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UNITS IN GROUP RINGS OF FREE PRODUCTS OF PRIME CYCLIC GROUPS

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ABSTRACT. Let G be a free product of cyclic groups of prime order. The structure of the unit group $U(\mathbb{Q}G)$ of the rational group ring $\mathbb{Q}G$ is given in terms of free products and amalgamated free products of groups. As an application, all finite subgroups of $U(\mathbb{Q}G)$, up to conjugacy, are described and the Zassenhaus Conjecture for finite subgroups in $\mathbb{Z}G$ is proved. A strong version of the Tits Alternative for $U(\mathbb{Q}G)$ is obtained as a corollary of the structural result.

1**. Introduction.** Let $U(\mathbb{Z}G)$ denote the unit group of the integral group ring $\mathbb{Z}G$ of a group G and let $U_1(\mathbb{Z}G)$ be the group of units of augmentation 1 in $\mathbb{Z}G$. Similar notation shall be used for the rational group algebra $\mathbb{Q}G$. The Conjecture of Zassenhaus, denoted (ZC3) [14], states that if G is finite and H is a finite subgroup of $U_1(\mathbb{Z}G)$ then H is conjugate in $U(\mathbb{Q}G)$ to a subgroup of G. A restricted version of this conjecture, denoted (ZC1) [14], says that every torsion unit of $U_1(\mathbb{Z}G)$ is conjugate in $U(\mathbb{Q}G)$ to an element of G. It is known that (ZC3) holds for finite nilpotent groups [16], [17], finite split metacyclic groups [12], [15] and some particular groups. However, (ZC3) is false in general and the counterexamples show that it does not hold for finite metabelian groups [7] and [13]. The Zassenhaus Conjecture restricted to finite p-subgroups of $U_1(\mathbb{Z}G)$ has been established for finite nilpotent-by-nilpotent groups G [4], for finite solvable groups G whose Sylow p-subgroups are either abelian or generalized quaternion [4] and for Frobenius groups G which cannot be mapped homomorphically onto S_5 [5]. More information on the Zassenhaus Conjecture and its various versions can be found in [3], [13], [14]. It is interesting to know which infinite groups satisfy (ZC3). In [11] an infinite nilpotent group is constructed which does not satisfy (ZC1) (compare with [2]). Problem 39 of [14] asks whether (ZC1) holds for a free product of finite cyclic groups.

Torsion units in integral group rings $\mathbb{Z}G$ where G is a free product of abelian groups were studied by A. I. Lichtman and S. K. Sehgal [10]. They proved that if $u \in U_1(\mathbb{Z}G)$ has order $m < \infty$ then one of the free factors of G contains an element h of order m. Moreover, if G is a free product of a finite number of finite abelian groups then u is conjugate to h in a large overing of $\mathbb{Q}G$ (Theorem 1 of [10]). In a particular case when G is the infinite dihedral group the conjugating element can be taken even in $\mathbb{Z}[\frac{1}{2}]G$, (see [9]).

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In this paper we study the free product $G = *G_\alpha$ $(\alpha \in I)$ of cyclic groups of prime order $|G_\alpha| = p_\alpha$ (the p_α's are not necessarily distinct and I may be infinite). In Section 2 by applying Gerasimov's Theorem [6] we prove that

$$\mathcal{U}(\mathbb{Q}G) = *_{\mathcal{U}(\mathbb{Q})}\big((A_\alpha * B_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)\big),$$

where $*_{\mathcal{U}(\mathbb{Q})}$ denotes the amalgamated free product over the multiplicative group $\mathcal{U}(\mathbb{Q})$ of $\mathbb{Q}$ and A_α, B_α are abelian groups isomorphic to the additive groups of some infinite dimensional vector spaces over $\mathbb{Q}$ (Theorem 2.3). As a consequence we prove that every nonabelian subgroup of $\mathcal{U}(\mathbb{Q}G)$ either contains a free noncyclic subgroup or is metabelian (Corollary 2.4). In Section 3 we use Theorem 2.3 to prove that every finite subgroup of $\mathcal{U}_1(\mathbb{Q}G)$ is conjugate in $\mathcal{U}(\mathbb{Q}G)$ to a subgroup of $\mathcal{U}_1(\mathbb{Q}G_\alpha)$ for some $\alpha \in I$ (Theorem 3.4). As a corollary the Zassenhaus Conjecture (ZC3) is proved for G (Corollary 3.5).

2**. The structure of the rational unit group.** Let K be an associative ring with identity and $G = *_H G_\alpha(\alpha \in I)$ be the free product of groups G_α with amalgamated subgroup H. It is easy to verify that KG is isomorphic to the coproduct $\coprod_{KH} KG_\alpha$, $(\alpha \in I)$ of rings KG_α over KH. In particular, if $G = *G_\alpha(\alpha \in I)$ is the free product of groups G_α, then $KG \cong \coprod_K KG_\alpha(\alpha \in I)$. Thus, Gerasimov's Theorem on units in coproducts of rings [6] can be used in the study of $\mathcal{U}(KG)$.

An element of KG of the form $1 + x\nu y$ where $x, y \in KG_\alpha$, $yx = 0$, $\nu \in KG$ is called a *KG_α-transvection*. Let $\mathbf{\Gamma}(KG_\alpha)$ be the subgroup of $\mathcal{U}(KG)$ generated by $\mathcal{U}(KG_\alpha)$ and all the KG_α-transvections of KG. A ring R with the identity element 1 is called *1-commutative* if $xy = 1$ implies $yx = 1$ $(x, y \in R)$. The following statement is an immediate consequence of Gerasimov's Theorem.

STATEMENT 2.1. Let $G = *G_\alpha(\alpha \in I)$ and K be a division ring. If each KG_α is 1-commutative then

$$\mathcal{U}(KG) \cong *_{\mathcal{U}(K)}\mathbf{\Gamma}(KG_\alpha), \quad (\alpha \in I),$$

where $\mathcal{U}(K)$ denotes the multiplicative group of K.

It is easy to see that the subgroup $T(KG_\alpha)$ generated by all the KG_α-transvections of KG is normal in $\mathbf{\Gamma}(KG_\alpha)$.

Suppose now that $K = \mathbb{Q}$ and that each $|G_\alpha| = p_\alpha$ is a prime $(\alpha \in I)$. The p_α's are not necessarily distinct and I may be infinite. Let S be the disjoint union of the $G_\alpha \setminus \{1\}$, $(\alpha \in I)$. We say that the product $g = g_1 \cdots g_n$, $(g_i \in S)$ is *reduced* if either $n = 1$ or $n \geq 2$ and no adjacent factors belong to the same G_α. In this case n is called the *length* of g and shall be denoted by $\ell(g)$.

Let β be a fixed index and $G_\beta = \langle c \rangle$. Take any ordering on each $G_\alpha \setminus \{1\}$, $(\alpha \neq \beta)$. Set $c^\imath < c^\jmath$ if and only if $\imath < \jmath$, $(0 < \imath, \jmath \leq p - 1, p = p_\beta)$.

Now take an ordering on I such that $\beta < \alpha$ for every $\alpha \in I$, $(\alpha \neq \beta)$ and assume that the identity element $1 \in G$ has length 0. This determines an ordering in S.

Suppose now that every element of G is given as a reduced product and order them first by their length and then lexicographically from left to right.

For a $\nu \in KG$ the leading term, lead(ν), of ν is the maximum of $\{g : g \in \mathrm{supp}(\nu)\}$, that is $\mathrm{lead}(\nu) \geq g$ for every g of the support of ν.

Let C_β be the $\mathbb{Q}$-subspace of $\mathbb{Q}G$ generated by all reduced products $c^\imath g_1 \cdots g_n$, ($g_\jmath \in S$, $n \geq 1$, $g_n \notin G_\beta$, $0 \leq \imath \leq p-2$) and let D_β be the $\mathbb{Q}$-subspace of $\mathbb{Q}G$ generated by all reduced products $g_1 \cdots g_n c^\imath$, ($g_\jmath \in S$, $n \geq 1$, $g_1 \notin G_\beta$, $0 \leq \imath \leq p-2$).

Set $\hat{c} = 1 + c + \cdots + c^{p-1}$ and consider the following maps: $\varphi: C_\beta \longrightarrow U(\mathbb{Q}G)$ and $\psi: D_\beta \longrightarrow U(\mathbb{Q}G)$ defined by $\varphi(\nu) = 1 + (1-c)\nu\hat{c}$, and $\psi(\nu) = 1 + \hat{c}\nu(1-c)$. It is easily seen that φ and ψ are homomorphisms from the additive groups C_β and D_β respectively into $T(\mathbb{Q}G_\beta)$.

LEMMA 2.2. *Set $A_\beta = \mathrm{Im}\,\varphi$ and $B_\beta = \mathrm{Im}\,\psi$. Then $T(\mathbb{Q}G_\beta) = \langle A_\beta, B_\beta \rangle$ and $\varphi: C_\beta \longrightarrow A_\beta$, $\psi: D_\beta \longrightarrow B_\beta$ are isomorphisms.*

PROOF. It is easily seen that if $xy = 0$ for some $x, y \in \mathbb{Q}G_\beta$ then one of these elements belongs to $(1-c)\mathbb{Q}G_\beta$ and the other to $\mathbb{Q}\hat{c}$. Hence $T(\mathbb{Q}G_\beta)$ is generated by all elements of the form $1 + (1-c)\nu\hat{c}$, $1 + \hat{c}\nu(1-c)$, $\nu \in \mathbb{Q}G$. Then it follows from the equality

$$(1-c)c^{p-1} = -(1-c)(1 + c + \cdots + c^{p-2})$$

that $T(\mathbb{Q}G_\beta)$ is generated by $\mathrm{Im}\,\varphi$ and $\mathrm{Im}\,\psi$. This proves the first statement. It remains to be shown that $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\psi = \{0\}$.

Let $0 \neq \nu \in C_\beta$ and $\mathrm{lead}(\nu) = c^\imath g_1 \cdots g_n$, ($n \geq 1, 0 \leq \imath \leq p-2$) be written as a reduced product.

Let $c^\jmath h_1 \cdots h_k \neq \mathrm{lead}(\nu)$ be a reduced product from the support of ν. Observe that since $\beta < \alpha$ for every $\alpha \in I$, ($\alpha \neq \beta$), we have that $k \leq n$. (Note that this observation will be used in (8)). Then either $k < n$ or $k = n$ and $\jmath < \imath$ or $k = n$, $\jmath = \imath$ and $h_1 \cdots h_n < g_1 \cdots g_n$.

It is easy to see that in all cases $c^{\imath+1} g_1 \cdots g_n > c^{\jmath+1} h_1 \cdots h_k$ and, consequently,

$$\mathrm{lead}\big((1-c)\nu\big) = c^{\imath+1} g_1 \cdots g_n,$$

$$\mathrm{lead}\big(\varphi(\nu)\big) = c^{\imath+1} g_1 \cdots g_n c^{p-1} = c\big(\mathrm{lead}(\nu)\big)c^{p-1}. \tag{1}$$

Thus, $\varphi(\nu) \neq 1$ and $\mathrm{Ker}\,\varphi = \{0\}$.

Let $0 \neq \nu \in D_\beta$ and for a reduced product $g = h_1 \cdots h_k c^\jmath$ from the support of ν, set $\omega(g) = h_1 \cdots h_k$. Let

$$g_1 \cdots g_n = \max\{\omega(g) : g \in \mathrm{supp}(\nu)\}$$

and

$$g_1 \cdots g_n c^\imath = \max\{g \in \mathrm{supp}(\nu) : \omega(g) = g_1 \cdots g_n\}.$$

If $h_1 \cdots h_k c^\jmath$ is any other reduced product from $\mathrm{supp}(\nu)$, then either $h_1 \cdots h_k < g_1 \cdots g_n$ or $k = n$, $h_1 \cdots h_k = g_1 \cdots g_n$ and $\jmath < \imath$. In both cases we have that $h_1 \cdots h_k c^{\jmath+1} < g_1 \cdots g_n c^{\imath+1}$, therefore,

$$\mathrm{lead}\big(\nu(1-c)\big) = g_1 \cdots g_n c^{\imath+1},$$

$$\mathrm{lead}\big(\psi(\nu)\big) = \mathrm{lead}\big(\hat{c}\nu(1-c)\big) = c^{p-1} g_1 \cdots g_n c^{\imath+1}. \tag{2}$$

Thus $\mathrm{Ker}\,\psi = \{0\}$. ∎

Now we shall prove the main result of this section.

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THEOREM 2.3. *Let $G = *G_\alpha$, $(\alpha \in I)$ where $|G_\alpha| = p_\alpha$ is a prime.Then*

$$U(\mathbb{Q}G) = *_{U(\mathbb{Q})}\big((A_\alpha * B_\alpha) \rtimes U(\mathbb{Q}G_\alpha)\big), \quad (\alpha \in I)$$

where $A_\alpha * B_\alpha = T(\mathbb{Q}G_\alpha)$ is the group generated by all $\mathbb{Q}G_\alpha$-transvections of $\mathbb{Q}G$, A_α and B_α are abelian groups isomorphic to the additive groups of some infinite dimensional vector spaces over $\mathbb{Q}$ (see Lemma 2.2).

PROOF. Fix $\beta \in I$. We shall use the notation and the ordering introduced above. By Statement 2.1 and Lemma 2.2 it suffices to prove that $T(\mathbb{Q}G_\beta) \cap U(\mathbb{Q}G_\beta) = \{1\}$ and $T(\mathbb{Q}G_\beta) = A_\beta * B_\beta$. We shall do this by calculating the leading term of an arbitrary element of $T(\mathbb{Q}G_\beta)$.

We shall say that two $\mathbb{Q}G_\beta$-transvections t_1 and t_2 have *the same type* if $t_1, t_2 \in A_\beta$ or $t_1, t_2 \in B_\beta$. A product of $\mathbb{Q}G_\beta$-transvections $u = t_1 \cdots t_n$ shall be called *reduced* if no adjacent factors have the same type. It is easy to see that an arbitrary reduced product u of transvections is a sum of the identity and elements of the form

$$(3)\quad 0 \neq w = [(1-c)\nu_0\hat{c}]^{\varepsilon_1}\hat{c}\nu_1(1-c)^2\nu_2\hat{c}\cdots\hat{c}\nu_{2n-1}(1-c)^2\nu_{2n}\hat{c}[\hat{c}\nu_{2n+1}(1-c)]^{\varepsilon_2},$$

where $\nu_0, \nu_2, \ldots, \nu_{2n} \in C_\beta$, $\nu_1, \nu_3, \ldots, \nu_{2n+1} \in D_\beta$, and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.

We shall proceed by finding the leading term of $\nu_i(1-c)^2\nu_j$ where $i < j$, $i < 2n+1$, i is odd and j is even. Write an arbitrary element $g \in G$ as $g = g_1\omega(g)g_2$ where $g_1, g_2 \in \langle c\rangle$ and $\omega(g)$ does not begin or end in a nonidentity element of $\langle c\rangle$. Set $t_k = \max\{\omega(g) : g \in \operatorname{supp}(\nu_k)\}$, $0 \le k \le 2n+1$. If k is odd write $\nu_k = t_kx_k + r_k$, where $x_k \in \mathbb{Q}\langle c\rangle$ and for every $g \in \operatorname{supp}(r_k)$, $\omega(g) < t_k$. For an even k, $0 \le k \le 2n$ write

$$(4)\qquad \nu_k = x_k^{(1)}t_k^{(1)} + \sum_{s=2}^{m} x_k^{(s)}t_k^{(s)} + r'_k,$$

where $x_k^{(1)}, \ldots, x_k^{(m)} \in \mathbb{Q}\langle c\rangle$, $t_k^{(1)} = t_k$, $\ell(t_k^{(s)}) = \ell(t_k)$, $(2 \le s \le m)$ and $\ell\big(\omega(g)\big) < \ell(t_k)$ for every $g \in \operatorname{supp}(r'_k)$.

Fix an odd i, $1 \le i < 2n+1$, and an even j, $j \le 2n$ such that $i < j$. Let

$$c^l = \max\Big\{g : g \in \bigcup_{s=1}^{m} \operatorname{supp}(x_i(1-c)^2x_j^{(s)})\Big\}$$

and

$$f_j = \max\big\{t_j^{(s)} : c^l \in \operatorname{supp}\big(x_i(1-c)^2x_j^{(s)}\big)\big\}$$

where $t_j^{(s)}$ is defined in (4). We claim that $c^l \neq 1$ and that

$$(5)\qquad \operatorname{lead}[\nu_i(1-c)^2\nu_j] = t_ic^lf_j.$$

In particular,

$$(6)\qquad \ell(\operatorname{lead}[\nu_i(1-c)^2\nu_j]) = \ell(t_i) + 1 + \ell(t_j) \ge 3.$$

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Let ζ be a primitive $p = p_\beta$-th root of unity and $\pi\colon \mathbb{Q}\langle c\rangle \longrightarrow \mathbb{Q}(\zeta)$ be the map determined by $\pi(c) = \zeta$. It follows from the definitions of C_β and D_β that c^{p-1} does not belong to the supports of x_i and $x_j^{(s)}$, $(1 \le s \le m)$, hence $\pi(x_i)\pi(x_j^{(s)}) \ne 0$, $(1 \le s \le m)$ and therefore $\pi\big(x_i(1-c)^2x_j^{(s)}\big) \ne 0$ $(1 \le s \le m)$. Consequently $x_i(1-c)^2x_j^{(s)} \ne 0$, $(1 \le s \le m)$ and as $(1-c)^2$ is not a unit in $\mathbb{Q}\langle c\rangle$ we see that $x_i(1-c)^2x_j^{(s)} \notin \mathbb{Q}$. Thus $\operatorname{supp}\big(x_i(1-c)^2x_j^{(s)}\big)$ contains a nonidentity element of $\langle c\rangle$ for every s, $(1 \le s \le m)$. In particular, $c^l \ne 1$.

Let $g_1c^ag_2$ be an arbitrary element from $\operatorname{supp}(\nu_i(1-c)^2\nu_j)$, where $g_1 = \omega(h_1)$, $g_2 = \omega(h_2)$ for some $h_1 \in \operatorname{supp}(\nu_i)$ and $h_2 \in \operatorname{supp}(\nu_j)$. It follows from the definitions of t_i and f_j that $g_1 \le t_i$ and $\ell(g_2) \le \ell(f_j)$.

If $g_1 < t_i$ then clearly $g_1c^ag_2 < t_ic^lf_j$. So let $g_1 = t_i$. If $\ell(g_2) < \ell(f_j)$ then

$$\ell(g_1c^ag_2) \le \ell(g_1) + 1 + \ell(g_2) < \ell(t_i) + 1 + \ell(f_j) = \ell(t_ic^lf_j)$$

and therefore again

$$g_1c^ag_2 = t_ic^ag_2 < t_ic^lf_j.$$

Thus we may suppose that $\ell(g_2) = \ell(f_j)$. Then $g_2 = t_j^{(s)}$ for some s, $(1 \le s \le m)$ and consequently $c^a \in \operatorname{supp}\big(x_i(1-c)^2x_j^{(s)}\big)$. Thus $c^l \ge c^a$ and since $c^l > c^a$ implies

$$g_1c^ag_2 = t_ic^at_j^{(s)} < t_ic^lf_j$$

we may suppose that $a = l$. But then $c^l \in \operatorname{supp}\big(x_i(1-c)^2x_j^{(s)}\big)$ and by the definition of f_j we get that $f_j \ge g_2$. Finally, as $g_2 < f_j$ implies

$$g_1c^ag_2 = t_ic^lg_2 < t_ic^lf_j,$$

we conclude that $t_ic^lf_j$ is indeed the leading term of $\nu_i(1-c)^2\nu_j$, proving our claim.

Now we obtain from (3) that

$$\begin{aligned} \operatorname{lead}(w) = \big(\operatorname{lead}[(1-c)\nu_0]\big)^{\varepsilon_1}\big[c^{p-1}\operatorname{lead}[\nu_1(1-c)^2\nu_2]c^{p-1}\big]\cdots \\ \big[c^{p-1}\operatorname{lead}[\nu_{2n-1}(1-c)^2\nu_{2n}]c^{p-1}\big]\big(\operatorname{lead}[\nu_{2n+1}(1-c)]\big)^{\varepsilon_2}. \end{aligned} \tag{7}$$

Clearly this product is reduced if all the leading terms are given as reduced products. In particular, $\operatorname{lead}(w) \notin G_\beta$ and consequently, $T(\mathbb{Q}G_\beta) \cap U(\mathbb{Q}G_\beta) = \{1\}$.

Applying (1) and (2) to ν_i and ν_j respectively, and keeping in mind the observation made in the proof of Lemma 2.2, we obtain

$$\begin{aligned} \ell\big(\operatorname{lead}[(1-c)\nu_j]\big) = 1 + \ell\big(\omega(\operatorname{lead}\nu_j)\big) = 1 + \ell(t_j) \ge 2, \\ \ell\big(\operatorname{lead}[\nu_i(1-c)]\big) = 1 + \ell(t_i) \ge 2. \end{aligned} \tag{8}$$

Comparing (5) and (6) we see that

$$\ell\big(\operatorname{lead}[\nu_i(1-c)^2\nu_j]\big) \ge \max\big\{\ell\big(\operatorname{lead}[(1-c)\nu_j]\big), \ell\big(\operatorname{lead}[\nu_i(1-c)]\big)\big\}. \tag{9}$$

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Note that (8) holds for arbitrary even j, $(0 \le j \le 2n)$ and for arbitrary odd i, $(1 \le i \le 2n+1)$. Observe that (7), (8) and (5) imply that

$$\ell\big(\mathrm{lead}(w)\big) \ge 3 \tag{10}$$

for all w as in (3).

Now suppose that $1, 0 \ne w'$ is obtained from w by dropping some consecutive factors $\hat{c}\nu_i(1-c), (1-c)\nu_j\hat{c}$. Then we can write $w = w_1w_2w_3$, $w' = w_1w_3$ where w_2 has the form (3) with less ν_k's involved. We shall prove that

$$\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w')\big). \tag{11}$$

Suppose first that one of w_1 or w_3 is 1. It is enough to treat the case $w_1 = 1$, since the other one is similar. So let $w_1 = 1$; then $w = w_2w_3, w' = w_3$. If w_2 ends in $\hat{c}$ then w_3 begins with $\hat{c}$ and by (7)

$$\ell\big(\mathrm{lead}(w)\big) = \ell\big(\mathrm{lead}(w_2)\big) + \ell\big(\mathrm{lead}(w_3)\big) - 1.$$

It follows from (10) that $\ell\big(\mathrm{lead}(w_2)\big) \ge 3$ and therefore $\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w_3)\big)$. Let w_2 be ending in $1-c$. Then w_3 begins with $1-c$ and we can write $w_2 = w_2'\hat{c}\nu_k(1-c)$ and $w_3 = (1-c)\nu_{k+1}\hat{c}w_3'$. Call

$$\lambda_j = \begin{cases} \ell\big(\mathrm{lead}(w_j')\big) & \text{if } w_j' \ne 1,\\ 1 & \text{otherwise.}\end{cases}$$

It follows from (7) and (6) that

$$\ell\big(\mathrm{lead}(w)\big) = \lambda_2 + \ell(t_k) + \ell(t_{k+1}) + 1 + \lambda_3.$$

By (7) and (8) we have

$$\ell\big(\mathrm{lead}(w')\big) = \ell(t_{k+1}) + 1 + \lambda_3$$

Consequently, $\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w')\big)$.

Now suppose that $w_1 \ne 1$ and $w_3 \ne 1$. If w_2 begins with $\hat{c}$ then w_1 ends in $\hat{c}$, and therefore w_3 begins with $\hat{c}$ and w_2 ends in $\hat{c}$. By (7) we get

$$\ell\big(\mathrm{lead}(w)\big) = \ell\big(\mathrm{lead}(w_1)\big) - 1 + \ell\big(\mathrm{lead}(w_2)\big) - 1 + \ell\big(\mathrm{lead}(w_3)\big)$$
$$\ell\big(\mathrm{lead}(w')\big) = \ell\big(\mathrm{lead}(w_1)\big) + \ell\big(\mathrm{lead}(w_3)\big) - 1,$$

hence

$$\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w')\big).$$

If w_2 begins with $1-c$ then w_1 ends in $1-c$, w_3 begins with $1-c$ and w_2 ends in $1-c$. Write $w_1 = w_1'\hat{c}\nu_k(1-c)$, $w_2 = (1-c)\nu_{k+1}\hat{c}w_2'\hat{c}\nu_s(1-c)$, $w_3 = (1-c)\nu_{s+1}\hat{c}w_3'$. Applying (7) and (6) we obtain

$$\ell\big(\mathrm{lead}(w)\big) = \lambda_1 + \ell(t_k) + 1 + \ell(t_{k+1}) + \lambda_2 + \ell(t_s) + 1 + \ell(t_{s+1}) + \lambda_3,$$
$$\ell\big(\mathrm{lead}(w')\big) = \lambda_1 + \ell(t_k) + \ell(t_{s+1}) + 1 + \lambda_3,$$

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and clearly $\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w')\big)$ which completes the proof of (11).

Now let

$$u = \big(1 + (1-c)\nu_0\hat{c}\big)^{\varepsilon_1}\prod_{i=1}^{n}\Big[\big(1+\hat{c}\nu_{2i-1}(1-c)\big)\big(1+(1-c)\nu_{2i}\hat{c}\big)\Big]\big(1+\hat{c}\nu_{2n+1}(1-c)\big)^{\varepsilon_2}$$

be an arbitrary reduced product of transvections. Assume that ε_i, ν_i are as in (3). Then $u = w + \sum_{w'\in J} w' + 1$ where each $w' \in J$ is obtained from w by dropping some factors $\hat{c}\nu_i(1-c)$, $(1-c)\nu_j\hat{c}$.

Fix a $w' \in J$. Then there exists a sequence of elements $w' = w'_1, \dots, w'_s = w$ such that each w'_k, $(1 \le k \le s-1)$ is obtained from w'_{k+1} by dropping some consecutive factors $\hat{c}\nu_i(1-c)$, $(1-c)\nu_j\hat{c}$.

It follows from (11) that $\ell\big(\mathrm{lead}(w)\big) > \ell\big(\mathrm{lead}(w'_{s-1})\big) > \cdots > \ell\big(\mathrm{lead}(w')\big)$. Thus, $\mathrm{lead}(u) = \mathrm{lead}(w)$ and since $\ell\big(\mathrm{lead}(w)\big) \ge 3$, $u \ne 1$. We conclude that $T(\mathbb{Q}G_\beta)$ is the free product of A_α and B_β and as $\beta \in I$ is arbitrary, the theorem is proved. ∎

As a corollary we obtain a strong version of the Tits Alternative for $\mathcal{U}(\mathbb{Q}G)$.

COROLLARY 2.4. *Let G be as in Theorem 2.3. Then every subgroup of $\mathcal{U}(\mathbb{Q}G)$ either contains a free noncyclic subgroup or is solvable of derived length at most* 2.

PROOF. Let H be a subgroup of $\mathcal{U}(\mathbb{Q}G)$ which does not contain a noncyclic free subgroup. As

$$\mathcal{U}(\mathbb{Q}G) = *_{\mathcal{U}(\mathbb{Q})}\big(T(\mathbb{Q}G_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)\big), \quad (\alpha \in I)$$

and $\mathcal{U}(\mathbb{Q})$ is central in $\mathcal{U}(\mathbb{Q}G)$, applying the Kurosh Subgroup Theorem [8, p. 17] to the factor group $\mathcal{U}(\mathbb{Q}G)/\mathcal{U}(\mathbb{Q})$ we conclude that, modulo $\mathcal{U}(\mathbb{Q})$, H is either infinite cyclic, or a free product of two cyclic groups of order 2, or is conjugate to a subgroup of $T(\mathbb{Q}G_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)$ for some α. In the first case H is obviously abelian, and in the second it is metabelian. In the third case we may suppose that H is a subgroup of $T(\mathbb{Q}G_\alpha) \rtimes \mathcal{U}(\mathbb{Q}G_\alpha)$.

Now $T(\mathbb{Q}G_\alpha) = A_\alpha * B_\alpha$ where A_α and B_α are torsion-free abelian groups. Since

$$H \cap T(\mathbb{Q}G_\alpha) \subseteq A_\alpha * B_\alpha$$

applying again the Kurosh Subgroup Theorem we see that $H \cap T(\mathbb{Q}G_\alpha)$ is abelian. But $H/\big(H \cap T(\mathbb{Q}G_\alpha)\big)$ is isomorphic to a subgroup of $\mathcal{U}(\mathbb{Q}G_\alpha)$ and therefore is abelian. Hence H is either abelian or metabelian. ∎

3**. The Zassenhaus conjecture.** Let G be a group, $G(i) = \{g \in G : o(g) = i\}$, and C_g be the conjugacy class of $g \in G$. For $u = \sum_{g\in G} u(g)g \in \mathbb{Q}G$ set $T^{(i)}(u) = \sum_{g\in G(i)} u(g)$ and $\tilde{u}(g) = \sum_{h\in C_g} u(h)$. We recall a result on generalized traces $T^{(i)}$:

LEMMA 3.1 (SEE [1, LEMMA 2.4]). *Let G be a group and p a prime. If $u \in \mathcal{U}_1(\mathbb{Z}G)$ is a torsion unit of order p^n then $T^{(p^n)}(u) \equiv 1 \pmod p$ and $T^{(p^i)}(u) \equiv 0 \pmod p$ for all $i < n$.*

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LEMMA 3.2. *Suppose that $u, w \in \mathbb{Q}G$ and $x^{-1}ux = w$ where $x \in \mathcal{U}(\mathbb{Q}G)$. Then $\tilde{u}(g) = \tilde{w}(g)$ for all $g \in G$.*

PROOF. Let $[\mathbb{Q}G, \mathbb{Q}G]$ be the $\mathbb{Q}$-submodule of $\mathbb{Q}G$, generated by all $gh - hg$ $(g, h, \in G)$. Then $y = x^{-1}ux - u = x^{-1}(ux) - (ux)x^{-1} \in [\mathbb{Q}G, \mathbb{Q}G]$, and therefore $\tilde{y}(g) = 0$ for all $g \in G$. The result follows. ∎

The next result is an adaptation of [14, Lemma 37.13] to the case of an infinite group G.

LEMMA 3.3. *Let G be a group, $t = 1 + x\nu y$ where $x, \nu, y \in \mathbb{Q}G$, $yx = 0$ and let tw $\big(w \in \mathcal{U}(\mathbb{Q}G)\big)$ be a torsion unit of order n such that*

$$(1 + x\mathbb{Q}Gy) \cap \langle w \rangle = \{1\}.$$

If w commutes with x and y, then the element

$$z = 1 + t + tt^w + tt^w t^{w^2} + \cdots + tt^w \cdots t^{w^{n-2}},$$

where $t^{w^i} = w^i t w^{-i}$, is invertible in $\mathbb{Q}G$, and $z^{-1}twz = w$.

PROOF. Since w commutes with x and y we see that $t^{w^j} \in 1 + x\mathbb{Q}Gy$, for all j. Therefore we get from $(tw)^n = 1$ that

$$w^n = tt^w t^{w^2} \cdots t^{w^{n-1}} = 1. \tag{12}$$

We have that

$$\begin{aligned} z &= 1 + (1 + x\nu y) + (1 + x\nu y)(1 + x\nu^w y) + \cdots + (1 + x\nu y)(1 + x\nu^w y) \cdots (1 + x\nu^{w^{n-2}} y) \\ &= n + x\bar{\nu} y \end{aligned}$$

for some $\bar{\nu} \in \mathbb{Q}G$. Thus, $z^{-1} = \frac{1}{n}(1 - \frac{1}{n}x\bar{\nu}y) \in \mathbb{Q}G$.

Now by (12) we get

$$\begin{aligned} twz &= tz^w w = t(1 + t^w + t^w t^{w^2} + \cdots + t^w t^{w^2} \cdots t^{w^{n-1}})w \\ &= (t + tt^w + tt^w t^{w^2} + \cdots + tt^w \cdots t^{w^{n-1}})w = zw. \end{aligned}$$

Hence, $z^{-1}twz = w$ as desired. ∎

Let G be the free product $G = *G_\alpha$, $(\alpha \in I)$ of cyclic groups of prime order $|G_\alpha| = p_\alpha$ (the p_α's are not necessarily distinct and I may be infinite). For $\alpha \in I$ fix a generator $c = c_\alpha$ of G_α and set

$$w_\alpha = \begin{cases} \frac{2}{p}\hat{c} - c & \text{if } p > 2 \\ c & \text{otherwise,} \end{cases}$$

where $p = p_\alpha$ and $\hat{c} = 1 + c + \cdots + c^{p-1}$. It is easy to see that $w_\alpha^2 = c^2$.

THEOREM 3.4. *A finite subgroup of $\mathcal{U}_1(\mathbb{Q}G)$ is conjugate in $\mathcal{U}(\mathbb{Q}G)$ to a subgroup of $\langle w_\alpha \rangle$ for some $\alpha \in I$.*

PROOF. Let $H \neq \{1\}$ be a finite subgroup of $U_1(\mathbb{Q}G)$. By Theorem 2.3,

$$U(\mathbb{Q}G) = *_{U(\mathbb{Q})}\big(T(\mathbb{Q}G_\alpha) \rtimes U(\mathbb{Q}G_\alpha)\big), \quad (\alpha \in I),$$

and $T(\mathbb{Q}G_\alpha) = A_\alpha * B_\alpha$ where A_α and B_α are the torsion-free abelian groups defined in Section 2.

Applying the Kurosh Subgroup Theorem to the factor group $U(\mathbb{Q}G)/U(\mathbb{Q})$ [8] or a subgroup theorem for amalgamated free products we get that H is conjugate in $U(\mathbb{Q}G)$ (and therefore in $U_1(\mathbb{Q}G)$) to a subgroup of $T(\mathbb{Q}G_\alpha) \rtimes U(\mathbb{Q}G_\alpha)$ for some $\alpha \in I$). Thus, replacing H by its conjugate we may assume that

$$H \subseteq T(\mathbb{Q}G_\alpha) \rtimes U(\mathbb{Q}G_\alpha).$$

Since every element of $T(\mathbb{Q}G_\alpha)$ has augmentation 1, we really have

$$H \subseteq T(\mathbb{Q}G_\alpha) \rtimes U_1(\mathbb{Q}G_\alpha).$$

Let u be a nonidentity element of H. Then $u = tw$ where $t \in T(\mathbb{Q}G_\alpha)$ and $w \in U_1(\mathbb{Q}G_\alpha)$. Moreover, since $T(\mathbb{Q}G_\alpha)$ is torsion-free, w is a torsion unit of the same order as u.

Take a p-th primitive root of unity ζ and consider the isomorphism $\phi\colon \mathbb{Q}\langle c\rangle \longrightarrow \mathbb{Q}\oplus\mathbb{Q}(\zeta)$ defined by $\phi(c) = (1,\zeta)$ and linearly extended. Since the torsion units of $\mathbb{Q}(\zeta)$ are of the form: ζ^k or $-\zeta^k$, we see that $\phi(w) = (1,\zeta^k)$ or $\phi(w) = (1,-\zeta^k)$. If $p > 2$ then $\phi^{-1}(1,-\zeta) = \frac{2}{p}\hat{c} - c = w_\alpha$. Thus, in any case $w \in \langle w_\alpha\rangle$.

Let $v \neq 1$ be another element of H. Similarly we can write $v = fw'$ where $f \in T(\mathbb{Q}G_\alpha)$ and $1 \neq w' \in \langle w_\alpha\rangle$. Replacing u or v by an appropriate power of it we may suppose that $v = fw^{-1}$. Then

$$uv = twfw^{-1} \in T(\mathbb{Q}G_\alpha) \cap H.$$

As $T(\mathbb{Q}G_\alpha)$ is torsion-free $T(\mathbb{Q}G_\alpha) \cap H = \{1\}$ and hence $v = u^{-1}$. It follows that H is cyclic whose order divides $2p$, and we may suppose that $H = \langle u\rangle$.

We shall now show that u is conjugate to w in $U(\mathbb{Q}G)$ and this will complete the proof of the theorem.

Using the fact that $T(\mathbb{Q}G_\alpha) = A_\alpha * B_\alpha$, write t as a reduced product $t_1 \cdots t_n$ of elements from A_α and B_α. Since the order of $u = t_1 \cdots t_n w$ divides $2p$ we have $(t_1 \cdots t_n w)^{2p} = 1$ which implies that

$$t_1 \cdots t_n(wt_1 \cdots t_n w^{-1})(w^2 t_1 \cdots t_n w^{-2}) \cdots (w^{2p-1} t_1 \cdots t_n w^{-(2p-1)}) = 1.$$

Note that $wA_\alpha w^{-1} \subseteq A_\alpha$ and $wB_\alpha w^{-1} \subseteq B_\alpha$, because w commutes with c. As the product $t_1 \cdots t_n$ is reduced and $T(\mathbb{Q}G_\alpha) = A_\alpha * B_\alpha$, we have that n is odd and that

$$t_n w t_1 w^{-1} = t_{n-2} w t_2 w^{-1} = \cdots = t_{\frac{n+1}{2}+1} w t_{\frac{n+1}{2}-1} w^{-1} = 1.$$

Thus,

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$$u = (t_1 t_2 \cdots t_{\frac{n+1}{2}-1}) t_{\frac{n+1}{2}} w (t^{-1}_{\frac{n+1}{2}-1} \cdots t_2^{-1} t_1^{-1}),$$

and u is conjugate by transvections to $t_{(n+1)/2}w$. Since $T(\mathbb{Q}G_\alpha) \cap H = \{1\}$ it follows from Lemma 3.3 that $t_{(n+1)/2}w$ is conjugate in $U(\mathbb{Q}G)$ to w. ∎

The theorem implies the Zassenhaus Conjecture (ZC3) for G:

COROLLARY 3.5. *Let G be as in Theorem 3.4. Then every nonidentity finite subgroup of $U_1(\mathbb{Z}G)$ is conjugate in $U(\mathbb{Q}G)$ to one of the G_α, $(\alpha \in I)$.*

PROOF. Let $H \neq \{1\}$ be a finite subgroup of $U_1(\mathbb{Q}G)$. By Theorem 3.4 $x^{-1}Hx \subseteq \langle w_\alpha \rangle$, for some $x \in U(\mathbb{Q}G)$ and some $\alpha \in I$. Thus H is cyclic; its order divides $2p_\alpha$ if p_α is odd and is equal to 2 otherwise. Obviously, in the last case $x^{-1}Hx = G_\alpha$, so we may assume that $p_\alpha > 2$.

Suppose that H contains an element u of order 2 and set $w = x^{-1}ux$. It follows from Lemma 3.1 that there exists an element $g \in G$ of order 2 such that $\tilde{u}(g) \neq 0$. Hence by Lemma 3.2, $\tilde{w}(g) \neq 0$ which is impossible as $w \in \mathbb{Q}G_\alpha$, where G_α has order $p_\alpha > 2$. We conclude that the order of H is p_α and that $x^{-1}Hx = G_\alpha$. ∎

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REFERENCES

1. A. A. Bovdi, *The Multiplicative Group of an Integral Group Ring*. Uzhgorod, Dep. UkrNIINTI, 24.09.87, 2712-Uk87, 1987.

2. A. A. Bovdi, Z. Marciniak and S. K. Sehgal, *Torsion Units in Infinite Group Rings*. J. Number Theory **47**(1994), 284–299.

3. M. A. Dokuchaev, *Finite Subgroups of Units in Integral group Rings*. Resenhas IME-USP (3) **2**(1996), 293–302.

4. M. A. Dokuchaev and S. O. Juriaans, *Finite Subgroups in Integral Group Rings*. Canad. J. Math. (6) **48**(1996), 1170–1179.

5. M. A. Dokuchaev, S. O. Juriaans and C. Polcino Milies, *Integral Group Rings of Frobenius Groups and the Conjectures of H. J. Zassenhaus*. Comm. Algebra (7) **25**(1997), 2311–2325.

6. V. N. Gerasimov, *The group of units of a free product of rings*. Math. USSR-Sb. (1) **62**(1989), 41–63.

7. L. Klinger, *Construction of a counterexample to a conjecture of Zassenhaus*. Comm. Algebra **19**(1993), 2303–2330.

8. A. G. Kurosh, *The theory of groups, II*. Chelsea Publ. Co., New York, 1956.

9. F. Levin and S. K. Sehgal, *Zassenhaus conjecture for the infinite dihedral group*. Proc Amer. Math. Soc., Special Session, Series in Algebra **1**, World Scientific, Singapore, 1993, 57–68.

10. A. I. Lichtman and S. K. Sehgal, *The elements of finite order in the group of units of group rings of free products of groups*. Comm. Algebra **17**(1989), 2223–2253.

11. Z. S. Marciniak and S. K. Sehgal, *Zassenhaus Conjecture and Infinite Nilpotent Groups*. J. Algebra **184**(1996), 207–212.

12. C. Polcino Milies, J. Ritter and S. K. Sehgal, *On a conjecture of Zassenhaus on torsion units in integral group rings II*. Proc. Amer. Math. Soc. (2) **97**(1986), 206–210.

13. K. W. Roggenkamp and M. J. Taylor, *Group Rings and Class Groups*. Birkhäuser Verlag, Basel, 1992.

14. S. K. Sehgal, *Units of Integral Group Rings*. Longman's, Essex, 1993.

15. A. Valenti, *Torsion Units in Integral Group Rings*. Proc. Amer. Math. Soc. (1) **120**(1994), 1–4.

16. A. Weiss, *Rigidity of p-adic torsion*. Ann. Math. **127**(1988), 317–332.
17. ______, *Torsion units in integral group rings*. J. Reine Angew. Math. **415**(1991), 175–187.

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