(\Sigma).$$

This implies that the L-parameter of $\mathrm{spin}_*(\sigma)$ is $1 + \phi^\flat$. On the other hand, it is also the case that

$$\mathrm{spin} \circ \iota \circ \mathcal{L}(\pi) = 1 + \phi^\flat.$$

Hence, $\iota \circ \mathcal{L}(\pi)$ is a Kret-Shin parameter for σ.

Suppose ϕ is another Kret-Shin parameter for σ. Then ϕ is a quadratic twist of $\iota \circ \mathcal{L}(\pi)$ and $\mathrm{spin} \circ \phi = 1 \oplus \phi^\flat$. It follows by [HKT, Lemma 4.6] (which is a group theoretic result based on [Gr95]) that $\iota \circ \mathcal{L}(\pi)$ and ϕ are in fact conjugate in $\mathrm{Spin}_7(\mathbb{C})$, i.e

$$\phi = \iota \circ \mathcal{L}(\pi) \in \Phi(\mathrm{PGSp}_6).$$

This completes the proof of the proposition. $\qquad\square$

Corollary 5.3. *One has the commutative diagram in (vi) of the Main Theorem.*

6. Surjectivity of $\mathcal{L}^\diamond$

Using the results of the previous two sections, we shall explain in this section how one can show the surjectivity of

$$\mathcal{L}^\diamond : \mathrm{Irr}_{ds}^\diamond(G_2) \longrightarrow \Phi_{ds}^\diamond(G_2).$$

The main result of this section is:

Proposition 6.1. *Let $\phi : WD_F \to G_2(\mathbb{C})$ be a discrete series parameter belonging to $\Phi_{ds}^\diamond(G_2)$. Then there exists a generic discrete series representation π of G_2 such that $\mathcal{L}(\pi) = \phi$.*

Proof. If the L-parameter ϕ is nontrivial on the Deligne SL_2 in $WD_F = W_F \times \mathrm{SL}_2(\mathbb{C})$, then the discussion in [GS23, §3.5] gives us a candidate non-supercuspidal generic discrete series $\pi \in \mathrm{Irr}(G_2)$ with L-parameter ϕ. The results on theta correspondence in [GS23] allow us to determine $\theta(\pi) \in \mathrm{Irr}(\mathrm{PGSp}_6)$ explicitly, as a non-supercuspidal discrete series representation. Using the knowledge of the LLC for Sp_6 (specifically the L-parameters of generic non-supercuspidal discrete series representations), one readily checks that $\mathcal{L}(\pi) = \phi$ as desired.

Henceforth, we focus on the case when $\phi : W_F \longrightarrow G_2(\mathbb{C})$ is trivial on the Deligne SL_2. Then the L-parameter $\iota \circ \phi$ is a supercuspidal L-parameter of $\mathrm{PGSp}_6(F)$. Likewise, $\phi^\flat := \mathrm{std} \circ \iota \circ \phi$ is a supercuspidal L-parameter of Sp_6 and its L-packet contains a unique generic supercuspidal representation $\sigma^\flat$ with trivial central character. Let σ be any irreducible generic supercuspidal representation of PGSp_6 such that the restriction to Sp_6 contains $\sigma^\flat$. The main issue is to show that we can pick σ so that it has nonzero theta lift to G_2.

Let ϕ_σ be a Kret-Shin parameter of σ (as provided by Proposition 5.1). Then one has

$$\mathrm{std} \circ \phi_\sigma = \mathrm{std} \circ \iota \circ \phi,$$

so that there is a quadratic character χ such that $\iota \circ \phi = \phi_\sigma \otimes \chi$. But by Proposition 4.1(ii), $\phi_\sigma \otimes \chi$ is a Kret-Shin parameter for $\sigma \otimes \chi$. Hence, after replacing σ by $\sigma \otimes \chi$, we may assume that $\iota \circ \phi$ is a Kret-Shin parameter for σ. In particular, we have the following identities of local L-functions:

$$L(s, \sigma, \mathrm{spin}) = L(s, \mathrm{spin}_*(\sigma), \mathrm{std}) = L(s, \mathrm{spin} \circ \iota \circ \phi)$$

with the first equality holding by Proposition 4.1(i) and the second holding by Proposition 5.1 (and the property of the LLC for GL_8). Since

$$\text{spin} \circ \iota \circ \varphi = 1 \oplus \text{std} \circ \iota \circ \varphi = 1 \oplus \phi^\flat$$

and thus fixes a line in the 8-dimensional representation, we see that

$$L(0, \sigma, \text{spin}) = L(0, \text{spin} \circ \iota \circ \phi) = \infty.$$

Now it follows by [SW, Prop. 4.6 and Thm. 5.3] that σ has nonzero theta lift π on G_2, so that $\theta(\pi) = \sigma$ and $\mathcal{L}(\pi) = \phi$. This completes the proof of the proposition. $\qquad\square$

7. Results of Bin Xu

After the results of the previous sections, the remaining issue is the parametrization of the fibers of $\mathcal{L}^\diamond$ over $\Phi_{ds}^\diamond(G_2)$. For this, we shall appeal to recent results of Bin Xu [Xu1, Xu2, Xu3] who studied the problem of extending the LLC for the isometry groups Sp_{2n} to the corresponding similitude groups GSp_{2n}, whose Langlands dual groups are $GSpin_{2n+1}(\mathbb{C})$. Let us briefly recall his results, specialized to the context of $n = 3$.

Take an L-parameter $\phi^\flat$ for Sp_6 with associated L-packet $\Pi_{\phi^\flat}$. Suppose that the representations in $\Pi_{\phi^\flat}$ have trivial central character (i.e. that $\phi^\flat$ can be lifted to $Spin_7(\mathbb{C})$). We consider the finite set

$$\tilde{\Pi}_{\phi^\flat} = \{\tilde{\pi} \in \text{Irr}(PGSp_6) : \text{rest}(\tilde{\pi}) \subset \Pi_{\phi^\flat}\} \subset \text{Irr}(PGSp_6). \tag{7.1}$$

Observe that the group $\text{Hom}(F^\times, \mu_2)$ of quadratic characters of $F^\times$ can be regarded as the group of quadratic characters of $PGSp_6$ (by composition with the similitude map) and acts naturally on $\tilde{\Pi}_{\phi^\flat}$ by twisting.

On the other hand, consider the (finite) set

$$\tilde{\Phi}_{\phi^\flat} := \{\phi \in \Phi(PGSp_6) : \text{std} \circ \phi = \phi^\flat\} \tag{7.2}$$

of lifts of ϕ to $Spin_7(\mathbb{C})$. The group $\text{Hom}(W_F, \mu_2)$ of quadratic characters of W_F acts transitively on $\tilde{\Phi}_{\phi^\flat}$ by twisting. In particular, one has a canonical identification of component groups for any two elements of $\tilde{\Phi}_{\phi^\flat}$.

In [Xu1, Xu2], Xu has obtained a partition of $\tilde{\Pi}_{\phi^\flat}$ into a disjoint union of finite subsets satisfying the following list of properties, which is a compilation of [Xu1, Prop. 6.27, Prop. 6.28 and Thm. 6.30], [Xu2, Prop. 4.4 and Thm. 4.6]) and [Xu3, Thm. 4.1]:

(a) If $\tilde{\Pi}_{\phi^\flat}^X \subset \tilde{\Pi}_{\phi^\flat}$ denotes one such subset, then all others are of the form $\tilde{\Pi}_{\phi^\flat}^X \otimes \chi$ for a quadratic character χ. We will call any of these subsets a Xu's packet.

(b) The natural restriction of representations of $PGSp_6$ to Sp_6 defines a bijection

$$\tilde{\Pi}_{\phi^\flat}^X \longrightarrow \Pi_{\phi^\flat}/PGSp_6.$$

(c) There is a natural bijection

$$\tilde{\Pi}_{\phi^\flat}^X \longleftrightarrow \text{Irr}(S_\phi/Z(Spin_7))$$

for any lift $\phi \in \tilde{\Phi}_{\phi^\flat}$ with component group S_ϕ.

(d) With respect to the parametrization above, the set $\tilde{\Pi}_{\phi,\nu}^X$ satisfies the stability properties and local character identities required by the theory of endoscopy.

(e) For any $\sigma \in \tilde{\Pi}^X_{\phi^b}$ and any $\phi \in \tilde{\Phi}_{\phi^b}$, the stabilizer of σ in $\mathrm{Hom}(F^\times, \mu_2)$ is equal to the stabilizer of ϕ in $\mathrm{Hom}(W_F, \mu_2)$ (identified via local class field theory). In particular,

$$\#\{\text{Xu's packets} \subset \tilde{\Pi}_{\phi^b}\} = \#\tilde{\Phi}_{\phi^b}$$

and the two sets above are (noncanonically) isomorphic as homogeneous sets under $\mathrm{Hom}(F^\times, \mu_2) = \mathrm{Hom}(W_F, \mu_2)$.

(f) Globally, Xu used these local L-packets to describe the tempered part of the automorphic discrete spectrum of PGSp_6 in the style of Arthur's conjecture, in terms of an Arthur multiplicity formula.

There is no doubt that Xu's packet $\tilde{\Pi}^X_{\phi^b}$ is an L-packet for PGSp_6 and its L-parameter should be an element of $\tilde{\Phi}_{\phi^b}$. Constructing an LLC for PGSp_6 compatible with the restriction of representations to Sp_6 amounts to constructing an equivariant bijection between the two homogeneous sets in (e) above. However, this problem was not resolved in [Xu1, Xu2, Xu3]. We shall see in §9 how this issue can be resolved if ϕ^b takes value in $G_2(\mathbb{C})$.

8. Fibers of $\mathcal{L}$

Let $\phi \in \Phi^\diamond_{ds}(G_2)$ be a discrete series parameter so that $\iota \circ \phi$ is a discrete series parameter of PGSp_6. Then we have shown in Proposition 6.1 that $\mathcal{L}^{-1}(\phi)$ contains a generic discrete series representation and is thus nonempty. In this section, we shall enumerate the fiber $\mathcal{L}^{-1}(\phi)$.

For ease of notation, let us set

$$\phi^b = \mathrm{std} \circ \iota \circ \phi \quad \text{and} \quad \phi^\# = \mathrm{spin} \circ \iota \circ \phi.$$

It is clear from the construction of $\mathcal{L}$ that

$$\mathcal{L}^{-1}(\phi) \subset \tilde{\Pi}_{\phi^b}$$

and as we discussed in the previous section, the latter set is the disjoint union of Xu's packets.

8.1. Uniqueness of generic member

We first show:

Proposition 8.1. *For $\phi \in \Phi^\diamond_{ds}(G_2)$, there is a unique generic representation in $\mathcal{L}^{-1}(\phi)$.*

Proof. Since the existence part has been shown in Proposition 6.1, the main issue here is the uniqueness. Suppose, for the sake of contradiction, that π and π' are two distinct generic representations of G_2 such that $\mathcal{L}(\pi) = \mathcal{L}(\pi') = \phi$. Then π and π' belongs to $\mathrm{Irr}^\diamond_{ds}(G_2)$ and we set

$$\sigma = \theta(\pi) \quad \text{and} \quad \sigma' = \theta(\pi').$$

Both σ and σ' are distinct generic discrete series representations of PGSp_6 (by Howe duality) belonging to $\tilde{\Pi}_{\phi^b}$ (see (7.1)). Hence, there is a quadratic character χ such that $\sigma' \cong \sigma \otimes \chi \not\cong \sigma$.

Now by Proposition 5.2, σ has (unique) Kret-Shin parameter $\iota \circ \phi$ which is an element of $\tilde{\Phi}_{\phi^b}$ (see (7.2)). On the other hand,

- by Proposition 4.1(ii), since $\sigma' \cong \sigma \otimes \chi$, σ' has Kret-Shin parameter $(\iota \circ \phi) \otimes \chi$;
- by Proposition 5.2, $\sigma' = \theta(\pi')$ has Kret-Shin parameter $\iota \circ \phi$.

By the uniqueness part of Proposition 5.2, we have

$$\iota \circ \phi = (\iota \circ \phi) \otimes \chi \in \tilde{\Phi}_{\phi^b}.$$

Hence the quadratic character χ stabilizes $\iota \circ \phi \in \tilde{\Phi}_{\phi^\flat}$ but does not stabilize $\sigma \in \tilde{\Pi}_{\phi^\flat}$. This contradicts statement (e) of §7.   □

8.2. A distinguished Xu's packet

In view of Proposition 8.1, we can now pick out a distinguished Xu's packet contained in $\tilde{\Pi}_{\phi^\flat}$. Namely, we set

$$\tilde{\Pi}^{X*}_{\phi^\flat} := \text{the Xu's packet containing the unique generic representation } \theta(\pi) \text{ with } \mathcal{L}(\pi) = \phi.$$

Another way to say this is:

$$\tilde{\Pi}^{X*}_{\phi^\flat} := \text{the Xu's packet containing the unique generic } \sigma \in \tilde{\Pi}_{\phi^\flat} \text{ with nonzero theta lift to } G_2.$$

We shall show:

Proposition 8.2. *Let $\phi \in \Phi^\diamond_{ds}(G_2)$. The local theta correspondence defines a bijection*

$$\mathcal{L}^{-1}(\phi) \longleftrightarrow \tilde{\Pi}^{X*}_{\phi^\flat}.$$

8.3. Parametrization of fiber of $\mathcal{L}$

Let us assume this proposition for a moment and explain how it leads to a proof of (vii) of the Main Theorem. It was shown in [GrS2, Section 4, Prop. 1.10] that the inclusion ι gives rise to an isomorphism

$$S_\phi \cong S_{\iota \circ \phi}/Z(\mathrm{Spin}_7),$$

and thus a bijection

$$\mathrm{Irr}(S_{\iota \circ \phi}/Z(\mathrm{Spin}_7)) \longleftrightarrow \mathrm{Irr}(S_\phi).$$

On the other hand, statement (c) of §7 gives a bijection

$$\tilde{\Pi}^{X*}_{\phi^\flat} \longleftrightarrow \mathrm{Irr}(S_{\iota \circ \phi}/Z(\mathrm{Spin}_7)).$$

Combining this with Proposition 8.2, we obtain a bijection

$$\mathcal{L}^{-1}(\phi) \longleftrightarrow \tilde{\Pi}^{X*}_{\phi^\flat} \longleftrightarrow \mathrm{Irr}(S_{\iota \circ \phi}/Z(\mathrm{Spin}_7)) \longleftrightarrow \mathrm{Irr}(S_\phi).$$

This finishes the proof of (vii) of the Main Theorem.

8.4. One in, all in

It remains to prove Proposition 8.2. The following is the key lemma:

Lemma 8.3. *Let $\tilde{\Pi}^X_{\phi^\flat} \subset \tilde{\Pi}_{\phi^\flat}$ be any Xu's packet. Then either all elements of $\tilde{\Pi}^X_{\phi^\flat}$ have nonzero theta lift to G_2 or none of them has.*

Proof. It suffices to show that if $\sigma \in \tilde{\Pi}^X_{\phi^\flat}$ has nonzero theta lift to G_2, then so does any other $\sigma' \in \tilde{\Pi}^X_{\phi^\flat}$. This will proceed by a global argument.

Suppose that $\sigma = \theta(\pi)$ for $\pi \in \mathrm{Irr}^\diamond_{ds}(G_2)$. Note that all non-supercuspidal representations in $\mathrm{Irr}^\diamond_{ds}(G_2)$ are generic (see [GS23, Thm. 15.3]). We shall globalize π to a cuspidal representation Π in one of the following two ways, depending on the type of π.

(i) Suppose that π is generic (this includes all non-supercuspidal π). Let k be a totally real number field with two places w_1 and w_2 such that $k_{w_1} \cong k_{w_2} \cong F$. Let $\mathbb{O}$ be the split octonion algebra

over k, and let $G = \mathrm{Aut}(\mathbb{O})$. Fix an additional place w and a generic supercuspidal representation δ of $G(k_w)$ such that $\theta(\delta)$ is a generic supercuspidal representation of $H(k_w)$. (Using [G] there are such representations δ of depth 0.) By Proposition 10.7 and Lemma 10.8, one can find a globally generic cuspidal automorphic representation $\Pi = \otimes_v \Pi_v$ of G such that

- Π_∞ is a discrete series representation, with the infinitesimal character sufficiently away from walls, for every archimedean place ∞;
- $\Pi_{w_1} \cong \Pi_{w_2} \cong \pi$;
- $\Pi_w = \delta$.

Then the global theta lift of Π to PGSp_6 is a globally generic cuspidal representation Σ satisfying

- Σ_∞ is a discrete series representation for every archimedean place ∞;
- $\Sigma_{w_1} \cong \Sigma_{w_2} \cong \sigma = \theta(\pi)$;
- the restriction of Σ to Sp_6 is attached to a generic A-parameter.

Here, the first item holds by the matching of infinitesimal characters [HPS] and the fact that, at infinitesimal characters of discrete series sufficiently away from walls, only discrete series representations are unitary. The third item is assured by the first.

(ii) Suppose that π is supercuspidal. Let k be a totally real number field as in (i). Let $\mathbb{O}$ be the totally definite octonion algebra over k, and let $G = \mathrm{Aut}(\mathbb{O})$. By Proposition 10.5 and Corollary 10.6, one can find a cuspidal automorphic representation Π such that:

- $\Pi_{w_1} \cong \Pi_{w_2} \cong \pi$;
- Π_u is the Steinberg representation for some other finite place u;
- Π has nonzero cuspidal global theta lift Σ on PGSp_6.

Thus, Σ is a nonzero cuspidal automorphic representation of PGSp_6 satisfying:

- $\Sigma_{w_1} \cong \Sigma_{w_2} \cong \sigma = \theta(\pi)$;
- Σ_u is the Steinberg representation;
- the restriction of Σ to Sp_6 is attached to a generic A-parameter.

Here the last item is a consequence of the second.

Now let σ' be a representation in $\tilde{\Pi}^X_{\phi^\flat}$. Then by the Arthur multiplicity formula for PGSp_6 established by Xu, there exists a cuspidal automorphic representation Σ' satisfying:

- $\Sigma'_v \cong \Sigma_v$ for all $v \neq w_1$ or w_2;
- $\Sigma'_{w_1} \cong \Sigma'_{w_2} \cong \sigma'$.

In addition, one has an equality of partial Spin L-functions:

$$L^S(s, \Sigma', \mathrm{spin}) = L^S(s, \Sigma, \mathrm{spin}) = \zeta^S(s) \cdot L^S(s, \Sigma, \mathrm{std}).$$

In particular, since $L^S(s, \Sigma, \mathrm{std})$ is nonzero at $s = 1$ (as $\mathrm{rest}(\Sigma)$ is associated with a generic A-parameter), we see that $L^S(s, \Sigma', \mathrm{spin})$ has a pole at $s = 1$. By [GS20, Thm. 1.1], Σ' has nonzero global theta lift to a form of G_2 over k. Specializing to the place w_1, we see that σ' has nonzero local theta lift to the split G_2, as desired. This concludes the proof. □

8.5. *Proof of Proposition 8.2*

Using Lemma 8.3, we can now give the proof of Proposition 8.2. Lemma 8.3 implies that all members of the distinguished Xu's packet $\tilde{\Pi}^{X*}_{\phi^\flat}$ has nonzero theta lift to G_2 (since the generic member does). On the other hand, for any other Xu's packet contained in $\tilde{\Pi}_{\phi^\flat}$, the generic member does not participate in theta correspondence with G_2 by Proposition 8.1. Hence, Lemma 8.3 implies that none of the members does. This proves Proposition 8.2.

9. Applications to Theta Correspondence

The starting point of this paper is the Howe duality and theta dichotomy result of [GS23] for the dual pairs

$$
\begin{array}{c}
\text{PGSp}_6 \\
G_2 \\
\text{PD}^\times \qquad \qquad \text{PGL}_3 \rtimes \mathbb{Z}/2\mathbb{Z}
\end{array}
$$
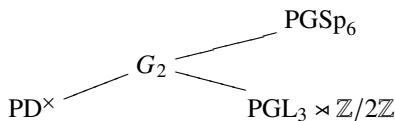
The investigation of these dual pair correspondences was begun by Gross and the second author [GrS1, GrS2] in the 1990's, and precise conjectures for the behaviour of these theta correspondences were formulated in [GrS1, GrS2] in terms of the conjectural LLC for the groups involved. Our construction of the (refined) LLC for G_2 in this paper is to a large extent guided by these conjectures. With the results established in this paper, we can turn the table around and establish these conjectures from [GrS1, GrS2].

For this, we first need to establish a partial LLC for PGSp_6. Namely, suppose that $\phi \in \Phi^\diamond_{ds}(G_2)$ and we set

$$\phi^\flat = \text{std} \circ \iota \circ \phi \in \Phi_{ds}(\text{Sp}_6).$$

We have noted in §7 that the two sets

$$\{\text{Xu's packets} \subset \tilde{\Pi}_{\phi^\flat}\} \quad \text{and} \quad \tilde{\Phi}_{\phi^\flat}$$

are noncanonically isomorphic as $\text{Hom}(F^\times, \mu_2)$-sets. However, we have seen in §8.2 that one has a distinguished Xu's packet $\tilde{\Pi}^{X_*}_{\phi^\flat}$ in the first set. Moreover, the second set contains the distinguished L-parameter $\iota \circ \phi$. The existence of these distinguished base points gives rise to a canonical isomorphism of $\text{Hom}(F^\times, \mu_2)$-sets.

In other words, for $\phi \in \Phi^\diamond_{ds}(G_2)$, we define the L-parameter of the distinguished Xu's packet $\tilde{\Pi}^{X_*}_{\phi^\flat}$ to be $\iota \circ \phi$. This definition is not really ad hoc. Indeed, the distinguished Xu's packet $\tilde{\Pi}^{X_*}_{\phi^\flat}$ is the unique one in $\tilde{\Phi}_{\phi^\flat}$ for which the Langlands-Shahidi Spin L-function of its unique generic member is equal to the local L-factor of $\text{spin} \circ \iota \circ \phi$ (and hence has a pole at $s = 0$). Further, by extending the spin lifting constructed in §4 and using the results of Kret-Shin recalled in §5.1, we construct in Appendix C below a weak LLC for PGSp_6, of which this particular L-packet is an instance.

In any case, with this definition of the L-packet of PGSp_6 associated to $\iota \circ \phi$, and the results of [GS23] as well as the LLC of G_2 established in this paper, we have:

Theorem 9.1. (i) *Conjecture 3.1 in [GrS1] holds when $p \neq 3$. More precisely,*

(a) *For $\tau \in \text{Irr}(\text{PGL}_3)$ with L-parameter ϕ_τ, the local theta lifts of τ to G_2 is the set of $\pi \in \text{Irr}(G_2)$ whose (enhanced) L-parameter (ϕ, η) satisfies*

$$\phi = \iota' \circ \phi_\tau \quad \text{and} \quad \eta \circ \iota'_* = 1,$$

where we recall that

$$\iota' : \text{SL}_3(\mathbb{C}) \longrightarrow G_2(\mathbb{C})$$

is the natural embedding which induces a map of component groups

$$\iota'_* : \pi_0(Z_{\text{SL}_3}(\phi_\tau)) \longrightarrow S_{\iota' \circ \phi_\tau} = \pi_0(Z_{G_2}(\iota' \circ \phi_\tau)).$$

(b) *If τ_D denotes the Jacquet-Langlands lift of τ to $D^\times$, then the set of local theta lifts of τ_D and $\tau_D^\vee$ to G_2 consists of those $\pi \in \text{Irr}(G_2)$ whose (enhanced) L-parameter (ϕ, η) satisfies*

$$\phi = \iota' \circ \phi_\tau \quad \text{and} \quad \eta \circ \iota'_* \neq 1,$$

(ii) *Conjecture 1.13 (as well as Conjectures 2.3 and 2.5) in* [GrS2, *Section 4, Pg. 188*] *hold. More precisely, suppose that $\pi \in \mathrm{Irr}(G_2)$ has (enhanced) L-parameter (ϕ, η), then its local theta lift $\theta(\pi) \in \mathrm{Irr}(\mathrm{PGSp}_6)$ has (enhanced) L-parameter (ϕ', η') satisfying*

$$\phi' = \iota \circ \phi \quad and \quad \eta = \eta' \circ \iota_*$$

where we recall that

$$\iota : G_2(\mathbb{C}) \longrightarrow \mathrm{Spin}_7(\mathbb{C})$$

is the natural embedding which induces a map of component groups

$$\iota_* : S_\phi \longrightarrow S_{\iota \circ \phi}.$$

Thus, in a sense, we have come full circle.

10. Appendix A: Some Globalization Results

In this appendix, we record some globalization results that were used in the main body of the paper.

10.1. A local non-vanishing result

Let F be a p-adic field and $\mathbb{O}$ be an octonion algebra over F, and let $G = \mathrm{Aut}(\mathbb{O})$. Let D be a quaternion algebra over F and fix an embedding $D \hookrightarrow \mathbb{O}$, unique up to conjugation by G. Let $G_D \subset G$ be the pointwise stabilizer of $D \subset \mathbb{O}$. Then G_D is isomorphic to the group of norm one elements in D.

Let $H = \mathrm{PGSp}_6(F)$, and $P = MN$ the Siegel maximal parabolic subgroup. Let $\hat{N}$ be the unitary dual of the unipotent radical N. The group M acts on $\hat{N}$ with open orbits parameterized by quaternion algebras D: the stabilizer in M of an element in the orbit parameterized by is isomorphic to $\mathrm{Aut}(D)$.

Recall that we have the dual pair $G \times H \subset E_7$ and the minimal representation Π of E_7 induces a local theta correspondence $\theta : \mathrm{Irr}^\heartsuit(G) \longrightarrow \mathrm{Irr}(H)$. Now we note:

Lemma 10.1. *Let π be an irreducible representation of G such that $\theta(\pi)$ is a tempered representation of H. Then there exists a quaternion algebra $D \subset \mathbb{O}$ such that π is a quotient of $\mathrm{ind}_{G_D}^G(1)$.*

Proof. We claim that there exists ψ_N, in one of the open orbits, such that $\theta(\pi)_{N,\psi_N} \neq 0$. If not then any summand of $\theta(\pi)$ restricted to $\mathrm{Sp}_6(F)$ is of Howe's N-rank 2, and thus a classical theta lift of a unitary representation of an orthogonal group $\mathrm{O}(2)$ [Li]. But these theta lifts are not tempered, hence the claim

Let Π be the minimal representation of the group E_7, so that $\pi \otimes \theta(\pi)$ is its quotient. From [GrS2, Section 5, Lemma 3.4] (case (4) in the proof) we have

$$\Pi_{N,\psi_N} = \mathrm{ind}_{G_D}^G(1)$$

where D is such that ψ_N belongs to the open orbit parameterized by D. The lemma follows. □

Corollary 10.2. *Let π be a supercuspidal representation of G such that $\theta(\pi) \neq 0$. Then there exists a quaternion algebra D such that π is a submodule of $\mathrm{ind}_{G_D}^G(1)$.*

Proof. Since θ preserves tempered representations, it follows from Lemma 10.1 that π is a quotient of $\mathrm{ind}_{G_D}^G(1)$ for some D. The corollary follows from projectivity of π. □

Lemma 10.3. *Let* St *be the Steinberg representation of G. Let D be a division quaternion algebra. Then* St *is a direct summand of $L^2(G_D \backslash G)$.*

Proof. Recall that $\theta(\text{St})$ is the Steinberg representation of H. Let U be a maximal unipotent subgroup of H containing N. By [CS, Prop. 5], the restriction of the Steinberg representation of H to U is the regular representation of U. It follows at once that $\theta(\text{St})_{N,\psi_N} \neq 0$ for any character ψ_N of N. Hence, arguing as in Lemma 10.1, St is a quotient of $\text{ind}_{G_D}^G(1)$ with D non-split. Since

$$0 \neq \text{Hom}_G(\text{ind}_{G_D}^G(1), \text{St}) \cong \text{Hom}_G(\text{St}, \text{Ind}_{G_D}^G(1)) \cong \text{Hom}_{G_D}(\text{St}, \mathbb{C})$$

and G_D is anisotropic, there exists a non-zero vector in St fixed by G_D. The corresponding matrix coefficient gives us

$$\text{St} \subset L^2(G_D\backslash G) \subset L^2(G).$$

$\square$

We have a complementary result for the compact Lie group $G_2^c(\mathbb{R})$ which is the automorphism group of the octonion division $\mathbb{R}$-algebra.

Lemma 10.4. *Let $\mathbb{H}$ be the quaternion division $\mathbb{R}$-algebra. For any irreducible (finite-dimensional) representation π of $G_2^c(\mathbb{R})$, one has $\pi^{G_{\mathbb{H}}(\mathbb{R})} \neq 0$, so that*

$$\pi \subset L^2(G_{\mathbb{H}}(\mathbb{R})\backslash G_2^c(\mathbb{R})).$$

Proof. One has $G_{\mathbb{H}}(\mathbb{R}) \subset \text{SU}_3 \subset G_2^c(\mathbb{R})$. By the Gelfand-Zetlin branching rule, any irreducible representation of SU_3 has nonzero vectors fixed by $G_{\mathbb{H}}(\mathbb{R}) \cong \text{SU}_2$. The proposition follows. $\square$

10.2. First globalization result

Let k be a totally real number field and $\mathbb{O}$ be a totally definite octonion algebra over k, so that $\mathbb{O}_\infty$ is non-split for all archimedean places ∞ of k, and set $G = \text{Aut}(\mathbb{O})$. For any place v of k, write $G_v = G(k_v)$ for simplicity. Then G_v is anisotropic for archimedean places v and split otherwise.

Let $D \subseteq \mathbb{O}$ be a quaternion subalgebra. Define

$$L_D^2(G(k)\backslash G(\mathbb{A})) \subset L^2(G(k)\backslash G(\mathbb{A}))$$

be the Hilbert subspace which is orthogonal to the span of all automorphic representations $\Pi = \otimes_v \Pi_v$ such that

$$\int_{G_D(k)\backslash G_D(\mathbb{A})} f(g)\, dg = 0$$

for all $f \in \Pi$. Hence, for any $\Pi \subset L_D^2(G(k)\backslash G(\mathbb{A}))$, the global period integral over G_D is nonzero on Π.

Proposition 10.5. *Let $v_1, \ldots, v_n$ be a (nonempty) set of places of k. For nonarchimedean v_i, let π_i be either the Steinberg representation St or a supercuspidal representation of G_{v_i} such that $\theta(\pi_i) \neq 0$. For a real v_i, let π_i be any irreducible (finite dimensional) representation. Then there exist a quaternion algebra $D \subseteq \mathbb{O}$ and an automorphic representation $\Pi = \otimes_v \Pi_v$ in $L_D^2(G(k)\backslash G(\mathbb{A}))$ such that $\Pi_{v_i} \cong \pi_i$ for $i = 1, \ldots, n$.*

Proof. Pick D such that for all $i = 1, \ldots, n$,

$$\pi_i \subset L^2(G_{D,v_i}\backslash G_{v_i}).$$

This is possible by Corollary 10.2, Lemma 10.3 and Lemma 10.4. We can now proceed following [SV, Theorem 16.3.2 and Remark 16.4.1]. Using Poincaré series arising from compactly supported functions

on $G_D \backslash G$, one shows that

$$\bigotimes_{i=1}^{n} L^2(G_{D,v_i} \backslash G_{v_i}) \quad \text{is weakly contained in} \quad L^2_D(G(k) \backslash G(\mathbb{A}))$$

in the sense of Definition 11.5. Hence, $\otimes_i \pi_i$ is weakly contained there as well. The proposition now follows from Corollary 11.7, since supercuspidal representations and St are isolated in the unitary dual of G_{v_i} by Proposition 11.3 (for the Steinberg representation). □

We consider the global theta correspondence for the dual pair $G \times H$ in the adjoint group of absolute type E_7 (and k-rank 3), that corresponds to the Albert algebra $J_3(\mathbb{O})$ via the Koecher-Tits construction, with respect to the minimal representation constructed in [HS, Theorem 6.4].

Corollary 10.6. *In the context of Proposition 10.5, the representation Π has nonzero global theta lift to $H = \mathrm{PGSp}_6$. Moreover, this global theta lift is cuspidal if one of the following conditions hold:*

- *at some nonarchimedean v_i, π_i is the Steinberg representation;*
- *at a real v_i, π_i has regular highest weight.*

Proof. The nonvanishing of the global theta lift follows by [GrS2, Chapter 5, Prop. 4.5]: it is for this global nonvanishing result that we insist on globalizing with a nonzero G_D-period in Proposition 10.5. The cuspidality of the global theta lift follows by [GrS2, Chapter 5, Cor. 4.9]. We note that though [GrS2] works over the base field $\mathbb{Q}$, the results from [GrS2] that we use in this proof hold over a general number field k with the same proofs given there. □

10.3. Second globalization result

Assume now that k is an arbitrary number field and G is a simple split group over k. Let U be the unipotent radical of a Borel subgroup of G, and fix a Whittaker character $\psi = \prod_v \psi_v : U(\mathbb{A})/U(k) \to \mathbb{C}^\times$. In particular, we have a notion of local and automorphic Whittaker functional (relative to ψ) and the notion of (Whittaker) generic representations, both locally and globally.

Fix a place w of k and a ψ_w-generic supercuspidal representation σ of $G(k_w)$. Let

$$L^2_{\psi-\mathrm{gen},\sigma}(G(k) \backslash G(\mathbb{A})) \subset L^2_{cusp}(G(k) \backslash G(\mathbb{A}))$$

be the Hilbert subspace of the automorphic discrete spectrum of G which is orthogonal to the span of all square-integrable automorphic representations $\Pi = \otimes_v \Pi_v$ such that either the automorphic ψ-Whittaker functional vanishes on Π or $\Pi_w \not\cong \sigma$. Then any $\Pi \subset L^2_{\psi-\mathrm{gen},\sigma}(G(k) \backslash G(\mathbb{A}))$ is globally (Whittaker) generic and $\Pi_w \cong \sigma$.

For any place v, let $L^2_{\psi_v}(U(k_v) \backslash G(k_v))$ be the space of square integrable Whittaker functions. Recall that all generic square integrable representations of $G(k_v)$ are direct summands of $L^2_{\psi_v}(U(k_v) \backslash G(k_v))$ [SV].

Proposition 10.7. *Let $v_1, \ldots, v_n$ be a set of (possibly archimedean) places of k, different from w. For $i = 1, \ldots, n$, let δ_i be a ψ_{v_i}-generic square integrable representation of $G(k_{v_i})$. Assume that these representations are isolated in $L^2_{\psi-\mathrm{gen},\sigma}(G(k) \backslash G(\mathbb{A}))$. Then there exists an irreducible cuspidal automorphic representation Π in $L^2_{\psi-\mathrm{gen},\sigma}(G(k) \backslash G(\mathbb{A}))$ such that $\Pi_{v_i} \cong \delta_i$ for $i = 1, \ldots, n$. Moreover, for all but one place different from w or v_i, we may assume that Π_v is a spherical representation.*

Proof. Again, using Poincaré series [SV, Theorem 16.3.2 and Remark 16.4.1], one proves that

$$L^2_{\psi_{v_1}}(U(k_{v_1}) \backslash G(k_{v_1})) \otimes \cdots \otimes L^2_{\psi_{v_n}}(U(k_{v_n}) \backslash G(k_{v_n}))$$

is weakly contained in $L^2_{\psi-\mathrm{gen},\sigma}(G(k) \backslash G(\mathbb{A}))$. In particular, $\delta_1 \otimes \cdots \otimes \delta_n$ is weakly contained there as well. The existence of the globalization Π now follows from Proposition 11.6 and the hypothesis

of isolation of the δ_i's. The extra control at the places different from w and v_i is achieved by using appropriate test functions at these other places as the input for the Poincaré series, see [GI, proof of Prop. A.2]. □

In applications of Proposition 10.7, one would need to verify the hypothesis of isolation in the proposition. For classical groups and their associated similitude groups, the isolation for all generic discrete series representations is a consequence of estimates towards the Ramanujan conjecture for globally generic cuspidal representations; see [ILM, Prop. A.5 and Lemma A.2]. The following verifies it for the group G_2 to the extent we shall need it in this paper.

Lemma 10.8. *Let $G = G_2$ and $H = \mathrm{PGSp}_6$. Let $u \neq w$ be two finite places. Assume that σ is a generic supercuspidal representation of $G(k_w)$ whose local theta lift to $H(k_w)$ is a generic supercuspidal representation (using [G], there are such representations of depth 0). Assume that δ is a generic discrete series representation of $G(k_u)$ such that $\theta(\delta)$ is a generic discrete series representation of $H(k_u)$. Then δ is isolated in $L^2_{\psi-\mathrm{gen},\sigma}(G(k)\backslash G(\mathbb{A}))$ with respect to the Fell topology.*

Proof. We argue by way of contradiction. Assume that there exists a sequence π_n of local components of automorphic representations in $L^2_{\mathrm{gen},\sigma}(G(k)\backslash G(\mathbb{A}))$ such that δ is a limit point of the sequence in Fell topology. Since discrete series representations are isolated in the tempered dual, the π_n's may be assumed to be nontempered for any n. Further, since any irreducible $\Pi \subset L^2_{\mathrm{gen},\sigma}(G(k)\backslash G(\mathbb{A}))$ has a nonzero globally generic global theta lift $\Sigma \subset L^2_{\mathrm{gen},\theta(\sigma)}(H(k)\backslash H(\mathbb{A}))$ (see [HKT, appendix]), the local theta lifts $\theta(\pi_n)$ are generic and unitarizable. We shall show that $\theta(\pi_i)$ converges to $\theta(\delta)$ and this will give the desired contradiction, since as we remarked before the lemma, $\theta(\delta)$ is known to be isolated in $L^2_{\mathrm{gen},\theta(\sigma)}(H(k)\backslash H(\mathbb{A}))$.

By [Ta1], the supercuspidal support of δ is the limit of the sequence of supercuspidal supports of π_n. For generic representations, the theta correspondence is continuous for supercupidal supports. Thus the supercuspidal support of $\theta(\delta)$ is the limit of the sequence of supercuspidal supports of $\theta(\pi_n)$. Now this should imply that $\theta(\delta)$ is a limit point of a subsequence of $\theta(\pi_n)$, since any sequence of generic unitary representations should have a generic representation as a cluster point. For example, this will be true if genericity can be detected by a certain K-type. Since we could not find a reference for this result, we proceed as follows, using Proposition 11.2 shown below.

By Proposition 11.2, on passing to a subsequence of the given π_n's, we may realize δ as a limit point of a sequence of non-tempered representations $\pi_n = \mathrm{Ind}_P^G(\tau_n)$ such that δ is a subquotient of $\mathrm{Ind}_P^G(\tau)$, where τ is an irreducible generic representation which is a limit point of the sequence τ_n. Now the local theta correspondence for non-tempered representation is known by [GS23]. In particular, there exists a parabolic subgroup $Q = LU \subset H$, depending on P, such that

$$\theta(\pi_n) = \mathrm{Ind}_Q^H(\nu_n),$$

where ν_n is irreducible and explicitly constructed from τ_n.

Let ν be the irreducible representation of L explicitly constructed from τ in the same way as ν_n is constructed from τ_n. Then it follows (by the continuity of induction) that the irreducible subquotients of $\mathrm{Ind}_Q^H(\nu)$ are limit points of the sequence $\mathrm{Ind}_Q^H(\nu_n) = \theta(\pi_n)$. Now the irreducible representation ν is generic (since τ is generic), so that $\mathrm{Ind}_Q^H(\nu)$ has a unique irreducible generic subquotient. Since the supercuspidal support of $\mathrm{Ind}_Q^H(\nu)$ is the same as the supercccuspidal support of $\theta(\delta)$, we see that $\theta(\delta)$ is the unique irreducible generic subquotient of $\mathrm{Ind}_Q^H(\nu)$. In particular, $\theta(\delta)$ is a limit point of $\theta(\pi_n)$. Thus we have proved that $\theta(\delta)$ is not isolated in $L^2_{\mathrm{gen},\theta(\sigma)}(H(k)\backslash H(\mathbb{A}))$, giving the desired contradiction. □

11. Appendix B: Topology of Unitary dual

Let G be a (simple) Chevalley group over a p-adic field F. Recall that a unitary representation π of G on a Hilbert space H_π is continuous if, for every $v \in H_\pi$, the map $g \mapsto \pi(g)v$, where $g \in G$, is

continuous. In that case, the subspace $V_\pi \subseteq H_\pi$ of smooth vectors is dense. All unitary representations are assumed to be continuous. We say that a smooth irreducible representation of G is unitarizable if it has a G-invariant positive-definite hermitian form, or equivalently if it is the space of smooth vectors in an irreducible unitary representation.

11.1. Steinberg is isolated

Using an argument due to Miličić, we prove that the Steinberg representation of G is isolated, with respect of the Fell topology, in the unitary dual of G. (We refer the reader to [BHV, Appendix F], for a nice summary of Fell topology, definitions and basic properties). Here we work with smooth representations.

We first recall some basic notions:

- An irreducible unitarizable representation π of G is a *limit point* of a sequence π_n of irreducible unitarizable representations π_n of G if for some (or equivalently for any) matrix coefficient $\langle \pi(g)v, v \rangle$ of π, there is a sequence of matrix coefficients $\langle \pi_n(g)v_n, v_n \rangle$ of π_n converging uniformly on all compact sets in G to $\langle \pi(g)v, v \rangle$.
- An irreducible unitarizable representation π of G is a *cluster point* of the sequence π_n if it is a limit point of a subsequence of π_n.

Let $L[\pi_n]$ and $C[\pi_n]$ denote the sets of limit and cluster points of π_n respectively, so that clearly,

$$L[\pi_n] \subseteq C[\pi_n].$$

However, since the unitary dual is not a Hausdorff space, a nonempty $L[\pi_n]$ does not have to be a singleton and the inclusion $L[\pi_n] \subseteq C[\pi_n]$ can be strict. However, as explained by Miličić, one can have the equality $L[\pi_n] = C[\pi_n]$ under some conditions.

More precisely, let χ_{π_n} be the character distribution of π_n. We say that a sequence χ_{π_n} is weakly converging if, for every $f \in C_c(G)$, the sequence of complex numbers $\chi_{\pi_n}(f)$ is convergent. We have the following result [Mi, Thm. 5]; see also [Ta1].

Theorem 11.1. *Let π_n be a sequence of irreducible unitarizable representations of G. Assume that the corresponding sequence of characters is weakly converging. Then there exists a finite set S of unitary representations such that*

$$\lim_{n\to\infty} \chi_{\pi_n}(f) = \sum_{\sigma \in S} n_\sigma \chi_\sigma(f)$$

for all $f \in C_c(G)$, where $n_\sigma > 0$ are natural numbers. Moreover, one has

$$S = L[\pi_n] = C[\pi_n].$$

We shall need the following proposition which, for real groups, was established by Vogan [Vo] in a greater generality.

Proposition 11.2. *Let G be a simple split group over F. Let δ be a generic unitarizable representation of G. Let π_n be a sequence of generic unitarizable representations of G such that δ is a limit point of the sequence π_n. Then, after passing to a subsequence of π_n if necessary, there exists a parabolic $P \subseteq G$ and a sequence of standard modules $\mathrm{Ind}_P^G(\tau_n)$ such that*

- *$\pi_n \cong \mathrm{Ind}_P^G(\tau_n)$ for all n;*
- *the character distributions of τ_n weakly converge to the character distribution of τ, where τ is a not necessarily irreducible representation;*
- *δ is a subquotient of $\mathrm{Ind}_P^G(\tau)$.*

In particular, the representation τ is generic.

Proof. Fix a Borel subgroup $B = TU$. Let Δ be the set of simple roots given by the choice of B. Let $P = MN$ be a parabolic subgroup of G corresponding to a subset $S \subset \Delta$. Let τ be an irreducible representation of M with a polar decomposition $\tau = \tau^\circ \cdot \chi$ where

- τ° is a tempered representation of M.
- $\chi : M \to \mathbb{R}_+^\times$ is an unramified multiplicative character, with associated "differential"

$$d\chi = \sum_{\alpha \in \Delta \setminus S} s_\alpha \varpi_\alpha$$

where $s_\alpha > 0$ and ϖ_α is the fundamental weight corresponding to α.

A standard module corresponding to the pair (M, τ) is the induced representation $\mathrm{Ind}_P^G(\tau)$. Recall from [HO] that any generic representation is isomorphic to a standard module $\mathrm{Ind}_P^G(\tau)$ for some pair (M, τ).

Thus any π_n is isomorphic to a standard module $\mathrm{Ind}_P^G(\tau_n)$ where the parabolic P depends on n. However, since there are finitely many parabolic subgroups in the standard position, after passing to a subsequence of π if necessary, we can assume that P is independent of n. This takes care of the first bullet.

Write $\tau_n = \tau_n^\circ \cdot \chi_n$, the polar decompostion. Then by [Ta1], there exists a subsequence of τ_n° such that the corresponding character distributions weakly converge. Replace the sequence τ_n° by that subsequence. Hence there exists a representation τ°, not necessarily irreducible, such that its character distribution is the weak limit of the character distributions of τ_n°. Since π_n are unitary, the differentials $d\chi_n$ must be bounded. Hence, after passing to a subsequence again, we can assume that χ_n converge to a character χ. Let $\tau = \tau^\circ \cdot \chi$. This takes care of the second bullet.

Finally, since the character distributions of τ_n weakly converge to the character distribution of τ, the character distributions of $\mathrm{Ind}_P^G(\tau_n)$ weakly converge to the character distribution of $\mathrm{Ind}_P^G(\tau)$. Hence, by Theorem 11.1, the limit points of the sequence π_n are precisely the subquotients of $\mathrm{Ind}_P^G(\tau)$. This completes the third bullet. ☐

Now we are ready to prove the main result in this section:

Proposition 11.3. *Let σ be the Steinberg representation of G. If the rank of G is at least 2, then σ is isolated in the unitary dual of G.*

Proof. We shall prove this by contradiction, so let's assume that σ is a limit point of a sequence π_n.

Let K be an open compact subgroup of G. Let τ be any smooth irreducible representation of K. Let f_τ be a smooth function on G, supported on K, and equal to the complex conjugate of the character of τ, divided by the volume of K, on K. Thus, for any unitary representation (π, V) of G, the operator $\pi(f_\tau)$ is the projector on the τ-isotypic subspace of V.

Lemma 11.4. *Let π be an irreducible unitarizable representation such that the K-type τ occurs in π. Assume that π is a limit point of a sequence of irreducible representations π_n. Then, for almost all n, the K-type τ occurs in π_n.*

Proof. By the assumption there exists a non-zero $v \in V$, such that $\langle \pi(f_\tau)v, v \rangle = \langle v, v \rangle \neq 0$. Let $\langle \pi_n(g)v_n, v_n \rangle$ be a sequence of matrix coefficients of π_n converging uniformly on all compact sets in G, in particular on K, to $\langle \pi(g)v, v \rangle$. Thus the limit of $\langle \pi_n(f_\tau)v_n, v_n \rangle$ is $\langle \pi(f_\tau)v, v \rangle$, and the lemma follows. ☐

Now assume that K is a special maximal compact subgroup. Then K has a quotient $G(\mathbb{F}_q)$ where $\mathbb{F}_q$ is the residual field of F. Let st_G be the inflation to K of the Steinberg representation of the finite group $G(\mathbb{F}_q)$. The restriction of σ to K contains st_G. By Lemma 11.4, after passing to a subsequence, we may assume that all π_n contain st_G. But this implies that π_n are generic [BM], hence we can apply Proposition 11.2. In particular, σ is a subquotient of some induced representation $\mathrm{Ind}_P^G(\tau)$ where all subquotients are unitarzable. Observe that P is a proper parabolic subgroup. Indeed, since square integrable representations are isolated in the tempered dual of G, we see that the π_n may be assumed

to be nontempered; in particular, $P \neq G$. Furthermore, since convergence in Fell topology implies convergence in Bernstein center [Ta2], all irreducible subquotients of $\operatorname{Ind}_P^G(\tau)$ are contained in any principal series $\operatorname{Ind}_B^G(\chi)$ containing σ.

Next, recall that st_G is the unique K-type of σ with Iwahori-fixed vectors. Observe that K-types of $\operatorname{Ind}_P^G(\tau)$ are easily determined from $K \cap M$-types of τ. Observe that τ is generic, and hence st_M is a type of τ. If the rank of G is greater than one, it is easy to see that there is an Iwahori type in $\operatorname{Ind}_P^G(\tau)$, different from st_G, that contains st_M. Hence $\operatorname{Ind}_P^G(\tau)$ and therefore $\operatorname{Ind}_B^G(\chi)$ contains a unitarizable representation different from σ and the trivial representation of G. We claim that there is no such representation. Indeed, by Rodier [Ro], $\operatorname{Ind}_B^G(\chi)$ has 2^r irreducible suqbquotients. Their exponents are known, and thus the asymptotics of their matrix coefficients. It follows that the matrix coefficients of subquotients, different from the Steinberg representation, are not decaying at infinity. Thus, by a theorem of Howe and Moore the only unitarizable subquotients of $\operatorname{Ind}_B^G(\chi)$ are the trivial representation and σ. □

11.2. Weak containment

Definition 11.5. Let π and σ be two unitary representations of G. We say that σ is weakly contained in π if every normalized matrix coefficient $\langle \sigma(g)v, v \rangle$ of σ, that is $||v|| = 1$, can be approximated uniformly, on compact subsets of G, by convex combinations of normalized matrix coefficients of π.

Proposition 11.6. *Let $\pi_1, \pi_2, \ldots$ be a sequence of irreducible unitary representations of G. Let $\pi = \hat{\oplus}_{i=1}^\infty \pi_i$. Let σ be an irreducible unitary representation of G. If σ is weakly contained in π then σ is a limit of a subsequence of π_n.*

Proof. Recall that $L^\infty(G)$ is the dual space of $L^1(G)$, and the unit ball in $L^\infty(G)$ is compact in the weak-$*$ topology. Let $\mathcal{C}$ be the weak-$*$ closure, in $L^\infty(G)$, of the convex hull of

$$\mathcal{F} = \{\langle \pi_n(g)v_n, v_n \rangle \mid v_n \in \pi_n, ||v_n|| = 1, n = 1, 2 \ldots\}.$$

Then $\mathcal{C}$ is closed and convex subset of the unit ball, in particular, it is compact by Alaoglu's Theorem [Co, page 130]. Recall that an element of a convex set is extremal if it is not a proper convex combination of two other elements of the set. By [BHV, Thm. C.5.2], elements of $\mathcal{F}$ are extremal in $\mathcal{C}$. (That theorem states that normalized matrix coefficients associated to an irreducible representation are extremal in a larger convex set.) By our assumption, $f(g) = \langle \sigma(g)v, v \rangle$, for $v \in \sigma$ of norm 1, is contained in $\mathcal{C}$. Thus it is also an extremal point of $\mathcal{C}$ since σ is irreducible. By the Krein-Milman Theorem [Co, page 141], any extremal point in $\mathcal{C}$ is in the closure of $\mathcal{F}$. In particular, f is a limit of a sequence $f_n(g) = \langle \pi_n(g)v_n, v_n \rangle$ in $\mathcal{F}$. Of course, the limit is in the weak-$*$ topology, however, by a theorem of Raikov [BHV, Thm. C.5.6], the sequence f_n converges uniformly on compact sets to f. This is banal for p-adic groups. Indeed, we can assume that v and all v_n are K-fixed for a small open compact subgroup of G. Thus these functions are constant on K-double cosets, and uniform convergence on compact sets is equivalent to point-wise convergence. Thus, if $g \in G$ and χ_g is the characteristic function of KgK divided by the volume of KgK volume, then we have

$$f_n(g) = \int_G f_n \cdot \chi_g \to \int_G f \cdot \chi_g = f(g)$$

since $f_n \to f$ in the weak-$*$ topology. □

Corollary 11.7. *Let $\pi_1, \pi_2, \ldots$ be a sequence of irreducible unitary representations of G. Let $\pi = \hat{\oplus}_{i=1}^\infty \pi_i$. Let σ be an irreducible unitary representation of G isolated in the unitary dual. If σ is weakly contained in π then $\sigma \cong \pi_i$ for some i.*

12. Appendix C: A Weak LLC for PGSp$_6$

We have mentioned that the ideas involved in the construction of the Spin lifting in §4 via an application of triality and the results of Kret-Shin allow one to construct a weak LLC for PGSp$_6$ refining the results of B. Xu in this specific case. In this appendix, let us give an outline of this, explaining in particular what "weak" means. The reader will notice that the general structure is very similar to our treatment of the LLC for G_2.

12.1. *Weak equivalence of L-parameters*

In §5.2, we have encountered the natural map

$$\text{std}_* \times \text{spin}_* : \Phi(\text{PGSp}_6) \longrightarrow \Phi(\text{Sp}_6) \times \Phi(\text{GL}_8).$$

As mentioned there, this map is not injective, because $\text{Spin}_7(\mathbb{C})$ is not acceptable in the sense of Larsen [CG1]. On the other hand, the standard and Spin L-functions of $\phi \in \Phi(\text{PGSp}_6)$ is completely determined by its image in $\Phi(\text{Sp}_6) \times \Phi(\text{GL}_8)$. This motivates us to define the coarser equivalence relation on L-parameters:

Definition 12.1. Two elements $\phi_1, \phi_2 \in \Phi(\text{PGSp}_6)$ are said to be *weakly equivalent* if

$$(\text{std}_*(\phi_1), \text{spin}_*(\phi_1)) = (\text{std}_*(\phi_2), \text{spin}_*(\phi_2)) \in \Phi(\text{Sp}_6) \times \Phi(\text{GL}_8).$$

We denote the set of weak equivalence classes by $\Phi_w(\text{PGSp}_6)$.

One has [CG1, §1]:

Lemma 12.2. *The following are equivalent for two elements ϕ_1 and ϕ_2 of $\Phi(\text{PGSp}_6)$:*

- *ϕ_1 and ϕ_2 are weakly equivalent;*
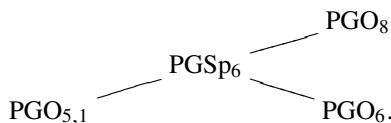- *ϕ_1 and ϕ_2 are elementwise conjugate.*

Our goal in this appendix is to construct a canonical map

$$\mathcal{L}_w : \text{Irr}(\text{PGSp}_6) \longrightarrow \Phi_w(\text{PGSp}_6)$$

satisfying some desirable properties.

12.2. *Theta dichotomy*

We shall make use of the similitude theta correspondence associated to the following dual pairs:

$$
\begin{array}{ccc}
 & & \text{PGO}_8 \\
 & \diagup & \\
 & \text{PGSp}_6 & \\
 & \diagdown & \\
\text{PGO}_{5,1} \diagup & & \text{PGO}_6.
\end{array}
$$

Here, the orthogonal similitude groups in the right tower are split, whereas the one on the left has F-rank 1 and is isomorphic to $\text{PGL}_2(D) \rtimes \mathbb{Z}/2\mathbb{Z}$ where D is the quaternion division F-algebra.

As a consequence of the conservation relation [SZ], any $\sigma \in \text{Irr}(\text{PGSp}_6)$ has a nonzero theta lift to exactly one of PGO_8 or $\text{PGO}_{5,1}$. This gives a partition

$$\text{Irr}(\text{PGSp}_6) = \text{Irr}^\heartsuit(\text{PGSp}_6) \bigsqcup \text{Irr}^\spadesuit(\text{PGSp}_6)$$

where the elements in $\mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6)$ are those with nonzero theta lifts to PGO_8. Among the representations in $\mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6)$, some will participate in the theta correspondence with the lower step of the tower, i.e. with $\mathrm{PGO}_6 = \mathrm{PGL}_4(F) \rtimes \mathbb{Z}/2\mathbb{Z}$. Thus, we have a further decomposition

$$\mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6) = \mathrm{Irr}^\diamond(\mathrm{PGSp}_6) \sqcup \mathrm{Irr}^\clubsuit(\mathrm{PGSp}_6)$$

where $\mathrm{Irr}^\clubsuit(\mathrm{PGSp}_6)$ denotes the subset of those representations which has nonzero theta lifting to PGO_6. It will be better to group $\mathrm{Irr}^\clubsuit(\mathrm{PGSp}_6)$ and $\mathrm{Irr}^\clubsuit(\mathrm{PGSp}_6)$ together, denoting their union by $\mathrm{Irr}^{\clubsuit\clubsuit}(\mathrm{PGSp}_6)$.

In short, the similitude theta correspondence gives maps:

$$\theta^{\clubsuit\clubsuit} : \mathrm{Irr}^{\clubsuit\clubsuit}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{PGL}_2(D))_{/\mathbb{Z}/2\mathbb{Z}} \sqcup \mathrm{Irr}(\mathrm{PGL}_4)_{/\mathbb{Z}/2\mathbb{Z}} \xrightarrow{\;\mathrm{JL}\;} \mathrm{Irr}(\mathrm{PGL}_4)_{/\mathbb{Z}/2\mathbb{Z}}$$

and

$$\theta^\diamond : \mathrm{Irr}^\diamond(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6) \xrightarrow{\;\theta^\heartsuit\;} \mathrm{Irr}(\mathrm{PGO}_8) \longrightarrow \mathrm{Irr}(\mathrm{PGSO}_8)_{/\mathbb{Z}/2\mathbb{Z}}.$$

The latter map is in fact injective and takes value in the $\mathbb{Z}/2\mathbb{Z}$-fixed points in $\mathrm{Irr}(\mathrm{PGSO}_8)$.

12.3. Spin lifting

We have defined the Spin lifting of generic representations of PGSp_6 in §4.3. In fact, this definition can be extended to the whole of $\mathrm{Irr}(\mathrm{PGSp}_6)$ to give:

$$\mathrm{spin}_* : \mathrm{Irr}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8).$$

We consider two cases, according to the decomposition of $\mathrm{Irr}(G)$.

(a) if $\sigma \in \mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6)$, we set:

$$f_2^*(\theta^\heartsuit(\pi)) \in \mathrm{Irr}(\mathrm{SO}_8)_{/\mathrm{PGSO}_8}$$

and

$$\mathrm{spin}_*(\sigma) = \mathcal{A}\left(f_2^*(\theta^\heartsuit(\pi))\right) \in \mathrm{Irr}(\mathrm{GL}_8).$$

Here, we refer to §4.1 for the definition of f_2. This definition of spin_* is the same as that given in §4.3 for generic representations.

(b) if $\sigma \in \mathrm{Irr}^{\clubsuit\clubsuit}(\mathrm{PGSp}_6)$, then recall that we have the theta lifting map

$$\theta^{\clubsuit\clubsuit} : \mathrm{Irr}^{\clubsuit\clubsuit}(G) \longrightarrow \mathrm{Irr}(\mathrm{PGL}_4)_{/\mathbb{Z}/2\mathbb{Z}}.$$

Moreover, one has a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Irr}(\mathrm{PGL}_4)_{/\mathbb{Z}/2\mathbb{Z}} & \longrightarrow & \Phi(\mathrm{PGL}_4)_{/\mathbb{Z}/2\mathbb{Z}} \\
{\scriptstyle \boxplus}\downarrow & & \downarrow{\scriptstyle \oplus} \\
\mathrm{Irr}(\mathrm{GL}_8) & \xrightarrow{\;\mathcal{L}_{\mathrm{GL}_8}\;} & \Phi(\mathrm{GL}_8),
\end{array}
$$

where the horizontal arrows are LLC maps, the first vertical arrow is the isobaric sum and the second vertical arrow is the map sending $(\phi, \phi^\vee)$ to $\phi \oplus \phi^\vee$. More precisely, $\boxplus$ is given concretely as:

$$\boxplus : (\tau, \tau^\vee) \mapsto \text{the unique constituent of } \mathrm{Ind}_P^{\mathrm{GL}_8}(\tau \boxtimes \tau^\vee) \text{ with } L - \text{parameter } \phi_\tau \oplus \phi_\tau^\vee,$$

where P is the maximal parabolic subgroup of GL_8 with Levi factor $\mathrm{GL}_4 \times \mathrm{GL}_4$.

We may now set:

$$\mathrm{spin}_*(\sigma) = \boxplus \circ \theta^{\spadesuit\clubsuit}(\sigma) \in \mathrm{Irr}(\mathrm{GL}_8).$$

The representations in $\mathrm{Irr}^{\clubsuit}(\mathrm{PGSp}_6)$ have been treated in both (a) and (b) above. One can check that the two definitions of spin_* agree on $\mathrm{Irr}^{\clubsuit}(\mathrm{PGSp}_6)$. Thus, we have now a well-defined map

$$\mathrm{spin}_* : \mathrm{Irr}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{GL}_8).$$

12.4. Kret-Shin parameters

Now a key result in the construction of our map $\mathcal{L}_w$ is the following:

Theorem 12.3. *Given $\sigma \in \mathrm{Irr}(\mathrm{PGSp}_6)$, there exists a unique $\phi \in \Phi_w(\mathrm{PGSp}_6)$ such that*

$$\mathrm{std} \circ \phi = \mathcal{L}_{\mathrm{Sp}_6}(\mathrm{rest}(\sigma))$$

and

$$\mathrm{spin} \circ \phi = \mathcal{L}_{\mathrm{GL}_8}(\mathrm{spin}_*(\sigma)).$$

Here we recall that $\mathrm{rest} : \mathrm{Irr}(\mathrm{PGSp}_6) \longrightarrow \mathrm{Irr}(\mathrm{Sp}_6)_{/\mathrm{PGSp}_6}$ *is the restriction of representations from* PGSp_6 *to* Sp_6.

Proposition 5.1 is a special case of this for $\sigma \in \mathrm{Irr}_{gen,ds}(\mathrm{PGSp}_6)$. Recall that we have called a ϕ as in the Theorem a Kret-Shin parameter of σ.

Proof. To prove the Theorem, we consider two cases:

- If $\sigma \in \mathrm{Irr}^{\spadesuit\clubsuit}(\mathrm{PGSp}_6)$, then our discussion in Case (b) in the previous subsection already gives the desired ϕ. In the notation there, we set

$$\phi = j \circ \phi_{\theta^{\spadesuit\clubsuit}(\sigma)},$$

  where

$$j : \mathrm{PGL}_4^{\vee} = \mathrm{Spin}_6(\mathbb{C}) \longrightarrow \mathrm{Spin}_7(\mathbb{C}).$$

  This satisfies $\mathrm{spin} \circ \phi = \mathcal{L}_{\mathrm{GL}_8}(\mathrm{spin}_*(\sigma))$. Moreover, by the theory of theta correspondence, it is easy to see that

$$\mathrm{std} \circ \phi = \mathcal{L}_{\mathrm{SO}_6}(\theta^{\spadesuit\clubsuit}(\sigma)) \oplus 1 = \mathcal{L}_{\mathrm{Sp}_{66}}(\mathrm{rest}(\sigma)).$$

- Suppose now that $\sigma \in \mathrm{Irr}^{\heartsuit}(\mathrm{PGSp}_6)$. If σ is not a discrete series representation, then one can determine its local theta lift to PGSO_8 completely (using the fact that the LLC for Levi subgroups of PGSp_6 and PGSO_8 are known). From this knowledge, one can write down the desired $\phi \in \Phi(\mathrm{PGSp}_6)$: it will factor through a Levi subgroup of $\mathrm{Spin}_7(\mathbb{C})$. We omit the details here.

  The key case is that of discrete series representations. Here the proof is along the same lines as that of Proposition 5.1, where we demonstrated this for generic discrete series representations. To carry out the same proof, for $\sigma \in \mathrm{Irr}_{ds}(\mathrm{PGSp}_6)$, one needs to globalize σ appropriately. We discuss this globalization below. □

12.5. Globalization

Let k be a totally real number field with adele ring $\mathbb{A}$ and a place v_0 such that $k_{v_0} = F$, and let $v_1 \neq v_0$ be another finite place. Consider the split group PGSp_6 over k. We have:

Lemma 12.4. *There is a cuspidal automorphic representation Σ of PGSp_6 satisfying:*

- *$\Sigma_{v_0} = \sigma$;*
- *$\Sigma_{v_1} = \mathrm{St}_{v_1}$ is the Steinberg representation;*
- *for each real place v_∞ of k, Σ_{v_∞} is a generic discrete series representation with very regular infinitesimal character so that it is spin-regular in the sense of [KS];*
- *Σ_v is a quadratic twist of an unramified representation for all other v.*

Proof. Using [S, Theorem 5.8], we can find a cuspidal automorphic representation Σ' of PGSp_6 satisfying:

- $\Sigma'_{v_0} = \pi$;
- $\Sigma'_{v_1} = \mathrm{St}_{v_1}$;
- for each real place v_∞, Σ'_{v_∞} is a discrete series representation with very regular infinitesimal character and is spin-regular;
- for all other places v, Σ'_v is unramified.

The only condition potentially missing from Σ' is the genericity of its archimedean components.

Now the restriction of Σ' to Sp_6 determines a near equivalence class of square-integrable automorphic representations with a generic A-parameter Ψ of Sp_6, because of the Steinberg local component. Moreover, again because of the Steinberg local component, the global component group of the A-parameter Ψ is trivial. As a consequence, every representation of $\mathrm{Sp}_6(\mathbb{A})$ in the global A-packet attached to Ψ occurs in the space of cusp forms by Arthur's multiplicity formula [A]. In particular, if we take an irreducible automorphic summand $\Sigma'^\flat$ of $\Sigma'|_{\mathrm{Sp}_6}$, we may replace the archimedean components of $\Sigma'^\flat$ by the generic discrete series representations in the local A-packets (which are just L-packets) and obtain a cuspidal automorphic representation $\Sigma^\flat$ in the same A-packet such that $\Sigma^\flat_f = \Sigma'^\flat_f$. Note also that the central character of $\Sigma^\flat$ is the same as that of $\Sigma'^\flat$, which is trivial.

Now let Σ be a cuspidal automorphic representation of PGSp_6 whose restriction to Sp_6 contains $\Sigma^\flat$. Then the archimedean components of Σ are generic discrete series which are spin-regular. Moreover, $\Sigma_{v_0} = \pi \otimes \chi_0$ and $\Sigma_{v_1} = \mathrm{St}_{v_1} \otimes \chi_1$ for some quadratic characters χ_0 and χ_1. Thus, after replacing Σ by its twist by an automorphic quadratic character χ such that $\chi_{v_0} = \chi_0$ and $\chi_{v_1} = \chi_1$, we obtain the desired Σ as in the Lemma. $\square$

12.6. Global Theta lift.

With the cuspidal automorphic representation Σ given by the lemma, we may now consider the global theta lift of Σ to the split group PGSO_8 over k. We have:

Lemma 12.5. *The global theta lift $\Theta(\Sigma)$ of Σ to PGSO_8 is a nonzero cuspidal automorphic representation.*

Proof. We first note that $\Theta(\Sigma)$ must be cuspidal because of the tower property of theta correspondence. Indeed, the Steinberg representation St_{v_1} of $\mathrm{PGSp}_6(k_{v_1})$ does not participate in the local theta correspondence with $\mathrm{PGO}_6(k_{v_1})$, so that the global theta lift of Σ to the split PGO_6 is zero.

By [GQT, Thm. 1.4(ii)], to show the nonvanishing of the global theta lift $\Theta(\Sigma)$, it suffices to show:

- For any place v of k, the local component Σ_v has nonzero theta lift to $\mathrm{PGSO}_8(k_v)$. This follows because unramified representations or generic representations have nonzero theta lifting to $\mathrm{PGSO}_8(k_v)$; this takes care of all places $v \neq v_0$, and for the place v_0, it follows by our assumption that $\Sigma_{v_0} = \sigma \in \mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6)$.

- The degree 7 standard L-function $L(s, \Sigma, std)$ is nonzero at $s = 1$. This follows because $\Sigma|_{\mathrm{Sp}_6}$ has a generic A-parameter.

  By the Rallis inner product formula [GQT, Thm. 1.4(ii)], we deduce that the global theta lift $\Theta(\Sigma)$ to PGSO_8 is nonzero and the lemma is proved. $\qquad\square$

We may thus consider the global Spin lifting of Σ:

$$\mathrm{spin}_*(\Sigma) = \mathcal{A}(f_2^*(\Theta(\Sigma))).$$

Then $\mathrm{spin}_*(\Sigma)$ is in fact a cuspidal representation of GL_8 (because of the spin-regularity of the infinitesimal character at infinite places). Applying the results of Kret-Shin, as recounted in §5.1, and arguing as in the proof of Proposition 5.1, we conclude the proof of Theorem 12.3.

12.7. Construction of $\mathcal{L}_w$

By Theorem 12.3, we have shown:

Theorem 12.6. *Let*

$$\mathcal{L}_w : \mathrm{Irr}(\mathrm{PGSp}_6) \longrightarrow \Phi_w(\mathrm{PGSp}_6)$$

be defined by

$$\mathcal{L}_w(\sigma) := \textit{the unique } \phi \textit{ associated to } \sigma \textit{ by Theorem 12.3}$$

for each $\sigma \in \mathrm{Irr}(\mathrm{PGSp}_6)$. Then $\mathcal{L}_w$ features in and is characterized by the following commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Irr}(\mathrm{PGSp}_6) & \xrightarrow{\ \mathcal{L}_w\ } & \Phi_w(\mathrm{PGSp}_6) \\
{\scriptstyle \mathrm{rest}\times\mathrm{spin}_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{std}_*\times\mathrm{spin}_*} \\
\mathrm{Irr}(\mathrm{Sp}_6)_{/\mathrm{PGSp}_6} \times \mathrm{Irr}(\mathrm{GL}_8) & \xrightarrow[\ \mathcal{L}_{\mathrm{Sp}_6}\times\mathcal{L}_{\mathrm{GL}_8}\ ]{} & \Phi(\mathrm{Sp}_6) \times \Phi(\mathrm{GL}_8).
\end{array}
$$

Moreover, the map $\mathcal{L}_w$ sends tempered (respectively discrete series) representations to (weak equivalence classes of) tempered (respectively discrete series) L-parameters and satisfies

$$\mathcal{L}_w(\sigma \otimes \chi) \cong \mathcal{L}_w(\sigma) \otimes \chi$$

for any quadratic character χ of $F^\times$.

Proof. It remains to verify the last statement: the compatibility of $\mathcal{L}_w$ with twisting by quadratic characters, It comes down to the fact that the similitude theta correspondence is compatible with such twisting. More precisely, if $\sigma \in \mathrm{Irr}^\heartsuit(\mathrm{PGSp}_6)$, then

$$\theta^\heartsuit(\sigma \otimes \chi) = \theta^\heartsuit(\sigma) \otimes \chi.$$

Here, it is important to note that the similitude character of PGSO_8 implicit on the right hand side of the equation is the one which is trivial on $f_1(\mathrm{SO}_8(F))$. In particular, it is not trivial on $f_2(\mathrm{SO}_8(F))$ (see §4.1 for f_1 and f_2). Thus, for any representation π of $\mathrm{PGSO}_8(F)$,

$$f_2^*(\pi \otimes \chi)) = f_2^*(\pi) \otimes \chi$$

where we regard χ as a character of $\mathrm{SO}_8(F)$ via composition with the spinor norm. Since the LLC for SO_8 is compatible with twisting by quadratic characters, we deduce that

$$\mathrm{spin}_*(\sigma \otimes \chi) = \mathrm{spin}_*(\sigma) \otimes (\chi \circ \det) \in \mathrm{Irr}(\mathrm{GL}_8)$$

For $\sigma \in \mathrm{Irr}^{\clubsuit\spadesuit}(G)$, one verifies this identity along similar lines; we omit the details.   □

12.8. Fibers of $\mathcal{L}_w$

Finally, we would like to understand the fibers of $\mathcal{L}_w$. We shall show:

Theorem 12.7. (i) *The map $\mathcal{L}_w$ is surjective. Moreover, the natural restriction map*

$$\mathrm{rest} : \mathcal{L}_w^{-1}(\phi) \longrightarrow \mathcal{L}_{\mathrm{Sp}_6}^{-1}(\mathrm{std} \circ \phi)_{/\mathrm{PGSp}_6}$$

is surjective.

(ii) *The fibers of $\mathcal{L}_w$ are unions of Xu's packets (introduced in §7).*

Proof. (i) For given $\phi \in \Phi_w(\mathrm{PGSp}_6)$, with $\phi^\flat := \mathrm{std} \circ \phi \in \Phi(\mathrm{Sp}_6)$, write

$$\Pi_{\phi^\flat} := \mathcal{L}_{\mathrm{Sp}_6}^{-1}(\mathrm{std} \circ \phi)$$

be the corresponding L-packet for Sp_6, which is non-empty. Pick $\sigma^\flat \in \Pi_{\phi^\flat}$. Because $\phi^\flat$ can be lifted to $\mathrm{Spin}_7(\mathbb{C})$, the representations in $\Pi_{\phi^\flat}$ have trivial central character. Hence we can pick $\sigma \in \mathrm{Irr}(\mathrm{PGSp}_6)$ such that $\sigma^\flat \subset \mathrm{rest}(\sigma)$. Then $\mathcal{L}_w(\sigma)$ is a lift of $\phi^\flat$, so that

$$\mathcal{L}_w(\sigma) = \phi \otimes \chi$$

for some quadratic character χ. Hence $\mathcal{L}(\sigma \otimes \chi) = \phi$. This proves the first assertion. Indeed, the proof shows that for any $\sigma^\flat \in \Pi_{\phi^\flat}$, there exists $\sigma \in \mathrm{Irr}(\mathrm{PGSp}_6)$ such that $\sigma^\flat \subset \mathrm{rest}(\sigma)$ and $\mathcal{L}_w(\sigma) = \phi$. This gives the second assertion.

(ii) It is not hard to reduce this issue to the case of a discrete series L-parameter ϕ. Moreover, we shall assume that ϕ does not factor through $\mathrm{Spin}_6(\mathbb{C})$ so that

$$\mathcal{L}_w^{-1}(\phi) \subset \mathrm{Irr}^\diamond(\mathrm{PGSp}_6).$$

The case when $\mathcal{L}_w^{-1}(\phi) \subset \mathrm{Irr}^{\clubsuit\spadesuit}(\mathrm{PGSp}_6)$ is checked along similar lines and we leave it to the reader.

Suppose then that $\sigma \in \mathcal{L}_w^{-1}(\phi)$. Writing $\phi^\flat = \mathrm{std} \circ \phi$, we know that σ lies in a unique Xu's packet $\tilde{\Pi}_{\phi^\flat}^X$, where we are using the notations from §7. Thus, we need to show that

$$\tilde{\Pi}_{\phi^\flat}^X \subset \mathcal{L}_w^{-1}(\phi).$$

By Lemma 12.4, we globalize σ to a cuspidal automorphic representation Σ over a number field k such that $\Sigma_{v_0} = \sigma$ and $\Sigma_{v_1} = \mathrm{St}_{v_1}$ is the Steinberg representation; moreover, the global theta lift $\Theta(\Sigma)$ of Π to PGSO_8 is a nonzero cuspidal representation by Lemma 12.5.

As we saw in the proof of Lemma 12.4, $\mathrm{rest}(\Sigma)$ gives rise to a generic A-parameter $\Psi^\flat$ with trivial global component group $S_{\Psi^\flat}$. Now we appeal to the global results of Xu [Xu3, Thm. 4.1]. Under these circumstances, Xu showed that Σ belongs to a global generic A-packet for PGSp_6 of the form

$$\tilde{\Pi}_{\Psi^\flat}^X = \otimes_v \tilde{\Pi}_{\Psi_v^\flat}^X$$

where the local packets $\tilde{\Pi}^X_{\Psi^\flat_v}$ were introduced in §7. The contribution of $\tilde{\Pi}^X_{\Psi^\flat}$ to the automorphic discrete spectrum is governed by the Arthur multiplicity formula. In particular, since the global component group $S_{\Psi^\flat}$ is trivial, [Xu3, Thm. 4.1] says that every element of $\tilde{\Pi}^X_{\Psi^\flat}$ is automorphic.

Now we know that $\sigma = \Sigma_{v_0} \in \Pi^X_{\Psi^\flat_{v_0}}$ and we would like to show that

$$\tilde{\Pi}^X_{\Psi^\flat_{v_0}} \subset \mathcal{L}^{-1}_w(\phi).$$

For any $\sigma' \in \tilde{\Pi}^X_{\Psi^\flat_{v_0}}$, we thus need to show that

$$\mathcal{L}_w(\sigma') = \mathcal{L}_w(\sigma) = \phi.$$

For this, let Σ' be the representation of $PGSp_6(\mathbb{A})$ such that

$$\Sigma'_{v_0} = \sigma' \quad \text{and} \quad \Sigma'_v = \Sigma_v \text{ for all } v \neq v_0.$$

As we noted above, Σ' is cuspidal automorphic as well. Moreover, as we showed in the proof of Lemma 12.5, the global theta lift $\Theta(\Sigma')$ to $PGSO_8$ is nonzero.

Now we have two cuspidal representations $\Theta(\Sigma)$ and $\Theta(\Sigma')$ on $PGSO_8$ which are nearly equivalent. Hence

$$\mathcal{A}(f_2^*(\Theta(\Sigma))) \quad \text{and} \quad \mathcal{A}(f_2^*(\Theta(\Sigma')))$$

give the same generic A-parameter on GL_8. In particular, their local component at v_0 are the same. Likewise, both $f_1^*(\Theta(\Sigma))$ and $f_1^*(\Theta(\Sigma'))$ have A-parameter $1 \oplus \Psi^\flat$. Hence, on extracting the v_0-component, it follows that $\mathcal{L}_w(\sigma') = \mathcal{L}_w(\sigma) = \phi$, as desired.   □

As one believes that Xu's packets are precisely the L-packets of $PGSp_6$, the map $\mathcal{L}_w$ is not the ultimate LLC map. However, it is a strict refinement of the results of Xu in the setting of $PGSp_6$. Indeed, as we showed in the main body of the paper (see Proposition 5.2), if $\phi \in \Phi(PGSp_6)$ is valued in $G_2(\mathbb{C})$, then $\mathcal{L}^{-1}_w(\phi)$ is a single Xu's packet and hence is an honest L-packet of $PGSp_6$. This fact plays a crucial role in our proof of the LLC for G_2.

References

[A] J. Arthur, *The Endoscopic Classification of Representations. Orthogonal and Symplectic Groups*, American Mathematical Society Colloquium Publications, Vol. 61 (American Mathematical Society, Providence, RI, 2013), xviii+590 pp.

[AB] J. Adams and D. Barbasch, 'Reductive dual pair correspondence for complex groups', *J. Funct. Anal.* **132**(1) (1995), 1–42.

[AMS] A. M. Aubert, A. Moussaoui and M. Solleveld, 'Generalizations of the Springer correspondence and cuspidal Langlands parameters', *Manuscripta Math.* **157**(1–2) (2018), 121–192.

[AX] A.M. Aubert and Y. J. Xu, 'The explicit local Langlands correspondence for', Preprint, 2022, arXiv, https://arxiv.org/pdf/2208.12391.pdf.

[BM] D. Barbasch and A. Moy, 'Whittaker models with an Iwahori-fixed vector', *Contemp. Math.* **177** (1994), 101–105.

[BG] K. Buzzard and T. Gee, 'The conjectural connections between automorphic representations and Galois representations', in *Automorphic Forms and Galois Representations*, Vol. 1, London Math. Soc. Lecture Note Ser., Vol. 414 (Cambridge Univ. Press, Cambridge, 2014), 135–187.

[BHV] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's Property (T)* (Cambridge Univ. Press, Cambridge, 2008).

[Ca] A. Caraiani, 'Local-global compatibility and the action of monodromy on nearby cycles', *Duke Math. J.* **161**(12) (2012), 2311–2413.

[CG] P.S. Chan and W.T. Gan, 'The local Langlands conjecture for GSp(4) III: Stability and twisted endoscopy', *J. Number Theory* **146** (2015), 69–133.

[CG1] G. Chenevier and W. T. Gan, 'Spin$_7$is unacceptable', Preprint, 2023, arXiv:2303.15897.

[CG2] G. Chenevier and W. T. Gan, 'Triality and functoriality', In preparation.

[CS] K. Y. Chan and G. Savin, 'Iwahori model of Bessel model spaces', *Proceedings of AMS* **148**(4) (2020), 1487–1497

[C] G. Chenevier, 'Subgroups of Spin(7) or SO(7) with each element conjugate to some element of G_2 and applications to automorphic forms', *Doc. Math.* **24** (2019), 95–161.

[CKPSS] J. W. Cogdell, H. Kim, I. Piatetski-Shapiro and F. Shahidi, 'Functoriality for the classical groups', *Publ. Math. IHES.* **99** (2004), 163–233.

[CZ1] R. Chen and J. L. Zou, 'Local Langlands correspondence for even orthogonal groups via theta lifts', *Selecta Math. (N.S.)* **27**(5) (2021), Paper No. 88, 71 pp.

[CZ2] R. Chen and J. L. Zou, 'Local Langlands correspondence for unitary groups via theta lifts', *Represent. Theory* **25** (2021), 861–896.

[Co] J. B. Conway, *A Course in Functional Analysis*, second edn., Graduate Texts in Mathematics, Vol. 96 (Springer-Verlag, New York, 1990).

[FS] L. Fargues and P. Scholze, 'Geometrization of the local Langlands correspondence', Preprint, 2021, arXiv:2102.13459.

[G] W. T. Gan, 'Exceptional Howe correspondences over finite fields', *Compositio Math.* **118**(3) (1999), 323–344.

[GI] W. T. Gan and A. Ichino, 'The Shimura–Waldspurger correspondence for M_{2n}', *Ann. Math. Second Series* **188**(3) (2018), 965–1016.

[GQT] W. T. Gan, Y. N. Qiu and S. Takeda, 'The regularized Siegel–Weil formula (the second term identity) and the Rallis inner product formula', *Invent. Math.* **198**(3) (2014), 739–831.

[GS20] W. T. Gan and G. Savin, 'An exceptional Siegel–Weil formula and poles of the Spin L-function of $PGSp6$', *Compos. Math.* **156**(6) (2020), 1231–1261.

[GS23] W. T. Gan and G. Savin, 'Howe duality and dichotomy for exceptional theta correspondence', *Invent. Math.* **232**(1) (2023), 1–78.

[GS24] W. T. Gan and G. Savin, 'A theory of γ-factors for $G_2 \times GL_r$', Preprint, 2023, arXiv:2308.12561.

[GT] W. T. Gan and S. Takeda, 'The local Langlands conjecture for GSp(4)', *Ann. of Math. (2)* **173**(3) (2011), 1841–1882.

[GTW] W. T. Gan and W. Tantono, 'The local Langlands conjecture for GSp(4), II: The case of inner forms', *Amer. J. Math.* **136**(3) (2014), 761–805.

[GeSh] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-functions*, Perspectives in Mathematics, Vol. 6 (Academic Press, Boston, 1988).

[Gr95] R. L. Griess, Jr., 'Basic conjugacy theorems for G_2', *Invent. Math.* **121**(2) (1995) 257–277.

[GrS1] B. H. Gross and G. Savin, 'The dual pair $PGL_3 \times G_2'$', *Canad. Math. Bull.* **40**(3) (1997), 376–384.

[GrS2] B. H. Gross and G. Savin, 'Motives with Galois group of type G2: An exceptional theta-correspondence', *Compos. Math.* **114** (1998), 153–217.

[HS] M. Hanzer and G. Savin, 'Eisenstein series arising from Jordan algebras', *Canad. J. Math.* **72**(1) (2020), 183–201.

[HKT] M. Harris, C. Khare and J. Thorne, 'A local Langlands parameterization for generic supercuspidal representations of p-adic G_2', *Annales de l' ENS* **56**(1) (2023), 257–286.

[HO] V. Heiermann and E. Opdam, 'On the tempered L-functions conjecture', *Amer. J. Math.* **135**(3) (2013), 777–799.

[He] G. Henniart, 'Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p-adique', *Invent. Math.* **139**(2) (2000), 439–455.

[HL] J. Hundley and B. Y. Liu, 'On automorphic descent from GL(7) *to* G_2', *J. European Math. Society*, To appear.

[HPS] J. S. Huang, P. Pandžić and G. Savin, 'New dual pair correspondences', *Duke Math. J.* **82** (1996), 447–471.

[HT] M. Harris and R. Taylor, *The Geometry and Cohomology of Some Simple Shimura Varieties*, Annals of Mathematics Studies, Vol. 151 (Princeton University Press, Princeton, NJ, 2001), viii+276 pp. With an appendix by Vladimir G. Berkovich.

[ILM] A. Ichino, E. Lapid and Z. Mao, 'On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups', *Duke Math. J.* **166**(7) (2017), 1301–1348.

[K1] T. Kaletha, 'Regular supercuspidal representations', *J. Amer. Math. Soc.* **32**(4) (2019), 1071–1170.

[K2] T. Kaletha, 'Supercuspidal L-packets', Preprint, 2019, arXiv:1912.03274.

[KMSW] T. Kaletha, A. Minguez, S. W. Shin and P. J. White, 'Endoscopic classification of representations: inner forms of unitary groups', Preprint, 2014, arXiv:1409.3731.

[KS]   A. Kret and S. W. Shin, 'Galois representations for general symplectic groups', *J. Eur. Math. Soc. (JEMS)* **25**(1) (2023), 75–152.

[Li]   J.-S. Li, 'On the classification of irreducible low rank unitary representations of classical groups', *Compositio Math.* **71**(1) (1989), 29–48.

[Mi]   D. Miličić, 'On C^*-algebras with bounded trace', *Glasnik Mat.* **8** (28) (1973), 7–22.

[MR]   C. Moeglin and D. Renard, 'Sur les paquets d'Arthur des groupes classiques et unitaires non quasi-déployés', in *Relative Aspects in Representation Theory, Langlands Functoriality and Automorphic Forms*, Lecture Notes in Math., Vol. 2221 (CIRM Jean-Morlet Ser., Springer, Cham, 2018), 341–361.

[M]   C. P. Mok, 'Endoscopic classification of representations of quasi-split unitary groups', *Mem. Amer. Math. Soc.* **235**(1108) (2015), vi+248 pp.

[Ro]   F. Rodier, 'Decomposition of principal series for reductive p-adic groups and the Langlands' classification', in *Operator Algebras and Group Representations*, *Vol. II (Neptun, 1980)*, Monogr. Stud. Math., Vol. 18 (Pitman, Boston, MA, 1984), 86–94.

[SV]   Y. Sakellaridis and A. Venkatesh, *Periods and Harmonic Analysis on Spherical Varieties*, Asterisque 396 (Math. Soc. France, Paris, 2017).

[SW]   G. Savin and M. Weissman, 'Dichotomy for generic supercuspidal representations of G_2', *Compositio Math.* (2010) 1–49.

[S]   S. W. Shin, 'Automorphic Plancherel density theorem', *Israel J. Math.* **192**(1) (2012), 83–120.

[SZ]   B. Y. Sun and C. B. Zhu, 'Conservation relations for local theta correspondence', *J. Amer. Math. Soc.* **28**(4) (2015), 939–983.

[Ta1]   M. Tadić, 'On limits of characters of irreducible unitary representations', *Glasnik Mat.* **23** (43) (1988), 15–25.

[Ta2]   M. Tadić, 'Geometry of dual spaces of reductive groups (non-Archimedean case)', *J. Analyse Math.* **51** (1988), 139–181.

[TY]   R. L. Taylor and T. Yoshida, 'Compatibility of local and global Langlands correspondences', *J. Amer. Math. Soc.* **20**(2) (2007), 467–493.

[Vo]   D. Vogan, 'Isolated unitary representations', *Automorphic Forms and Applications*, IAS/Park City Math. Ser., Vol. 12 (Amer. Math. Soc., Providence, RI, 2007), 379–398.

[Xu1]   B. Xu, 'On lifting problem of L–packets', *Compositio Math.* **152**(9) (2016), 1800–1850.

[Xu2]   B. Xu, 'L–packets of quasisplit GSp(2n) and GO(2n)', *Math. Ann.* **370**(1–2) (2018), 71–189.

[Xu3]   B. Xu, 'Global L-packets of quasi-split GSp($2n$) and GSO($2n$)', *American J. of Math.*, To appear. arXiv:2103.15300.