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ON THE SYMMETRIC SQUARES OF COMPLEX AND QUATERNIONIC PROJECTIVE SPACE

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Abstract. The problem of computing the integral cohomology ring of the symmetric square of a topological space has long been of interest, but limited progress has been made on the general case until recently. We offer a solution for the complex and quaternionic projective spaces $\mathbb{K}P^n$, by utilising their rich geometrical structure. Our description involves generators and relations, and our methods entail ideas from the literature of quantum chemistry, theoretical physics, and combinatorics. We begin with the case $\mathbb{K}P^\infty$, and then identify the truncation required for passage to finite n. The calculations rely upon a ladder of long exact cohomology sequences, which compares cofibrations associated to the diagonals of the symmetric square and the corresponding Borel construction. These incorporate the one-point compactifications of classic configuration spaces of unordered pairs of points in $\mathbb{K}P^n$, which are identified as Thom spaces by combining Löwdin's symmetric orthogonalisation (and its quaternionic analogue) with a dash of *Pin* geometry. The relations in the ensuing cohomology rings are conveniently expressed using generalised Fibonacci polynomials. Our conclusions are compatible with those of Gugnin mod torsion and Nakaoka mod 2, and with homological results of Milgram.

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1. Introduction. The study of cyclic and symmetric powers has a long and varied history, and has remained active throughout the development of algebraic topology. At first, symmetric squares of smooth manifolds were associated mainly with critical point theory [**32**], but by the 1950s cyclic powers of simplicial complexes had come to underlie Steenrod's construction of mod p cohomology operations [**39**] and related work on the homology and cohomology of the symmetric groups [**35**]. These interactions were extended to stable splittings of classifying spaces during the 1980s [**31**]. More recently [**23**], symmetric powers of closed manifolds M have been viewed as compactifications of configuration spaces of finite sets of distinct points on M, and also as important examples of orbifolds [**1**]. Since 2000 or so, the latter perspective has gained popularity within theoretical physics [**9**], and led to the theory of *cyclic* and *permutation orbifolds*.

For any topological space X, its *cyclic* or *symmetric square* $SP^2(X)$ is the orbit space $(X \times X)/C_2$ under the action of the cyclic group C_2, generated by the involution ι that interchanges factors. Its elements are the unordered pairs $\{x, y\}$ for all $x,\ y$ in X, and the fixed points of ι determine the *diagonal* subspace $\Delta = \Delta(X)$, which contains all pairs $\{x, x\}$ and is homeomorphic to X. If X is a CW complex of finite type, then so is $SP^2(X)$, and this condition will hold for every space considered below. The quotient map $q\colon X \times X \to SP^2(X)$ is a *ramified covering* in the sense of [**38**], and its off-diagonal restriction

$$q'\colon (X \times X) \setminus \Delta \longrightarrow SP^2(X) \setminus \Delta$$

is a genuine double covering. By definition, the codomain $SP^2(X) \setminus \Delta$ is the *configuration space* $\mathcal{C}_2(X)$ of unordered pairs of distinct points of X.

In a small number of special cases, $SP^2(X)$ may be identified with some other familiar space. For example, there are homeomorphisms

$$SP^2(S^1) \;\cong\; M, \quad SP^2(\mathbb{R}P^2) \;\cong\; \mathbb{R}P^4 \quad \text{and} \quad SP^2(S^2) \;\cong\; \mathbb{C}P^2,$$

where M is the closed Möbius band. In these cases, the integral cohomology ring $H^*(SP^2(X)_+)$ follows immediately, but for arbitrary X its evaluation is more challenging, and few complete calculations appear in the literature.

Since work on this manuscript began, however, two independent advances have been made. First, Gugnin [**18**] computed the torsion-free quotient ring $H^*(SP^2(X))/\operatorname{Tor}$ for any CW complex of finite type, using the ring structure of $H^*(X)$ and the transfer homomorphism [**38**] associated to the projection q. Second, Totaro [**40**, Theorem 1.2] converted Milgram's description [**28**] of the integral *homology* groups $H_*(SP^2(X))$ into dualisable form, at least for finite connected CW complexes whose integral cohomology groups are torsion free and zero in odd dimensions (such an X may be called *even*). It will emerge below that our calculations are compatible with these developments, as well as with Nakaoka's classic papers [**33**] and [**34**], which focus mainly on $\mathbb{Z}/2$ coefficients.

The simplest examples are the spheres S^n, studied by Morse in [**32**, Chapter VI, Section 11], and in several papers by Nakaoka [**34**]. More generally, it is natural to consider CW complexes with richer intrinsic geometry, such as the complex and quaternionic projective spaces $\mathbb{K}P^n$. Our main aim is therefore to focus on the associated geometry of the spaces $SP^2(\mathbb{K}P^n)$, and, by way of application, to define generators and relations for their integral cohomology rings. As noted in [**6**], they differ from those obtained by following [**18**], and seem more suitable for extending to generalised cohomology theories, where analogues of the transfer may not exist for ramified coverings [**2**].

The results are stated in Theorem 8.9, and summarised in Theorem 1.1. The dimensions of the generators depend on the dimension d of $\mathbb{K}$ over $\mathbb{R}$; thus $d = 2$ for $\mathbb{K} = \mathbb{C}$ or 4 for $\mathbb{K} = \mathbb{H}$, and g, h, and $u_{i,j}$ have respective dimensions d, $2d$, and $2i + jd + 1$.

THEOREM 1.1. *For any $n \geq 1$, the ring $H^*(SP^2(\mathbb{K}P^n)_+)$ is isomorphic to*

$$\mathbb{Z}[h^p/2^{p-1},\; g^q h^s/2^s,\; u_{i,j}] \,/\, J_n\,,$$

where $p, q \geq 1$, $s \geq 0$ and $0 < i < jd/2$; the ideal J_n is given by

$$(2u_{i,j},\; u_{i,j}u_{k,l},\; u_{i,j}h^p/2^{p-1},\; u_{i,j}g^q h^s/2^s,\; r_t,\; u_{i,t} : t > n)\,,$$

for certain homogeneous polynomials $r_t = r_t(g, h)$ of dimension td.

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REMARKS 1.2. The torsion elements of this ring all have order 2, and the constraints on the integers i, j, and t ensure that they are finite in number; the highest dimensional example is $u_{(nd-2)/2,n}$, in dimension $2nd - 1$.

The torsion-free product structure is indicated by the notation for the generators; for example, $g \cdot h = 2 \cdot gh/2$ in dimension $3d$, for any $n \geq 2$. The ring is finitely generated because the polynomial relations ensure that the monomial $h^n/2^{n-1}$ is top dimensional, and generates a single $\mathbb{Z}$ in dimension $2nd$. The generators r_t of J_n are redundant for $t > n+2$, by Remarks 8.2.

The cases $n = 2$ and 3 are made explicit in Example A.1.

An alternative perspective on Theorem 1.1 is given by exploiting the canonical embedding of $SP^2(\mathbb{K}P^n)$ into the infinite symmetric product $SP^\infty(\mathbb{K}P^n)$, which is a product of Eilenberg–MacLane spaces [**14**]. However, the presence of torsion in their cohomology rings adds complications to the integral situation; and even with $\mathbb{Z}/2$ coefficients the resulting generators differ from ours. Such a viewpoint is taken in [**24**, Section 11], for example, and has the advantage of leading to a more systematic understanding of the action of the mod 2 Steenrod algebra than is provided by [**6**].

In principle, the results of [**18**] and [**40**] may be combined with those of [**34**] to show that Theorem 1.1 is in some sense generic, at least for even X. The reasons are purely algebraic and will be explored in [**8**], whose goal is to reconcile the algebra with our geometrical approach in certain less familiar cases.

A crucial supporting rôle is played by the homotopy theoretic analogue of $SP^2(X)$, namely the *Borel construction* $S^\infty \times_{C_2} (X \times X)$, where C_2 acts antipodally, and therefore freely, on the contractible sphere S^∞. It contains the diagonal subspace $\mathbb{R}P^\infty \times X$, sometimes written as $\widehat{\Delta} = \widehat{\Delta}(X)$ below. There is a *Borel bundle*

$$X \times X \xrightarrow{k} S^\infty \times_{C_2} (X \times X) \xrightarrow{r} \mathbb{R}P^\infty, \tag{1}$$

in which $\pi_1(\mathbb{R}P^\infty) \cong C_2$ acts on the fibre by ι. The integral cohomology ring $H^*(S^\infty \times_{C_2} (X \times X))$ may often be computed via the Leray–Serre spectral sequence of (1), as carried out for $X = \mathbb{K}P^n$ in Theorem 8.3.

There is also a canonical projection map

$$\pi \colon S^\infty \times_{C_2} (X \times X) \longrightarrow SP^2(X), \tag{2}$$

which identifies $\widehat{\Delta}$ with $\pi^{-1}(\Delta)$; its off-diagonal restriction

$$\pi' \colon S^\infty \times_{C_2} (X \times X) \setminus \widehat{\Delta} \longrightarrow SP^2(X) \setminus \Delta \tag{3}$$

is a fibration with contractible fibres, and hence a homotopy equivalence. The Vietoris–Begle theorem confirms that π induces an isomorphism of integral cohomology with $\mathbb{Z}[1/2]$ coefficients, so the main difficulties in proving Theorem 8.9 involve 2-torsion and 2-primary product structure. Our assumptions on X ensure that the map of quotients

$$\pi'' \colon S^\infty \times_{C_2} (X \times X)/\widehat{\Delta} \longrightarrow SP^2(X)/\Delta \tag{4}$$

is also a homotopy equivalence, allowing cohomological properties of the Borel construction to be related directly to those of $SP^2(X)$, following Bredon [**10**, Chapter VII].

For $X = \mathbb{K}P^n$, this relationship brings *Pin* geometry into play. Because $\mathbb{K}P^n$ is closed, compact and admits a suitable metric, Δ is a canonical deformation retract of a certain singular open neighbourhood [**25**, Corollary 4.2]. It turns out that $SP^2(\mathbb{K}P^n) \setminus \Delta$ contains an analogous closed, compact submanifold Γ_n, which is a deformation retract; it may be imagined as an *anti-diagonal*. The retraction is canonical because $SP^2(\mathbb{K}P^n) \setminus \Delta$ is diffeomorphic to the total space of a $Pin^{\ddagger}(d)$ vector bundle θ_n over Γ_n, where $\ddagger$ stands for c when $d = 2$, or $-$ when $d = 4$ [**19**, Section 2]. In terms of (4), $SP^2(\mathbb{K}P^n)/\Delta$ is therefore the Thom space of θ_n, whose cohomological structure helps to unlock the 2-primary information. The relationship between $Pin^-(4)$ and $Pin^+(4)$ has been clarified in the physics literature [**5**, Section 5.3], but continues to cause confusion elsewhere.

Our results are inspired by two sources. One is James, Thomas, Toda, and Whitehead [**22**], who describe $SP^2(S^n)/\Delta$ as a Thom space over $\mathbb{R}P^n$; our approach applies equally well to their situation, and recovers $H^*(SP^2(S^n))$ for $n \geq 1$. The other is Yasui [**41**], who introduces Stiefel manifolds into his determination of $H^*(\mathcal{C}_2(\mathbb{C}P^n))$, and builds on Feder's mod 2 calculations [**16**]. Recently, Dominguez, Gonzalez, and Landweber [**15**] have computed $H^*(\mathcal{C}_2(\mathbb{R}P^n))$, so the quaternionic case of Theorem 8.1 completes the trio.

The computations of [**15**] do not lead directly to $H^*(SP^2(\mathbb{R}P^n))$, because $\mathbb{R}P^n$ is neither simply connected nor even. Nevertheless, the dihedral group D_8 of [**15**, Definition 2.5] is the precise analogue of $Pin^{\ddagger}(d)$ for $d = 1$, and work is currently in progress (J. Gonzalez, Private communication) to extend our methods to the real case.

Since $H^*(\mathbb{K}P^n) \cong \mathbb{Z}[z]/(z^{n+1})$ is the truncation of $H^*(\mathbb{K}P^\infty)$, the possibility suggests itself of calculating $H^*(SP^2(\mathbb{K}P^\infty))$ and expressing $H^*(SP^2(\mathbb{K}P^n))$ as an appropriate quotient. This is indeed feasible, and occupies the second half of Theorem 8.9. The fact that colim Γ_n (denoted by Γ below) is a model for the classifying space $BPin^{\ddagger}(d)$ provides a convenient point of departure for Section 2.

There are seven subsequent sections, as follows:

Section 2: Classifying spaces and $Pin(d)$ bundles;
Section 3: Orthogonalisation and cofibre sequences;
Section 4: Characteristic classes;
Section 5: Mod 2 cohomology;
Section 6: Integral cohomology;
Section 7: Mod 2 truncation; and
Section 8: Integral truncation.

Appendix A makes the outcomes explicit in cases $n = 2$ and 3, and Appendix B records the first appearances of crucial notation.

The results for $\mathbb{H}P^n$ are taken from the first author's thesis [**6**], and were presented in August 2014 in Seoul. They were originally intended for application to quaternionic cobordism; that work remains in progress, together with extensions to other cohomology theories and higher cyclic powers.

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2. Classifying spaces and $Pin(d)$ **bundles.** This section establishes notation that allows the complex and quaternionic cases to be treated simultaneously, and collates background information on certain low dimensional compact Lie groups. The aim is to contextualise $Pin^c(2)$ and $Pin^{\pm}(4)$; the latter requires special care, having been the subject of several ambiguities in the literature. Background sources include [**36**] for quaternionic linear algebra, [**4**] for the accidental isomorphisms from a quaternionic viewpoint, and [**41**] for aspects of the complex case.

Henceforth, $\mathbb{K}$ denotes the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$, and d stands for their respective dimensions 2 or 4 over $\mathbb{R}$. Scalars act on the *right* of vector spaces over $\mathbb{K}$, unless otherwise stated; thus $GL(n, \mathbb{K})$ acts on the *left* of $\mathbb{K}^n$, whose elements are column vectors. By definition, the compact Lie group $O_{\mathbb{K}}(n) < GL(n, \mathbb{K})$ consists of all matrices Q that preserve the Hermitian inner product $u^\star v = \sum_{i=1}^n \bar{u}_i v_i$ on $\mathbb{K}^n$, where * denotes conjugate transpose. These are characterised by the property that $Q^{-1} = Q^\star$; thus $O_{\mathbb{K}}(n)$ is the unitary group $U(n)$ or the symplectic group $Sp(n)$ over $\mathbb{C}$ or $\mathbb{H}$, respectively.

When $n = 2$ there exist important *accidental isomorphisms*

$$U(2) \;\cong\; Spin^c(3) \qquad \text{and} \qquad Sp(2) \;\cong\; Spin(5). \tag{5}$$

They may be understood in terms of the real $(d+1)$-dimensional vector space

$$\mathfrak{H}_2^0(\mathbb{K}) \;=\; \left\{ \begin{pmatrix} r & k \\ \bar{k} & -r \end{pmatrix} : (r, k) \in \mathbb{R} \times \mathbb{K} \right\}$$

of 2×2 trace 0 Hermitian matrices Z, on which $O_{\mathbb{K}}(2)$ acts by

$$Q \cdot Z \;=\; QZQ^\star. \tag{6}$$

This defines the action of $Spin^c(3)$ on $\mathbb{R}^3$ or $Spin(5)$ on $\mathbb{R}^5$, although the former is often given by the equivalent action on *skew*-Hermitian matrices.

For any $n \geq 2$, let $V_{n+1,2}$ denote the Stiefel manifold of orthonormal 2-frames in $\mathbb{K}^{n+1}$; it is a closed compact manifold of dimension $(2n+1)d - 2$, whose elements are specified by $(n+1) \times 2$ matrices $(u\ v)$ over $\mathbb{K}$, with orthonormal columns. The group $O_{\mathbb{K}}(2)$ acts freely on the *right*, and has orbit space the Grassmannian $Gr_{n+1,2}$. For $n = \infty$, the colimit $V_2 := V_{\infty,2}$ is contractible, and $Gr_2 := Gr_{\infty,2}$ serves as a classifying space $BO_{\mathbb{K}}(2)$. In view of the accidental isomorphism (5), $BO_{\mathbb{K}}(2)$ is $BSpin^c(3)$ or $BSpin(5)$, and the associated real $(d+1)$-plane bundle χ is induced by the action (6). Of course $BO_{\mathbb{K}}(2)$ also admits the standard universal $\mathbb{K}^2$ bundle ω_2, which is written as ζ_2 over $BU(2)$ or ξ_2 over $BSp(2)$ in Section 4.

Note that $O_{\mathbb{K}}(1)$ is the unit sphere and multiplicative subgroup $S^{d-1} < \mathbb{K}^\times$, and $BO_{\mathbb{K}}(1)$ is the projective space $\mathbb{K}P^\infty$, with tautological line bundle $\omega = \omega_1$; the latter is written as ζ over $\mathbb{C}P^\infty$ or ξ over $\mathbb{H}P^\infty$. Thus, $S^{d-1} \times S^{d-1} < O_{\mathbb{K}}(2)$ is the subgroup of diagonal matrices, and isomorphic to $Spin^c(2)$ or $Spin(4)$. The points of $\mathbb{K}P^\infty$ are the 1-dimensional subspaces $[u]$ of $\mathbb{K}^\infty$, each of which is an equivalence class of unit vectors u.

DEFINITIONS 2.1. The subgroup $P^d < O_{\mathbb{K}}(2)$ consists of all matrices

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} : a, b \in S^{d-1} \right\};$$

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the subgroups $F^d := C_2 \times S^{d-1}$ and $S^{d-1} \times S^{d-1} < P^d$ consist of matrices

$$\left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in S^{d-1}\right\} \quad \text{and} \quad \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in S^{d-1}\right\}$$

respectively, where C_2 is generated by $\tau := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

REMARKS 2.2. The group P^d is the wreath product $S^{d-1} \wr C_2$, and the normalizer of $S^{d-1} \times S^{d-1}$ in $O_{\mathbb{K}}(2)$; so it is isomorphic to the compact Lie group $Pin^{\ddagger}(d)$, where $\ddagger$ stands for c when $d = 2$, or $-$ when $d = 4$ [**19**, Section 2]. This motivates the notation, and holds because $Pin^{\ddagger}(d)$ is the normalizer of $Spin^{\ddagger}(d)$ in $Spin^{\ddagger}(d+1)$, and $Spin^- = Spin$. The quotient epimorphism $P^d \to C_2$ is the composition of the determinant map with the double covering $Pin^{\ddagger}(d) \to O(d)$.

The left action (6) of P^d on $\mathbb{R} \times \mathbb{K}$ is made explicit by

$$\tau \cdot (r, k) \;=\; (-r, \bar{k}) \qquad \text{and} \qquad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot (r, k) \;=\; (r, ak\bar{b})\,. \tag{7}$$

This splits as the product of the actions on $\mathbb{R}$ by det, and on $\mathbb{R}^d \cong \mathbb{K}$ by

$$\tau \cdot k \;=\; \bar{k} \qquad \text{and} \qquad \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot k \;=\; ak\bar{b}\,. \tag{8}$$

PROPOSITION 2.3. *There are homeomorphisms of left coset spaces*

$$(1)\ O_{\mathbb{K}}(2)/(S^{d-1} \times S^{d-1}) \;\cong\; S^d, \quad (2)\ O_{\mathbb{K}}(2)/P^d \;\cong\; \mathbb{R}P^d,$$
$$(3)\ P^d/F^d \;\cong\; S^{d-1} \quad \textit{and} \quad (4)\ P^d/(S^{d-1} \times S^{d-1}) \;\cong\; C_2.$$

Proof. For (1), observe that the left action (6) of $O_{\mathbb{K}}(2)$ on $S^d \subset \mathbb{R} \times \mathbb{K}$ is transitive, and $(1, 0)$ has isotropy subgroup $S^{d-1} \times S^{d-1}$. The induced action of $O_{\mathbb{K}}(2)$ on $\mathbb{R}P^d$ is also transitive, and (2) holds because $\tau \cdot (1, 0) = (-1, 0)$, so $[1, 0]$ has isotropy subgroup P^d. For (3), note that the left action (8) of P^d on $S^{d-1} \subset \mathbb{K}$ is transitive, and 1 has isotropy subgroup F^d. Finally, (4) is the isomorphism of topological groups induced by the quotient epimorphism. □

Any closed subgroup H of a compact Lie group G gives rise to a bundle

$$G/H \xrightarrow{i} BH \xrightarrow{p} BG,$$

in which p is modelled by the natural projection $EG/H \to EG/G$; see [**29**, Chapter II], for example. So Proposition 2.3 has the following consequences.

COROLLARY 2.4. *There exist bundles*

$$(1)\ S^d \longrightarrow B(S^{d-1} \times S^{d-1}) \xrightarrow{p_1} BO_{\mathbb{K}}(2), \quad (2)\ \mathbb{R}P^d \xrightarrow{i_2} BP^d \xrightarrow{p_2} BO_{\mathbb{K}}(2),$$
$$(3)\ S^{d-1} \longrightarrow BF^d \xrightarrow{p_3} BP^d, \quad (4)\ C_2 \longrightarrow B(S^{d-1} \times S^{d-1}) \xrightarrow{p_4} BP^d,$$

where (1) *is the sphere bundle of* χ, (2) *is the projectivisation of* χ, (3) *is the sphere bundle of the universal* $Pin^{\ddagger}(d)$ *vector bundle* θ, *and* (4) *is the double covering associated to the determinant line bundle* $\det(\theta)$.

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Proof. The homeomorphisms of Proposition 2.3 associate the left actions of $O_{\mathbb{K}}(2)$ on the coset spaces (1) and (2) to their left actions on S^d and $\mathbb{R}P^d$, as given by (6); and the left actions of P^d on the coset spaces (3) and (4) to their left actions on S^{d-1} and C_2, as given by Remarks 2.2 and (8). □

REMARKS 2.5. The proof of Corollary 2.4 uses the orbit space

$$\Gamma \ := \ V_2/P^d \ = \ \{[u],[v] \,:\, u,v \in S^\infty,\ u^\star v = 0\}$$

as a model for BP^d. Its elements are unordered pairs of orthogonal lines in $\mathbb{K}^\infty$, so Γ may be interpreted as a subspace of $SP^2(\mathbb{K}P^\infty)$. The corresponding models for $B(S^{d-1}\times S^{d-1})$ and BF^d are $V_2/(S^{d-1}\times S^{d-1})$ and V_2/F^d, respectively.

REMARKS 2.6. The universal $Pin^\ddagger(d)$ structure on θ corresponds to the induced $Spin^\ddagger(d+1)$ structure on $\det(\theta)\oplus\theta$, via the isomorphism

$$p_2^*(\chi) \ \cong \ \det(\theta)\oplus\theta \tag{9}$$

associated to the splitting (8) of (7). In the quaternionic case, this exemplifies [**26**, Lemma 1.7]; moreover, $Pin^-(4)$ and $Pin^+(4)$ are isomorphic [**5**, Section 5.3], so θ must not be confused with the universal $Pin^+(4)$ bundle over Γ. The latter involves modifying (8) by $\tau\cdot k = -\bar{k}$, for any k in $\mathbb{H}$ [**6**].

In either case, the bundle of Corollary 2.4(2) arises from the bundle (1) by factoring out the action of τ, so the double covering (4) is also the S^0-bundle of the tautological real line bundle λ over $\mathbb{R}P(\chi)$. Thus, (9) coincides with the standard splitting of $p_2^*(\chi)$ as $\lambda\oplus\lambda^\perp$.

The universal $Pin^\ddagger(d)$ disc bundle may be displayed as the diagram

$$S(\theta) \xrightarrow{\subset} D(\theta) \underset{\supset}{\overset{e}{\rightleftarrows}} \Gamma, \tag{10}$$

where e is the projection and the zero-section its left inverse. By (8) this is homeomorphic to the diagram

$$V_2\times_{P^d} S^{d-1} \xrightarrow{\subset} V_2\times_{P^d} D^d \underset{\supset}{\overset{e}{\rightleftarrows}} V_2/P^d, \tag{11}$$

where P^d acts on the unit disc $D^d\subset\mathbb{K}$ by $\tau\cdot q=\bar{q}$ and $\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)\cdot q = aq\bar{b}$. Then Proposition 2.3(3) identifies $S(\theta)$ as a model for BF^d.

The *open* disc bundle associated to θ has fibre the open unit disc $D_o^d\subset\mathbb{K}$, and total space $D_o(\theta) = V_2\times_{P^d} D_o^d$.

DEFINITIONS 2.7. For any $n\geq 1$, the smooth manifold $\Gamma_n\subset\Gamma$ is the orbit space $V_{n+1,2}/P^d$, of dimension $(2n-1)d$. Its elements are unordered pairs of orthogonal lines in $\mathbb{K}^{n+1}$, so Γ_n may be interpreted as a subspace of $SP^2(\mathbb{K}P^n)$; it also admits the $Pin^\ddagger(d)$ bundle θ_n, obtained by restricting θ.

By construction, the inclusion $\Gamma\subset SP^2(\mathbb{K}P^\infty)$ is the colimit of the inclusions $\Gamma_n\subset SP^2(\mathbb{K}P^n)$, and (10) is the colimit of the disc bundle diagrams

$$S(\theta_n) \xrightarrow{\subset} D(\theta_n) \underset{\supset}{\overset{e_n}{\rightleftarrows}} \Gamma_n\,.$$

Similarly, $D_o(\theta)$ is the colimit of the total spaces $D_o(\theta_n)$.

3. Orthogonalisation and cofibre sequences. Inspiration for this section is provided by [22], where the cofibre sequence

$$S^n \xrightarrow{i_\Delta} SP^2(S^n) \xrightarrow{b_\Delta} Th(\eta_n^\perp) \longrightarrow \dots \tag{12}$$

is introduced; i_Δ denotes the inclusion of the diagonal S^n, and $Th(\eta_n^\perp)$ is the Thom space of the complement of the tautological line bundle η_n over $\mathbb{R}P^n$, for $n \geq 1$. Here, an analogue of (12) is developed for $SP^2(\mathbb{K}P^n)$, and $Th(\theta_n)$ is identified with the 1-point compactification of the configuration space $\mathcal{C}_2(\mathbb{K}P^n)$.

Henceforth, it is convenient to denote $SP^2(\mathbb{K}P^n)$ by SP^2_n. Following Section 2, $\mathbb{K}P^n$ is taken to be the quotient space $S^{(n+1)d-1}/S^{d-1}$; so for any element $\{[w],[x]\}$ of SP^2_n, the real number $|w^\star x|$ is well-defined and varies continuously between 0 and 1. As referenced in Section 1, the diagonal $\mathbb{K}P^n = \Delta_n \subset SP^2_n$ has an open neighbourhood associated to the tangent bundle $\tau_{\mathbb{K}P^n}$, with fibre the open cone on $\mathbb{R}P^{nd-1}$. Our immediate aim is to describe a complementary relationship, between an open neighbourhood of Γ_n and the bundle θ_n.

A brief diversion is required before proceeding to Theorem 3.2.

Definition 3.1. For any $n \geq 1$, the *non-compact* Stiefel manifold

$$\widetilde{V}_{n+1,2} := \left\{(w\ x) : \|w\| = \|x\| = 1,\ |w^\star x| < 1\right\}$$

consists of all normalised 2-frames over $\mathbb{K}$, topologised as a subspace of $\mathbb{K}^{2(n+1)}$; it contains $V_{n+1,2}$ as a natural subspace.

The right action of P^d on $V_{n+1,2}$ clearly extends to $\widetilde{V}_{n+1,2}$ by

$$(w\ x)\cdot\tau \ =\ (x\ w) \qquad \text{and} \qquad (w\ x)\cdot\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix} \ =\ (wa\ xb), \tag{13}$$

and the orbit space $\widetilde{V}_{n+1,2}/P^d$ may be identified with the subspace $SP^2_n \setminus \Delta_n$ of SP^2_n under the homeomorphism that maps each orbit $(w\ x)P^d$ to $\{[w],[x]\}$. On $V_{n+1,2}$, this restricts to the inclusion $\Gamma_n \subset SP^2_n$ of Definition 2.7.

Theorem 3.2. *For any $n \geq 1$ (including ∞), there is a homeomorphism*

$$m\colon (D_o(\theta_n), \Gamma_n) \longrightarrow (SP^2_n \setminus \Delta_n, \Gamma_n)$$

of pairs, which induces a deformation retraction of $SP^2_n \setminus \Delta_n$ onto Γ_n.

Proof. For any h in $D^d_o \subset \mathbb{K}$, define the real number R and the positive definite Hermitian matrix M_h by

$$R := (1+|h|)^{1/2} + (1-|h|)^{1/2} \qquad \text{and} \qquad M_h \ :=\ \begin{pmatrix} R^2/2 & h \\ \bar{h} & R^2/2\end{pmatrix}R^{-1}.$$

Straightforward manipulation shows that

$$M_h^2 = \begin{pmatrix} 1 & h \\ \bar{h} & 1\end{pmatrix} \quad \text{and} \quad M_h^{-1} = \begin{pmatrix} R^2/2 & -h \\ -\bar{h} & R^2/2\end{pmatrix}R^{-1}\left(1-|h|^2\right)^{-1/2}. \tag{14}$$

So $(u_h\ v_h) := (u\ v)M_h$ lies in $\widetilde{V}_{n+1,2}$ for any $(u\ v)$ in $V_{n+1,2}$, because $u_h^\star v_h = h$.

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The formula $f'((w\ x), h) := (w_h\ x_h)$ determines a function

$$f' \colon V_{n+1,2} \times D_o^d \longrightarrow \widetilde{V}_{n+1,2},$$

which is continuous because M_h varies continuously with h. The equations

$$M_h \tau \;=\; \tau M_{\bar{h}} \qquad \text{and} \qquad M_h \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;=\; \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} M_{\bar{a}hb}$$

hold for all a, b in S^{d-1}, so f' is equivariant with respect to the free P^d-actions of (11) on $V_{n+1,2} \times D_o^d$ and (13) on $\widetilde{V}_{n+1,2}$. Moreover, $f''(w\ x) := ((w\ x)M_k^{-1}; k)$ defines an equivariant inverse, where $k := w^\star x$ satisfies $|k| < 1$. So f' induces a homeomorphism $f \colon D_o(\theta_n) \to SP_n^2 \setminus \Delta_n$ of P^d-orbit spaces. It acts as the identity on the subspace of elements $((w\ x), 0)$, which descends to the zero section Γ_n of $D_o(\theta_n)$.

Since $|u_h^\star v_h| = t$ if and only if $|h| = t$, the required retraction is defined by $l_t := f \circ t_\bullet \circ f^{-1}$, where $t_\bullet$ denotes fibrewise multiplication by t in $D_o(\theta_n)$ for $0 \leq t \leq 1$; in particular, $l_0 = f \circ e_n \circ f^{-1}$ since $0_\bullet$ is projection.

For both statements, the case $n = \infty$ is obtained by taking colimits. □

Remark 3.3. The homeomorphism f'' exemplifies Löwdin's *symmetric orthogonalisation* procedure [**27**], which originally arose in the literature of quantum chemistry, albeit over $\mathbb{C}$. There seem to be no explicit references over $\mathbb{H}$, but the proof remains valid because quaternionic matrices have polar forms [**42**]. For any normalised 2-frame $(w\ x)$, where $w^\star x = k$ and $|k| < 1$, the 2-frame

$$(w^k\ x^k) \;:=\; (w\ x)M_k^{-1}$$

is orthonormal, and its construction is invariant with respect to interchanging vectors in each frame. The procedure works because M_k is the unique positive definite square root of $(w\ x)^\star(w\ x)$, as confirmed by (14).

Corollary 3.4. *For any $n \geq 1$, the configuration space $C_2(\mathbb{K}P^n)$ is homeomorphic to $D_o(\theta_n)$, and homotopy equivalent to Γ_n.* □

Since the quotient space SP_n^2/Δ_n is homeomorphic to the one-point compactification of $C_2(\mathbb{K}P^n)$, Corollary 3.4 shows that both may be identified with the Thom space $Th(\theta_n)$, for any $n \geq 1$.

Now recall the canonical projection $\pi \colon B_n \to SP_n^2$ of (2), where B_n denotes the Borel construction $S^\infty \times_{C_2} (\mathbb{K}P^n \times \mathbb{K}P^n)$. Following (3), its restriction to the diagonal $\widehat{\Delta}_n$ is the projection $\pi_2 \colon \mathbb{R}P^\infty \times \mathbb{K}P^n \to \mathbb{K}P^n$ onto the second factor. For convenience, $\mathbb{R}P^\infty \times \mathbb{K}P^n$ will usually be abbreviated to RK^n.

Proposition 3.5. *For every $n \geq 1$, the map π induces a commutative ladder*

$$\begin{array}{ccccccc} RK^n & \xrightarrow{i_n} & B_n & \xrightarrow{b_n} & Th(\theta_n) & \longrightarrow & \dots \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle id} & & \\ \mathbb{K}P^n & \xrightarrow{i_\Delta} & SP_n^2 & \xrightarrow{b_\Delta} & Th(\theta_n) & \longrightarrow & \dots \end{array} \tag{15}$$

of homotopy cofibre sequences.

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Proof. Both i_n and i_Δ are CW inclusions, and π is an off-diagonal homotopy equivalence, as noted in (3). It therefore induces a map

$$\begin{tikzcd} RK^n \arrow[r, "i_n"] \arrow[d, "\pi_2"'] & B_n \arrow[r, "q_n"] \arrow[d, "\pi"] & B_n/\widehat{\Delta}_n \arrow[r] \arrow[d, "\pi''"] & \dots \\ \mathbb{K}P^n \arrow[r, "i_\Delta"] & SP^2_n \arrow[r, "q_\Delta"] & SP^2_n/\Delta_n \arrow[r] & \dots \end{tikzcd} \tag{16}$$

of cofibre sequences, where π'' is a homotopy equivalence. To complete the proof, define b_n to be the composition $\pi'' \circ q_n$. □

REMARK 3.6. The lower row of (15) is our promised analogue of (12).

The corresponding ladder of colimits is given by

$$\begin{tikzcd} RK^\infty \arrow[r, "i"] \arrow[d, "\pi_2"'] & B \arrow[r, "b"] \arrow[d, "\pi"] & MP^d \arrow[r] \arrow[d, "id"] & \dots \\ \mathbb{K}P^\infty \arrow[r, "i_\Delta"] & SP^2 \arrow[r, "b_\Delta"] & MP^d \arrow[r] & \dots \end{tikzcd} , \tag{17}$$

where MP^d (more commonly written as $MPin^\ddagger(d)$) denotes the Thom space of the $Pin^\ddagger(d)$ bundle θ. Ladder (17) also commutes, and its rows are cofibre sequences; it is the primary input for the cohomology calculations of the remaining sections. If so preferred, the crucial properties of (16) may be deduced directly from Bredon's results [**10**, Chapter VII], using Čech cohomology.

The upper row of (17) may be replaced by the homotopy cofibre sequence

$$BF^d \xrightarrow{p_3} BP^d \xrightarrow{j} MP^d \longrightarrow \dots, \tag{18}$$

where MP^d is homeomorphic to the mapping cone of p_3, and j denotes the inclusion of the zero section. The resulting ladder is homotopy commutative, because there are homotopy equivalences h_1 and h_2 for which the square

$$\begin{tikzcd} BF^d \arrow[r, "p_3"] \arrow[d, "h_1"'] & BP^d \arrow[d, "h_2"] \\ RK^\infty \arrow[r, "i"] & B \end{tikzcd} \tag{19}$$

is homotopy commutative. The existence of h_1 and h_2 involves standard manipulations with models for classifying spaces, following [**30**], for example. Similar arguments with h_2 lead to a homotopy equivalence between fibrations

$$B(S^3 \times S^3) \xrightarrow{p_4} BP^d \longrightarrow \mathbb{R}P^\infty \quad \text{and} \quad \mathbb{K}P^\infty \times \mathbb{K}P^\infty \xrightarrow{k} B \longrightarrow \mathbb{R}P^\infty ; \tag{20}$$

the former arises by classifying Corollary 2.4(4) and the latter is the Borel bundle (1).

4. Characteristic classes. In this section, characteristic classes are determined for various of the vector bundles introduced above. The results are expressed in the notation of Section 2, and play an important part in the final calculations.

The integral cohomology rings

$$H^*(BO_{\mathbb{K}}(1)_+) \;\cong\; \mathbb{Z}[z] \qquad \text{and} \qquad H^*(BO_{\mathbb{K}}(2)_+) \;\cong\; \mathbb{Z}[l_1, l_2] \tag{21}$$

are standard, as are the properties of their generators. In particular, z has dimension d, and is the 1st Chern class $c_1(\zeta)$ or the 1st symplectic Pontryagin class $p_1(\xi)$; similarly, l_1 and l_2 have dimensions d and $2d$, and are the 1st and 2nd Chern classes of ζ_2 or the 1st and 2nd symplectic Pontryagin classes of ξ_2, respectively. Tensoring with $\mathbb{Z}/2$ yields the corresponding rings with mod 2 coefficients, so z, l_1, and l_2 may be confused with their mod 2 reductions by allowing the context to distinguish between them. With this convention, the non-zero Stiefel–Whitney classes of the stated bundles are given by

$$w_d(\omega) \;=\; z, \quad w_d(\omega_2) \;=\; l_1, \quad w_{2d}(\omega_2) \;=\; l_2 \quad \text{and} \quad w_d(\chi) = l_1$$

in $H^*(BO_{\mathbb{K}}(1); \mathbb{Z}/2)$ and $H^*(BO_{\mathbb{K}}(2); \mathbb{Z}/2)$ respectively.

Our calculations also involve $\mathbb{R}P^\infty$, and its tautological real line bundle η. The integral and mod 2 cohomology rings are given by

$$H^*(\mathbb{R}P^\infty_+) \;\cong\; \mathbb{Z}[c]/(2c) \qquad \text{and} \qquad H^*(\mathbb{R}P^\infty_+; \mathbb{Z}/2) \;\cong\; \mathbb{Z}/2[a], \tag{22}$$

where $c = c_1(\eta_{\mathbb{C}})$ has dimension 2 and $a = w_1(\eta)$ has dimension 1. In this case, reduction mod 2 satisfies $\rho(c) = a^2$. Because $H^*(\mathbb{K}P^\infty)$ is torsion free, it follows from the Künneth formula that the integral and mod 2 cohomology rings of RK^∞ are given by

$$H^*(RK^\infty_+) \;\cong\; \mathbb{Z}[c, z]/(2c) \qquad \text{and} \qquad H^*(RK^\infty_+; \mathbb{Z}/2) \;\cong\; \mathbb{Z}/2[a, z], \tag{23}$$

where a, c, and z are pullbacks of their namesakes along the projections.

The cohomology rings of $\mathbb{K}P^n$, $Gr_{n+1,2}$, $\mathbb{R}P^n$, and $\mathbb{R}P^\infty \times \mathbb{K}P^n$ are obtained from (21)–(23) by appropriate truncation.

Remarks 2.2 identify the epimorphism $P^d \to C_2$ with det, and Remarks 2.6 show that the line bundle λ is classified by the induced map

$$B\mathrm{det}\colon BP^d \longrightarrow \mathbb{R}P^\infty,$$

which will also be denoted by λ. In particular, $w_1(\lambda)$ is given by $\lambda^*(a)$ in $H^1(BP^d; \mathbb{Z}/2)$, and $c_1(\lambda_{\mathbb{C}})$ by $\lambda^*(c)$ in $H^2(BP^d)$. By definition, λ restricts to η_d over the fibre $\mathbb{R}P^d$ of Corollary 2.4(2), so $i_2^*(w_1(\lambda)) = a$ in $H^1(\mathbb{R}P^d; \mathbb{Z}/2)$ and $i_2^*(c_1(\lambda_{\mathbb{C}})) = c$ in $H^2(\mathbb{R}P^d)$. Henceforth, $w_1(\lambda)$ and $c_1(\lambda_{\mathbb{C}})$ are written as a and c, respectively.

REMARK 4.1. The epimorphism det restricts to the identity on the subgroup $C_2 < P^d$, so $H^*(\mathbb{R}P^\infty; R)$ is a summand of $H^*(BP^d; R)$ for coefficients $R = \mathbb{Z}$ or $\mathbb{Z}/2$. All powers of the elements a and c are therefore non-zero.

Now consider the pullback of ω_2 along the projection $p_2\colon BP^d \to BO_{\mathbb{K}}(2)$ of Corollary 2.4(2), and its characteristic classes $x := p_2^*(l_1)$ and $y := p_2^*(l_2)$. Equating the total Stiefel–Whitney classes of (9) gives

$$1 + x \;=\; (1 + a)\, w(\theta).$$

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in $H^*(BP^d;\mathbb{Z}/2)$. Since $w_i(\theta)=0$ for $i>d$, it follows that

$$w(\theta) \;=\; 1+a+\cdots+a^d+x, \tag{24}$$

and hence that $a^{d+j}=a^jx$ in $H^{d+j}(BP^d;\mathbb{Z}/2)$ for every $j\geq 1$.

REMARKS 4.2. These formulae highlight the importance of the integral cohomology class $m:=c^{d/2}+x$, whose mod 2 reduction a^d+x will also be written as m. Then equation (24) confirms that $w_2(\theta)=m$ is the reduction of an integral class when $d=2$, and that $w_2(\theta)=w_1^2(\theta)$ (because both are equal to a^2) when $d=4$. These are defining properties for $Pin^{\ddagger}(d)$-bundles [**19**, Section 2].

The map p_2 imposes an $H^*(BO_{\mathbb{K}}(2)_+;R)$-algebra structure on $H^*(BP^d_+;R)$. For $R=\mathbb{Z}/2$, the Leray–Hirsch theorem proves that p_2^* injects $\mathbb{Z}/2[l_1,l_2]$ as a direct summand [**20**], and that $H^*(BP^d_+;\mathbb{Z}/2)$ has basis $1, a, \ldots, a^d$ thereover. Combined with (24), this identifies $H^*(BP^d_+;\mathbb{Z}/2)$ as the $\mathbb{Z}/2$ algebra

$$G^* \;:=\; \mathbb{Z}/2[a,m,y]/(am). \tag{25}$$

The integral characteristic classes of $Pin^{\ddagger}(d)$-bundles are equally important.

THEOREM 4.3. *The integral cohomology ring $H^*(BP^d_+)$ is isomorphic to*

$$Z^* \;:=\; \mathbb{Z}[c,m,y]/(2c,cm).$$

Proof. Consider the Leray–Serre spectral sequence

$$E_2^{p,q} \;:=\; H^p(BO_{\mathbb{K}}(2)_+;H^q(\mathbb{R}P^d_+)) \;\Longrightarrow\; H^{p+q}(BP^d_+)$$

of the fibration p_2, noting that $BO_{\mathbb{K}}(2)$ is simply connected. Since $H^*(BO_{\mathbb{K}}(2))$ is free abelian and even dimensional, there are isomorphisms

$$E_\infty^{*,*} \;\cong\; E_2^{*,*} \;\cong\; H^*(BO_{\mathbb{K}}(2)_+)\otimes H^*(\mathbb{R}P^d_+);$$

thus p_2^* is monic, and $i_2^*(c)=c$ in $H^2(\mathbb{R}P^d_+)$. So any additive generator $x^iy^j\otimes c^k$ of $E_\infty^{*,*}$ is represented by $x^iy^jc^k$ in $H^*(BP^d_+)$, and there is an isomorphism $E_\infty^{*,*}\cong H^*(BP^d_+)$ of $H^*(BO_{\mathbb{K}}(2)_+)_+$-modules, on generators $1, c, \ldots, c^{d/2}$.

The multiplicative structure is described by the single relation $c^{d/2+1}=cx$, or $cm=0$, which follows from Remark 4.1 and the fact that $cx\neq 0$. □

REMARKS 4.4. The homotopy commutative square (19) may be used to transport the cohomology classes $c,\ z$, and a to $H^*(BF^d)$, and $c,\ a,\ x$, and y to $H^*(B)$ (supressing h_1^* and h_2^* from the notation). The homotopy equivalence of (20) ensures that x and y are then characterised by the facts that they satisfy

$$k^*(x)=z_1+z_2 \quad\text{and}\quad k^*(y)=z_1z_2$$

in $H^*(\mathbb{K}P^\infty\times\mathbb{K}P^\infty)$ and are pullbacks from $H^*(BO_{\mathbb{K}}(2))$. These conventions lead to isomorphisms $H^*(BF^d_+)\cong\mathbb{Z}[c,z]/(2c)$ and $H^*(B_+)\cong Z^*$.

Similarly, bundles η and ω are defined over BF^d as pullbacks from RK^∞.

Now recall Corollary 2.4 and consider the bundle

$$O_{\mathbb{K}}(2)/F^d \longrightarrow BF^d \xrightarrow{p_5} BO_{\mathbb{K}}(2),$$

whose projection p_5 factorises as $p_2 \circ p_3$. The pullback $\gamma := p_5^*(\omega_2)$ is a complex or quaternionic 2-plane bundle, and is induced by a representation of F^d on $\mathbb{K}^2$, given by Definitions 2.1. In terms of the basis $\{(1, \pm 1)^t\}$, the representation is equivalent to a sum $\alpha \oplus \beta$ of 1-dimensionals, where

$$\alpha(\tau, a) \;=\; \cdot\, a \qquad \text{and} \qquad \beta(\tau, a) \;=\; \cdot\,(-a)$$

respectively, for any a in S^{d-1}. In other words, there is an isomorphism

$$\gamma \;\cong\; \omega \oplus (\eta \otimes_{\mathbb{R}} \omega). \tag{26}$$

LEMMA 4.5. *The characteristic classes of γ are given by*

$$l_1(\gamma) = c^{d/2} + 2z \textit{ and } l_2(\gamma) = (c^{d/2} + z)z \textit{ in } H^*(BF^d), \textit{ and}$$
$$w_d(\gamma) = a^d \textit{ and } w_{2d}(\gamma) = (a^d + z)z \textit{ in } H^*(BF^d; \mathbb{Z}/2).$$

Proof. By (26), the total Chern or symplectic Pontryagin class is given by

$$l(\gamma) \;=\; (1+z)(1+c^{d/2}+z) \;=\; 1 + c^{d/2} + 2z + c^{d/2}z + z^2.$$

This makes l_1 and l_2 explicit, and determines w_d and w_{2d} by applying ρ. $\square$

Also, consider the real line bundle $p_3^*(\lambda)$ over BF^d. By analogy with Remark 4.1, the composition $C_2 < F^d < P^d$ is the standard inclusion; so there is an isomorphism $p_3^*(\lambda) \cong \eta$, whence

$$w_1(p_3^*(\lambda)) \;=\; a \quad \text{and} \quad c_1(p_3^*(\lambda_{\mathbb{C}})) \;=\; c \tag{27}$$

in $H^1(BF^d; \mathbb{Z}/2)$ and $H^2(BF^d)$, respectively.

LEMMA 4.6. *The homomorphism $p_3^*\colon Z^* \to \mathbb{Z}[c, z]/(2c)$ is determined by*

$$p_3^*(m) = 2z, \quad p_3^*(y) = (c^{d/2} + z)z, \quad \textit{and} \quad p_3^*(c) = c.$$

Proof. It suffices to combine Lemma 4.5 with (27). $\square$

Over $\mathbb{Z}/2$, the formulae become $p_3^*(m) = 0$, $p_3^*(y) = (a^d + z)z$ and $p_3^*(a) = a$.

5. Mod 2 cohomology. In this section, the homotopy commutative ladder (17) is exploited to compute the cohomology ring $H^*(SP^2; \mathbb{Z}/2)$ in terms of $H^*(B; \mathbb{Z}/2)$ and the Thom isomorphism. Some results are specific cases of those of Nakaoka [**33**].

To ease the calculations, it is convenient to identify the upper cofibre sequence with (18), and use Remarks 4.4. The long exact sequence

$$\dots \xleftarrow{\delta} H^*(RK_+^\infty; \mathbb{Z}/2) \xleftarrow{i^*} H^*(B_+; \mathbb{Z}/2) \xleftarrow{b^*} H^*(MP^d; \mathbb{Z}/2) \xleftarrow{\delta} \dots \tag{28}$$

becomes the Gysin sequence for the bundle θ, and may be rewritten as

$$\dots \xleftarrow{\delta} \mathbb{Z}/2[a, z] \xleftarrow{p_3^*} G^* \xleftarrow{\cdot m} G^{*-d} \xleftarrow{\delta} \dots, \tag{29}$$

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where $\cdot m$ denotes multiplication by the Euler class $m = w_d(\theta)$. The Thom isomorphism identifies $H^*(MP^d; \mathbb{Z}/2)$ with the free G^*-module $G^*\langle t\rangle$ on a single d-dimensional generator t, otherwise known as the Thom class, which satisfies $b^*(t) = m$. The multiplicative structure of $H^*(MP^d; \mathbb{Z}/2)$ is determined by the relation $t^2 = mt$, and may be encoded as an isomorphism

$$H^*(MP^d; \mathbb{Z}/2) \;\cong\; tG^*[t]/(t^2 + mt) \tag{30}$$

of algebras over $\mathbb{Z}/2$, where $tG^*[t]$ represents the ideal $(t) \subset G^*[t]$. The implication that $p_3^*(m) = 0$ is confirmed by the mod 2 version of Lemma 4.6.

So the key to calculation is held by δ, and its values on z^k for $k > 0$. For notational simplicity, it is convenient to let $R = \mathbb{Z}$ or $\mathbb{Z}/2$, and write

$$\delta_k \;:=\; \delta(z^k) \tag{31}$$

in $H^{kd+1}(MP^d; R)$. The Thom isomorphism is defined by relative cup product, and the following basic property [**13**, (8.13)] is required.

LEMMA 5.1. *For any u in $H^*(BP^d; R)$ and v in $H^*(BF^d; R)$, the equation*

$$\delta(p_3^*(u)v) \;=\; u\delta(v)$$

holds in $H^*(MP^d; R)$. □

Lemma 4.6 and (30) then imply that $\delta(a^j z^k) = a^j \delta_k$ for any $j \geq 0$ and $k \geq 1$, and that the second order recurrence relation

$$\delta_{k+2} \;=\; a^d \delta_{k+1} + y\delta_k \quad \text{with} \quad \delta_0 = 0,\; \delta_1 = at \tag{32}$$

holds in $H^*(MP^d; \mathbb{Z}/2)$. So δ may be found in terms of a, y and t by standard techniques from the theory of *generalised Fibonacci polynomials* [**3**].

Over any commutative ring Q with identity, these polynomials lie in $Q[x_1, x_2]$, and are specified by the recurrence relation

$$q_{k+2} = x_1 q_{k+1} + x_2 q_k \quad \text{with} \quad q_0 = 0,\; q_1 = 1; \tag{33}$$

when $x_2 = 1$, they reduce to the Fibonacci polynomials [**21**]. Alternative choices of q_0 and q_1 create new sequences of polynomials, which may be written as sums of monomials by adapting the methods of [**3**, Section 2]. When applied to (32) with $Q = H^*(MP^d; \mathbb{Z}/2)$, they lead to the following.

PROPOSITION 5.2. *For any $k > 0$, the equation*

$$\delta_k \;=\; \sum_{0\leq i\leq (k-1)/2} \binom{k-1-i}{i} a^{(k-1-2i)d+1} y^i t \tag{34}$$

holds in $H^{kd+1}(MP^d; \mathbb{Z}/2)$. □

Equation (34) may also be read off from [**17**, p. 252]. Only the parity of the binomial coefficients is relevant, and by [**39**, Lemma 2.6] $\binom{a}{b}$ is odd precisely when the 1s in the dyadic expansion of b form a subset of the 1s in the dyadic expansion of a. For example, $\delta_2 = a^{d+1} t$ and $\delta_3 = a^{2d+1} t + ayt$.

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REMARK 5.3. An immediate consequences of Proposition 5.2 is that

$$\delta_k \equiv a^{(k-1)d+1} t \mod (y)t$$

for all $k > 0$, and therefore that δ_k is non-zero.

These results may be used to understand the long exact cohomology sequence of the lower cofiber sequence of (17), for which the commutative square

$$\begin{tikzcd} H^*(MP^d;\mathbb{Z}/2) & H^{*-1}(RK^\infty;\mathbb{Z}/2) \arrow[l, "\delta"'] \\ H^*(MP^d;\mathbb{Z}/2) \arrow[u, "id"] & H^{*-1}(\mathbb{K}P^\infty;\mathbb{Z}/2) \arrow[l, "\delta"'] \arrow[u, "\pi_2^*"'] \end{tikzcd}$$

is crucial. The composition $\delta \circ \pi_2^*$ is monic, because $\pi_2^*\colon \mathbb{Z}/2[z] \to \mathbb{Z}/2[a, z]$ is the natural inclusion and δ_k is never 0. The lower δ is therefore also monic, and Nakaoka's short exact sequence [**33**, p 668]

$$0 \longleftarrow H^*(SP^2;\mathbb{Z}/2) \xleftarrow{b_\Delta^*} H^*(MP^d;\mathbb{Z}/2) \xleftarrow{\delta} H^{*-1}(\mathbb{K}P^\infty;\mathbb{Z}/2) \longleftarrow 0 \tag{35}$$

emerges immediately. Of course, (35) splits as vector spaces over $\mathbb{Z}/2$.

THEOREM 5.4. *There is an isomorphism*

$$H^*(SP^2;\mathbb{Z}/2) \cong tG^*[t]/(t^2 + mt,\ \delta_k : k > 0),$$

where t and m have dimension d, and δ_k has dimension $kd + 1$.

Proof. It follows from (35) that there is an isomorphism

$$H^*(SP^2;\mathbb{Z}/2) \cong H^*(MP^d;\mathbb{Z}/2)/\delta(H^{*-1}(\mathbb{K}P^\infty;\mathbb{Z}/2))$$

of rings, so it suffices to note that $H^*(\mathbb{K}P^\infty;\mathbb{Z}/2)$ is generated by the z^k. □

REMARKS 5.5. Since $\delta_k t = 0$ in $H^*(MP^d;\mathbb{Z}/2)$ for every $k > 0$, the summand $\mathbb{Z}\langle\delta_k\rangle$ coincides with the ideal (δ_k). Only monomials of the form $m^i y^j t$ multiply non-trivially in $H^*(SP^2;\mathbb{Z}/2)$, subject to the relation $t^2 = mt$.

The isomorphism of Theorem 5.4 may be clarified by importing the values of δ_k from Proposition 5.2. For example, $\delta_3 = 0$ gives $a^5 t = ayt$ in the complex case and $a^9 t = ayt$ in the quaternionic case, in dimensions 7 and 13, respectively.

6. Integral cohomology. In this section, the commutative ladder (17) is exploited to compute the integral cohomology rings $H^*(MP^d)$ and $H^*(SP^2)$. Input is provided by the geometric and mod 2 cohomology results of Sections 3–5. As in earlier sections, integral cohomology classes and their mod 2 reductions may be written identically when the context is sufficient to distinguish between them.

The integral version of (28) is the long exact sequence

$$\ldots \xleftarrow{\delta} H^*(RK_+^\infty) \xleftarrow{i^*} H^*(B_+) \xleftarrow{b^*} H^*(MP^d) \xleftarrow{\delta} \ldots. \tag{36}$$

The bundle θ has no integral Euler class, because $w_1(\theta) = a$ by (24); so (29) has no analogue over $\mathbb{Z}$. Nevertheless, almost all the required information can be deduced

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directly from (36), by separating the cohomology groups into even and odd dimensional summands.

By Theorem 4.3 and Remarks 4.4, $H^{od}(B)$ and $H^{od}(RK^\infty)$ are zero. So (36) reduces to a collection of exact sequences of length 4, which together imply that b^* and δ induce isomorphisms

$$H^{ev}(MP^d) \cong \operatorname{Ker} i^* \qquad \text{and} \qquad H^{od}(MP^d) \cong \operatorname{Cok} i^* \tag{37}$$

respectively. The relative cup product invests $H^*(MP^d)$ with a natural module structure over Z^*, and the first isomorphism is of Z^*-algebras; it interacts with δ as in Lemma 5.1. The second isomorphism is of graded abelian groups, and shifts dimension by 1.

LEMMA 6.1. *The kernel of i^* is the principal ideal $(m^2 - 4y)$ in Z^*.*

Proof. Lemma 4.6 implies that $i^*(m^2 - 4y) = 0$, so $(m^2 - 4y) \subseteq \operatorname{Ker} i^*$.
On the other hand, let $f(m, y) + g(c, y)$ denote an arbitrary element

$$\sum_{i=0}^{k} f_i\, m^{2(k-i)} y^i \;+\; \sum_{j=0}^{k-1} g_j\, c^{(k-j)d} y^j \tag{38}$$

of Z^{2kd}, and suppose that it is annihilated by i^*; then

$$f(2z, (c^{d/2} + z)z) + g(c, (c^{d/2} + z)z) \;=\; 0$$

in $\mathbb{Z}[c, z]/(2c)$. Equating coefficients of $c^{kd}, c^{(k-1/2)d} z, \dots, c^{(k-(k-1)/2)d} z^{k-1}$ shows that $g_0, g_1, \dots, g_{k-1} \equiv 0 \bmod 2$, and hence that $g(c, y) = 0$. Equating coefficients of z^{2k} shows that $f^+ := \sum_{i=0}^{k} 2^{2(k-i)} f_i = 0$, and hence that

$$f(m, y)|_{m^2=4y} \;=\; f^+ y^n \;=\; 0.$$

Thus, $m^2 - 4y$ divides $f(m, y)$. Similar arguments apply to elements of $Z^{2(kd+q)}$ for $1 \le q < d$, and confirm that $\operatorname{Ker} i^* \subseteq (m^2 - 4y)$ in Z^*. □

REMARK 6.2. To describe the image of i^*, it therefore suffices to work modulo $(m^2 - 4y)$; moreover, every element of the ideal takes the form $f(m, y)(m^2 - 4y)$ for some polynomial f in $\mathbb{Z}[m, y]$, because $c(m^2 - 4y) = 0$ in Z^{2d+2}.

LEMMA 6.3. *The cokernel of i^* is isomorphic to the $\mathbb{Z}/2$-module on basis elements $c^i z^j$ for $0 \le i < jd/2$.*

Proof. First observe that $i^*(2y^k) = 2z^{2k}$ and $i^*(my^k) = 2z^{2k+1}$ in $H^*(RK_+^\infty)$, for every $k > 0$. So $\operatorname{Cok} i^*$ is a $\mathbb{Z}/2$-module, spanned by those elements $c^i z^j$ for which $i \ge 0$ and $j > 0$. There are relations amongst them because

$$i^*(y^j) \;=\; (c^{d/2} + z)^j z^j \;=\; 0 \tag{39}$$

in $\operatorname{Cok} i^*$, and (39) may also be multiplied by any c^i. The resulting expression

$$c^{jd/2} z^j \;=\; \sum_{0<i\le j} \binom{j}{i} c^{(j-i)d/2} z^{j+i} \tag{40}$$

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describes $c^{jd/2}z^j$ as a homogeneous linear combination of monomials $c^a z^b$ for which $0 \leq a < bd/2$. By iteration, a similar description arises for *any* $c^i z^j$; this iteration terminates, because every step reduces the powers of c that occur.

Linear independence of the $c^a z^b$ follows by assuming that

$$\sum_{0<i\leq k} \epsilon_i c^{(k-i)d/2} z^{k+i} \;=\; f(2z, (c^{d/2}+z)z) + g(c, (c^{d/2}+z)z)$$

in $H^{2kd}(RK^\infty)$, with $\epsilon_i = 0$ or 1 for $i < k$. Adapting the proof of Lemma 6.1 shows that $g_0, g_1, \ldots, g_{k-1}, f_k \equiv 0 \bmod 2$, so $\epsilon_i = 0$ for $i < k$ and $\epsilon_k \equiv 0 \bmod 2$, as required. Similar arguments apply in dimensions $2(kd+q)$, for $1 \leq q < d$. □

To illustrate the iteration, note that (40) gives $c^{3d/2}z^3 = c^d z^4 + c^{d/2}z^5 + z^6$ for $j = 3$; so $c^{5d/2}z^3 = c^{2d}z^4 + c^{3d/2}z^5 + c^d z^6$. Importing $c^{2d}z^4 = z^8$ yields

$$c^{5d/2}z^3 \;=\; c^{3d/2}z^5 + c^d z^6 + z^8,$$

which is of the required format.

Remark 6.4. Lemma 6.3 shows that the coefficient homomorphism $2\colon \mathbb{Z} \to \mathbb{Z}$ induces the zero homomorphism on $H^{od}(MP^d)$, and the associated long exact cohomology sequence proves that mod 2 reduction is monic.

Theorem 6.5. *As a Z^*-module, $H^*(MP^d)$ is generated by a $2d$-dimensional element s of infinite order, and the $jd+1$-dimensional elements δ_j of order 2, for $j > 0$. The module structure is determined by $cs = m\delta_j = 0$,*

$$c^{jd/2}\delta_j = \sum_{0<i\leq j} \binom{j}{i} c^{(j-i)d/2}\delta_{j+i}, \quad \textit{and} \quad y\delta_j = c^{d/2}\delta_{j+1} + \delta_{j+2};$$

the algebra structure is determined by

$$s^2 \;=\; (m^2 - 4y)s \quad \textit{and} \quad \delta_i\delta_j \;=\; s\delta_i \;=\; 0,$$

for every $i > 0$.

Proof. Define s by $b^*(s) = m^2 - 4y$, and δ_j by (31). The abelian group structure then follows from Lemmas 6.1 and 6.3, and Remark 6.2. The relation $cs = 0$ holds because $b^*(cs) = c(m^2 - 4y) = 0$ by Theorem 4.3, and b^* is monic; also, $m\delta_j = \delta(2z^{j+1}) = 0$ for any j, by Lemma 5.1. The formula for $c^{jd/2}\delta_j$ arises by applying δ to (40), and the formula for $y\delta_j$ is the integral analogue of (32).

The relation $s^2 = (m^2 - 4y)s$ holds because b^* is monic, and $\delta_i\delta_j = 0$ because $H^{ev}(MP^d)$ is torsion free. Finally, the mod 2 reduction $\rho(s\delta_i) = \rho(s)\delta_i$ is zero by Remarks 5.5, so $s\delta_i = 0$, by Remark 6.4. □

Corollary 6.6. *There is an isomorphism*

$$H^{od}(MP^d) \;\cong\; \mathbb{Z}/2\langle c^i\delta_j : 0 \leq i < jd/2\rangle$$

of graded abelian groups.

By analogy with the mod 2 case, Theorem 6.5 may now be applied to study the lower row of the cohomology ladder

$$\begin{tikzcd}
\dots & H^*(RK^\infty_+) \arrow[l, "\delta"'] & H^*(B_+) \arrow[l, "i^*"'] & H^*(MP^d) \arrow[l, "b^*"'] & \dots \arrow[l, "\delta"'] \\
\dots & H^*(\mathbb{K}P^\infty_+) \arrow[l, "\delta"'] \arrow[u, "\pi_2^*"'] & H^*(SP^2_+) \arrow[l, "i^*_\Delta"'] \arrow[u, "\pi^*"'] & H^*(MP^d) \arrow[l, "b^*_\Delta"'] \arrow[u, "id"'] & \dots \arrow[l, "\delta"']
\end{tikzcd}$$

associated to (17).

REMARKS 6.7. Since $\pi_2^*\colon \mathbb{Z}[z] \to \mathbb{Z}[c, z]/(2c)$ is the canonical inclusion, Theorem 6.5 shows that the lower δ acts by $\delta(z^j) = \delta_j$ in $H^{kd+1}(MP^d)$ for every $j > 0$. Since $2\delta_j = 0$, the kernel of δ is the principal ideal $(2z)$, and is additively generated by $2z^j$ in each dimension jd.

These observations lead to additive descriptions of $H^{od}(SP^2)$ and $H^{ev}(SP^2)$.

PROPOSITION 6.8. *There is an isomorphism*

$$H^{od}(SP^2) \;\cong\; \mathbb{Z}/2\langle u_{i,j} : 0 < i < jd/2\rangle$$

of graded abelian groups, where $u_{i,j}$ has dimension $2i + jd + 1$; in particular, $H^{od}(SP^2) = 0$ only in dimensions 1, 3, *and* 5, *together with* 9 *when $\mathbb{K} = \mathbb{H}$.*

Proof. Remarks 6.7 imply that $b^*_\Delta(\delta_j) = 0$ in $H^{jd+1}(SP^2)$, and that

$$u_{i,j} \;:=\; b^*_\Delta(c^i\delta_j) \tag{41}$$

is non-zero and has order 2 in $H^{2i+jd+1}(SP^2)$, for every $0 < i < jd/2$. By Corollary 6.6, these elements exhaust the image of b^*_Δ, which is epic in odd dimensions because $H^{od}(\mathbb{K}P^\infty)$ is zero. $\square$

Note that $u_{jd/2,j} = \sum_{0<i\leq j} \binom{j}{i} u_{(j-i)d/2,j+i}$ in $H^{2jd+1}(SP^2)$, for any $j > 0$.

REMARK 6.9. Proposition 6.8 shows that mod 2 reduction is monic on $H^{od}(SP^2)$, by analogy with Remark 6.4.

Remarks 6.7 confirm the existence of elements n_p in $H^{pd}(SP^2)$ such that $i^*_\Delta(n_p) = 2z^p$, for every $p > 0$; they are well-defined up to the image of b^*_Δ.

PROPOSITION 6.10. *The graded abelian group $H^{ev}(SP^2)$ is torsion-free, and admits a canonical choice of generator n_p in each dimension pd.*

Proof. By Lemma 6.1, b^* injects $H^{ev}(MP^d)$ into $H^{ev}(B)$ as the ideal $(m^2 - 4y)$. Thus, b^*_Δ injects $H^{ev}(MP^d)$ as a subring Q^* of $H^{ev}(SP^2)$, and π^* maps Q^* isomorphically to $(m^2 - 4y)$. So there is a split short exact sequence

$$0 \longleftarrow \mathbb{Z}\langle 2z^p\rangle \xleftarrow{i^*_\Delta} \mathbb{Z}\langle n_p\rangle \oplus Q^{pd} \xleftarrow{b^*_\Delta} \mathbb{Z}\langle m^q y^r s\rangle \longleftarrow 0 \tag{42}$$

for each $p \geq 1$, where q and r range over $q,\ r \geq 0$ such that $q + 2r + 2 = p$. As p varies, (42) confirms that $H^{ev}(SP^2)$ is torsion free.

Since π^* is a $\mathbb{Z}[1/2]$ isomorphism, it is monic on $H^{ev}(SP^2)$. In dimension $2kd$, therefore, n_{2k} may be specified uniquely by $\pi^*(n_{2k}) = 2y^k$. For, if no such element exists, then $\pi^*(n_{2k}) - 2y^k = \alpha$ for some nonzero α in $(m^2 - 4y)$; hence $\pi^*(\alpha') = \alpha$ for

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some α' in Q^{2kd}, and $\pi^*(n_{2k} - \alpha') = 2y^k$, which is a contradiction. The same argument works in dimension $(2k-1)d$, after replacing $2y^k$ by my^{k-1}. In either dimension n_p is an additive generator. □

Henceforth, n_p is assumed to be chosen in this way, for any p. The cases

$$g := n_1 \quad \text{and} \quad h := n_2 \tag{43}$$

in $H^d(SP^2)$ and $H^{2d}(SP^2)$ are sufficiently important to merit special notation.

THEOREM 6.11. *There is an isomorphism*

$$H^*(SP^2_+) \cong \mathbb{Z}[h^p/2^{p-1},\ g^q h^r/2^r,\ u_{i,j}]\,/\,I,$$

where $p,\ q \geq 1$*,* $r \geq 0$ *and* $0 < i < jd/2$*, and* I *denotes the ideal*

$$(2u_{i,j},\ u_{i,j}u_{k,l},\ u_{i,j}h^p/2^{p-1},\ u_{i,j}g^q h^r/2^r)\,;$$

the classes g*,* h*, and* $u_{i,j}$ *are those of* (43) *and* (41) *respectively.*

Proof. Since π^* embeds $H^{ev}(SP^2)$ in $H^{ev}(B;\mathbb{Z}[1/2])$, the canonical additive generators g and h may be identified with their images m and $2y$, respectively; similarly, the generators n_{2k-1} and n_{2k} may be identified with my^{k-1} and $2y^k$, and therefore with $gh^{k-1}/2^{k-1}$ and $h^k/2^{k-1}$. In particular, $b^*_\Delta(s) = g^2 - 2h$.

From this viewpoint, the short exact sequence (42) prescribes additive bases

$$h^k/2^{k-1},\ h^{k-1}(g^2-2h)/2^{k-1},\ \ldots,\ g^{2k-2r-2}h^r(g^2-2h)/2^r,\ \ldots,\ g^{2k-2}(g^2-2h)$$

in dimensions $2kd$, and

$$gh^k/2^k,\ gh^{k-1}(g^2-2h)/2^{k-1},\ \ldots,\ g^{2k-2r-1}h^r(g^2-2h)/2^r,\ \ldots,\ g^{2k-1}(g^2-2h)$$

in dimensions $(2k+1)d$, where $0 \leq r < k$. These are equivalent to bases

$$h^k/2^{k-1}, \ldots, g^{2k-2r}h^r/2^r, \ldots, g^{2k} \quad \text{and} \quad gh^k/2^k, \ldots, g^{2k-2r+1}h^r/2^r, \ldots, g^{2k+1},$$

respectively, which exhibit the stated multiplicative structure on $H^{ev}(SP^2)$.

For the products in I, note that $u_{i,j}u_{k,l} = 0$ because $H^{ev}(SP^2)$ is torsion free, by Proposition 6.10. Furthermore, $\rho(u_{i,j}) = b^*_\Delta(a^{2i}\delta_j)$ in $H^{2i+jd+1}(SP^2;\mathbb{Z}/2)$, so $\rho(u_{i,j}n_p) = \rho(u_{i,j}g^q h^r/2^r) = 0$ by Remarks 5.5. Finally, Remark 6.9 shows that $u_{i,j}n_p = u_{i,j}g^q h^r/2^r = 0$, as required. □

7. Mod 2 truncation. In the final two sections, it remains to describe how the cohomology rings of Γ_n, B_n, and SP^2_n are obtained from the calculations above, using the restriction homomorphisms induced by inclusion. All the resulting truncations are valid for any $n \geq 1$, although the case $n = 1$ is degenerate. For notational simplicity, cohomology classes and their restrictions are written identically in both sections.

This section focuses mainly on mod 2 cohomology.

The first space to consider is Γ_n. As described in Definition 2.7, it is the total space $\mathbb{R}P(\chi_n)$ of the projectivisation of the restriction to $Gr_{n+1,2}$ of the universal $Spin(d+1)$ bundle over $BO_{\mathbb{K}}(2)$. By Corollary 3.4, it is also homotopy equivalent to the configuration space $C_2(\mathbb{K}P^n)$, whose one-point compactification is the Thom space $Th(\theta_n)$.

The cohomology ring of the Grassmannian is given by the truncation

$$H^*((Gr_{n+1,2})_+;R) \;\cong\; R[l_1,l_2]/(\sigma_n,\sigma_{n+1})$$

of $H^*(BO_{\mathbb{K}}(2);R)$, where $R=\mathbb{Z}$ or $\mathbb{Z}/2$ and the σ_k satisfy

$$\sigma_{k+2} = -l_1\sigma_{k+1} - l_2\sigma_k \quad \text{with} \quad \sigma_0 = 1,\ \sigma_1 = -l_1 \tag{44}$$

in $R[l_1,l_2]$. Thus, (44) fits into the framework of (33), and leads to

$$\sigma_k \;=\; \sum_{0\le i\le k/2} (-1)^{k-i}\binom{k-i}{i} l_1^{k-2i} l_2^i \tag{45}$$

in dimension kd, for $k\ge 1$; this agrees with Yasui [**41**, (4.7)]. Proceeding as for (25), $H^*((\Gamma_n)_+;\mathbb{Z}/2)$ may then be identified with the $\mathbb{Z}/2$-algebra

$$G_n^* \;:=\; \mathbb{Z}/2[a,m,y]/(am,\nu_n,\nu_{n+1}),$$

where $\nu_k=\sigma_k(a^d+m,y)$ in $H^{kd}(\Gamma;\mathbb{Z}/2)$, for $k\ge 1$.

The analogues of (29) and (30) confirm the existence of an isomorphism

$$H^*(Th(\theta_n);\mathbb{Z}/2) \;\cong\; tG_n^*[t]/(t^2+mt), \tag{46}$$

where t is pulled back from the universal Thom class in $H^d(MP^d;\mathbb{Z}/2)$. The associated Euler class is m, as before.

REMARK 7.1. Care is required to work with the restriction homomorphism on $H^*(MP^d;\mathbb{Z}/2)$. Its kernel is the ideal $(\nu_n,\nu_{n+1})t$, so (46) may be rewritten as

$$H^*(Th(\theta_n);\mathbb{Z}/2) \;\cong\; H^*(MP^d;\mathbb{Z}/2)/(\nu_n,\nu_{n+1})t,$$

for any $n\ge 1$. For example, $\nu_3=x^3$ in G_2^{3d}, so a^2x^3t lies in $(\nu_3)t$ and is 0 in $H^{4d+2}(Th(\theta_3);\mathbb{Z}/2)$; however, a^2x^3t does not lie in the ideal $(\nu_3 t)$.

In order to complete the calculation of $H^*(SP_n^2;\mathbb{Z}/2)$, it suffices to apply restriction to (35), and consider the resulting exact sequence

$$\ldots \longleftarrow H^*(SP_n^2;\mathbb{Z}/2) \xleftarrow{b_\Delta^*} H^*(Th(\theta_n);\mathbb{Z}/2) \xleftarrow{\delta} H^{*-1}(\mathbb{K}P^n;\mathbb{Z}/2) \longleftarrow \ldots. \tag{47}$$

Of course, Nakaoka's results show that (47) is also short exact. In the current context, this follows from the fact that restriction truncates $H^*(\mathbb{K}P^\infty;\mathbb{Z}/2)$ and $H^*(MP^d;\mathbb{Z}/2)$ by the ideals (z^{n+1}) and $(\nu_n,\nu_{n+1})t$, respectively. So restriction is an isomorphism on $H^*(MP^d;\mathbb{Z}/2)$ in dimensions $\le (n+1)d-1$, and maps δ_k to its non-zero namesake in $H^{kd+1}(Th(\theta_n);\mathbb{Z}/2)$ for each $1\le k\le n$. Hence δ in (47) is monic, as required.

THEOREM 7.2. *For any $n\ge 1$, there are isomorphisms*

$$\begin{aligned} H^*(SP_n^2;\mathbb{Z}/2) \;&\cong\; tG_n^*[t]/(t^2+mt,\ \delta_k : 0<k\le n) \\ &\cong\; H^*(SP^2;\mathbb{Z}/2)\big/(\nu_n,\nu_{n+1})t/(a\nu_{n+l}t : l\ge 0) \end{aligned}$$

of graded $\mathbb{Z}/2$-algebras.

Proof. The first isomorphism arises by adapting the proof of Theorem 5.4 to the short exact sequence (47). In particular, restriction induces an epimorphism onto

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$H^*(SP^2_n;\mathbb{Z}/2)$, with kernel $b^*_\Delta((\nu_n,\nu_{n+1})t)$. Comparing (34) with (45) and using the relation $ax = a^{d+1}$ from (24), gives $\delta_{r+1} = a\nu_r t$ in $H^{(r+1)d+1}(MP^d;\mathbb{Z}/2)$ for any $r \geq 1$; so b^*_Δ has kernel $(a\nu_{n+l}t : l \geq 0)$, and the second isomorphism holds. □

For example, the classes $\nu_2 t = (a^{2d} + m^2 + y)t$ and $\nu_3 t = (a^{3d} + m^3)t$ are non-zero in $(\nu_2,\nu_3)t/(a\nu_{2+l}t : l \geq 0)$. In general, $(a\nu_{n+l}t : l \geq 0)$ is simply a copy of $\mathbb{Z}/2 < (\nu_n,\nu_{n+1})t$ in each dimension $(n+l+1)d+1$.

8. Integral truncation. Our final section completes the calculations, by focusing on the integral cohomology rings of Γ_n, B_n, and SP^2_n. An important intermediate step is the study of $H^*(Th(\theta_n))$, whose global structure is best described in the spirit of [**11**], using local coefficients. That approach will, however, be pursued elsewhere, and attention will be restricted here to the applications.

As in the mod 2 case, the first space to consider is the configuration space $\Gamma_n \simeq \mathcal{C}_2(\mathbb{K}P^n)$, whose integral cohomology follows from the Leray–Serre spectral sequence for the fibration $\mathbb{R}P^d \to \Gamma_n \to Gr_{n+1,2}$, by restricting the cohomology of the base in the proof of Theorem 4.3.

THEOREM 8.1. *For any $n \geq 1$, there are isomorphisms*

$$H^*((\Gamma_n)_+) \;\cong\; \mathbb{Z}[c,m,y]/(2c,cm,\nu_n,\nu_{n+1}) \;\cong\; H^*(\Gamma_+)/(\nu_n,\nu_{n+1})$$

of graded rings, where $\nu_k = \sigma_k(c^{d/2}+m,\,y)$ in $H^{kd}(\Gamma)$ for every $k \geq 1$.

Our next task is to study $H^*(Th(\theta_n))$. The crucial geometric input is the commutative diagram (15) (whose upper row no longer arises from a 3-sphere bundle for any finite n). Applying $H^*(-)$ gives the commutative ladder

$$\begin{tikzcd}
\dots & H^*(RK^n_+) \arrow[l, "\delta"'] & H^*((B_n)_+) \arrow[l, "i_n^*"'] & H^*(Th(\theta_n)) \arrow[l, "b_n^*"'] & \dots \arrow[l, "\delta"'] \\
\dots & H^*(\mathbb{K}P^n_+) \arrow[l, "\delta"'] \arrow[u, "\pi_2^*"] & H^*((SP^2_n)_+) \arrow[l, "i_\Delta^*"'] \arrow[u, "\pi^*"'] & H^*(Th(\theta_n)) \arrow[l, "b_\Delta^*"'] \arrow[u, "id"'] & \dots \arrow[l, "\delta"']
\end{tikzcd} \tag{48}$$

for any $n \geq 1$. Elements c, x, m, and y are defined in $H^*(B_n)$ via Remarks 4.4, such that $H^*(RK^n_+)$ is the truncation $\mathbb{Z}[c,z]/(2c,z^{n+1})$ and $H^{od}(RK^n) = 0$. To describe $H^*(B_n)$ more precisely, a brief diversion is required.

For any $k \geq 0$, consider the *power sum* polynomial

$$r_k(e_1,e_2) \;=\; z_1^k + z_2^k\,, \tag{49}$$

where $e_1 = z_1 + z_2$ and $e_2 = z_1 z_2$. The r_k lie in $\mathbb{Z}[e_1,e_2]$, and satisfy

$$r_{k+2} = e_1 r_{k+1} - e_2 r_k \quad \text{with} \quad r_0 = 2,\; r_1 = e_1. \tag{50}$$

They fit into the framework of (33), and are given by

$$r_k \;=\; \sum_{0\leq i\leq k/2} (-1)^i \left(2\binom{k-i}{i} - \binom{k-i-1}{i}\right) e_1^{k-2i} e_2^i\,. \tag{51}$$

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REMARKS 8.2. Iterating (50) shows that r_{k+t} lies in the ideal (r_k, r_{k+1}) for any $t \geq 0$, and that $r_k \equiv (-1)^{k/2} 2 e_2^{k/2} \bmod (e_1)$ for every even k. The expression

$$e_2^j r_k \;=\; \sum_{0 \leq i \leq j} (-1)^i \binom{j}{i} e_1^{j-i} r_{k+j+i} \tag{52}$$

holds for any $j, k \geq 0$, by rewriting e_2 as $z_1(e_1 - z_1) = z_2(e_1 - z_2)$.

The polynomials r_k help to extend unpublished work of Roush [**37**] on $H^*(B)$.

THEOREM 8.3. *The integral cohomology ring $H^*((B_n)_+)$ is isomorphic to*

$$Z_n^* := \mathbb{Z}[c, m, y]/(2c,\, cm,\, r_{n+1},\, r_{n+2},\, y^{n+1}) \;\cong\; H^*(B_+)/(r_{n+1}, r_{n+2}, y^{n+1})$$

where $r_k = r_k(m, y)$ has dimension kd for every $k \geq 0$.

Proof. Consider the Leray–Serre spectral sequence

$$E_2^{p,q} \;:=\; H^p(\mathbb{R}P^\infty_+; \mathcal{H}^q((\mathbb{K}P^n \times \mathbb{K}P^n)_+) \;\Longrightarrow\; H^{p+q}((B_n)_+)$$

of the Borel bundle for B_n, where $\mathcal{H}^*(\ )$ denotes cohomology twisted by the involution ι. Since ι^* acts on $\mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$ by interchanging z_1 and z_2, the ring of invariants is isomorphic to $\mathbb{Z}[e_1, e_2]/\operatorname{Ker}\mu$, where μ is the projection of $\mathbb{Z}[e_1, e_2]$ into $\mathbb{Z}[z_1, z_2]/(z_1^{n+1}, z_2^{n+1})$. It follows from (50) that $\operatorname{Ker}\mu$ is the ideal generated by e_2^{n+1}, r_{n+1}, and r_{n+2}, and hence that

$$E_2^{0,*} \;=\; \mathbb{Z}[e_1, e_2]/(e_2^{n+1}, r_{n+1}, r_{n+2}).$$

Moreover, there is an isomorphism $E_2^{*,0} \cong H^*(\mathbb{R}P^\infty_+)$. So the standard cochain complex (as constructed in [**12**, Chapter XII Section 7], for example) leads to a multiplicative isomorphism

$$E_2^{*,*} \;\cong\; (\mathbb{Z}[c]/(2c)) \otimes_{\mathbb{Z}} E_2^{0,*} \big/ (c \otimes e_1), \tag{53}$$

where $c \otimes e_1 = 0$ holds because $E_2^{2,d} = 0$. This, in turn, is a consequence of the equation $e_1 = z_1 + \iota^*(z_1)$ in $\mathbb{Z}\langle e_1 \rangle / \operatorname{Im}(1 + \iota^*) \cong E_2^{2,d}$.

Every differential is zero for dimensional reasons, so (53) actually describes the E_∞ term. Furthermore, m and y in $H^*(B_n)$ represent e_1 and e_2 in $E_\infty^{0,*}$ respectively, and c in $H^2(B_n)$ represents c in $E_\infty^{2,0}$. Thus, $c^i f(m, y)$ in $H^{2i+jd}(B_n)$ represents $c^i \otimes f(e_1, e_2)$ in $E_\infty^{2i,jd}$ for any homogeneous polynomial $f(m, y)$; all extension problems are therefore trivial and (53) is additively isomorphic to $H^*(B_n)$. Since cm represents $c \otimes e_1$ and both are zero, the isomorphism is also multiplicative, and the result follows. □

COROLLARY 8.4. *For any $n \geq 1$, the monomials $m^i y^j$ form a basis for a maximal torsion-free summand of $H^{ev}(B_n)$, where $1 \leq i + j \leq n$; also, $H^{od}(B_n) = 0$.*

Proof. It suffices to work with e_1 and e_2 in $E_2^{0,*}$, which is zero when $*$ is odd.

When $* \leq nd$ is even, the monomials $e_1^i e_2^j$ already form a basis. In dimension $(n+1)d$ and higher, the relations take effect; for example, e_1^{n+1} is divisible by e_2 modulo $\operatorname{Ker}\mu$. Similarly, by repeated appeal to (52), $e_1^{n+1-s} e_2^s$ is divisible by e_2^{s+1} modulo $\operatorname{Ker}\mu$ in each dimension $(n+1+s)d$ for which $0 \leq s \leq n$, and $e_2^{n+1} = 0$. These relations, together with their multiples by powers of e_1, show that $E_2^{0,td}$ is spanned by those $e_1^i e_2^j$

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for which $i+2j=t$ and $i+j\leq n$, where $n<t\leq 2n$; they form a basis because their projections are linearly independent under μ. All monomials of higher dimension lie in $\operatorname{Ker}\mu$. □

REMARK 8.5. The highest dimensional non-torsion elements are in $H^{2nd}(B_n)$, where the monomial y^n generates a single summand $\mathbb{Z}$.

By analogy with (37), the upper row of (48) induces isomorphisms

$$H^{ev}(Th(\theta_n^d)) \;\cong\; \operatorname{Ker} i_n^* \qquad \text{and} \qquad H^{od}(Th(\theta_n^d)) \;\cong\; \operatorname{Cok} i_n^*, \tag{54}$$

where the first is of algebras over the ring Z_n^* of Theorem 8.3. These isomorphisms are best discussed in the context of the commutative ladder

$$\begin{array}{ccccccccc} \cdots & \xleftarrow{\delta} & H^*(RK_+^n) & \xleftarrow{i_n^*} & H^*((B_n)_+) & \xleftarrow{b_n^*} & H^*(Th(\theta_n)) & \xleftarrow{\delta} & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \xleftarrow{\delta} & H^*(RK_+^\infty) & \xleftarrow{i^*} & H^*(B_+) & \xleftarrow{b^*} & H^*(MP^d) & \xleftarrow{\delta} & \cdots \end{array} \tag{55}$$

which arises by restriction to B_n.

PROPOSITION 8.6. *For $n\geq 2$, the kernel of i_n^* is the principal ideal (m^2-4y), and for $n=1$, it is $(2y)$; in dimensions $>nd$ the monomials m^iy^j and $2y^k$ form an additive basis, where $i\geq 1$, $i+j\leq n<i+2j$ and $n/2<k\leq n$.*

Proof. Truncation has no effect in dimensions $\leq nd$, so Lemma 6.1 continues to hold; furthermore, (55) confirms that $i_n^*(m^2-4y)=0$ for any $n\geq 1$.

Now let α be such that $i_n^*(\alpha)=0$ in $H^{(n+s)d}(RK^n)$, for some $s\geq 1$. Restriction is epic, so α lifts to an element α' in $H^{(n+s)d}(B)$ whose expansion (38) has $f_i=g_j=0$ for $i,j>n$. Since $i^*(\alpha')\equiv 0 \bmod (z^{n+1})$ in $H^{(n+s)d}(RK^\infty)$, the proof of Lemma 6.1 may then be modified to show that $i^*(\alpha')=2\lambda z^{n+s}$ for some integer λ. Moreover, $i^*(\lambda r_{n+s})=2\lambda z^{n+s}$ by (49), so $\alpha'-\lambda r_{n+s}$ lies in $\operatorname{Ker} i^*$, and therefore in (m^2-4y) by Lemma 6.1. Restriction then confirms that α lies in (m^2-4y), as required. This argument also works for $n=1$ because $r_2=m^2-2y$, so the relation $r_2=0$ gives $m^2=2y$ in $H^{2d}(B_1)$.

In terms of Corollary 8.4, the monomials that lie in dimensions between $(n+1)d$ and $2nd$ clearly satisfy $i_n^*(m^iy^j)=0$ whenever $i\geq 1$; on the other hand, $i_n^*(y^k)\equiv c^{kd/2}z^k \bmod (cz^{k+1})$ is of order 2, but non-zero. It follows that $\operatorname{Ker} i_n^*$ has the stated basis in this range. □

REMARKS 8.7. By Remarks 8.2, the polynomials $r_k(m,y)$ are $\equiv(-1)^{k/2}2y^{k/2}$ or 0 mod (m) in $H^{kd}(B)$, for even or odd k respectively. So by Theorem 6.11, they lie in the image of π^*, and may therefore be rewritten as $r_k(g,h)$ in $H^{kd}(SP^2)$ without ambiguity. In this context, (52) confirms that every element $g^ah^jr_k/2^j$ belongs to the ideal $R^{ev}_{k+j-1}:=(r_\ell(g,h):\ell>k+j-1)$ of $H^{ev}(SP^2)$. Examples are (1) $h^jr_k/2^j$ when $a=0$, $k>1$, (2) $g^{a+1}h^j/2^j$ when $k=1$, and (3) $h^j/2^{j-1}$ when $a=k=0$.

COROLLARY 8.8. *The cohomology groups $H^{ev}(SP_n^2)$ are torsion free; in dimensions $>nd$ the monomials $g^qh^s/2^s$ and $h^p/2^{p-1}$ form an additive basis, where $q\geq 1$, $q+s\leq n<q+2s$ and $n/2<p\leq n$.*

Proof. Combine Proposition 8.6 with (43), (54), and the fact that b_Δ^* is an isomorphism for dimensions $>nd$ in diagram (48). □

To conclude the calculations, consider the commutative square

$$\begin{array}{ccc} H^*(B) & \longrightarrow & H^*(B_n) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ H^*(SP^2) & \longrightarrow & H^*(SP^2_n) \end{array}$$

induced by restriction, for any $n \geq 1$. The homomorphisms π^* are monic in even dimensions, because they induce isomorphisms over $\mathbb{Z}[1/2]$. The upper restriction is epic by Theorem 8.3, and the lower by Corollary 8.8; it is convenient to denote their kernels by K^*_n and L^*_n, respectively.

THEOREM 8.9. *For any* $n \geq 1$, *there are isomorphisms*

$$\begin{aligned} H^*((SP^2_n)_+) &\cong \mathbb{Z}[h^p/2^{p-1},\ g^q h^s/2^s,\ u_{i,j}]\,/\,J_n \\ &\cong H^*(SP^2_+)/(r_t,\ u_{i,t} : t > n) \end{aligned}$$

of graded rings, where $p, q \geq 1$, $s \geq 0$ *and* $0 < i < jd/2$*; the ideal* J_n *is given by*

$$(2u_{i,j},\ u_{i,j}u_{k,l},\ u_{i,j}h^p/2^{p-1},\ u_{i,j}g^q h^s/2^s,\ r_t,\ u_{i,t} : t > n)\,,$$

and the polynomials $r_t(g, h)$ *by* (51).

Proof. In odd dimensions, Proposition 6.8 leads to an isomorphism

$$H^{od}(SP^2_n) \ \cong\ \mathbb{Z}/2\langle u_{i,j} : 0 < i < jd/2,\ j \leq n\rangle,$$

by truncating $H^*(\mathbb{K}P^\infty)$. It therefore remains to check that L^{ev}_n and R^{ev}_n (as introduced in Remarks 8.7) coincide in $H^{ev}(SP^2)$.

By Theorem 8.3 and Remark 8.2, r_t lies in K^{ev}_n for every $t > n$. So by Remarks 8.7, $r_t(g, h)$ lies in L^{ev}_n, and $R^{ev}_n \subseteq L^{ev}_n$. For the reverse inclusion, consider r in L^{kd}_n for any $k \geq 1$; then $\pi^*(r)$ lies in both K^{kd}_n and $\operatorname{Im}\pi^*$. Since $K^*_n = (r_{n+1}, r_{n+2}, y^{n+1})$, an expression of the form

$$\pi^*(r) \ =\ v_1 r_{n+1} + v_2 r_{n+2} + v_3 y^{n+1}$$

must hold for some polynomials $v_j = v_j(m, y)$ in Z^*, where $j = 1, 2$, or 3. But $v_1 r_{n+1} + v_2 r_{n+2}$ lies in $\pi^*(R^{kd}_n)$ by Remarks 8.7, so $v_3 y^{n+1}$ lies in $\operatorname{Im}\pi^*$ as well. Thus, $v_3 y^{n+1} \equiv 2\lambda y^{kd/2}$ or $0 \bmod (m)$ for even or odd k, respectively, and therefore lies in $\pi^*(R^{kd}_n)$, by Remarks 8.7(2) and (3). So $\pi^*(r)$ lies in $\pi^*(R^{kd}_n)$, and r lies in R^{kd}_n; hence $L^{ev}_n \subseteq R^{ev}_n$, as sought. □

Combining (51) and Remarks 8.7 gives $r_{2k}(g, h) \equiv (-1)^k h^k/2^{k-1} \bmod (g)$ and $r_{2k+1}(g, h) \equiv (-1)^k(2k+1)gh^k/2^k \bmod (g^2)$, for $k \geq 0$. Corollary 8.8 identifies $H^*(SP^2_n)$ as $\mathbb{Z}\langle h^n/2^{n-1}\rangle$ in dimension $2nd$, and 0 above. The rest of Remarks 1.2 follow similarly.

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Appendix A. These tables use the notation of (41) and (43), as in Appendix B below. The additive generators exhibit the multiplicative structure in each dimension, and monomials in g and h have infinite order; all others have order 2.

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EXAMPLE A.1. By Theorem 8.9, $H^*(SP^2(\mathbb{C}P^2))$ is given by

0	2	4	6	7	8
1	g	g^2, h	$gh/2$	$u_{1,2}$	$h^2/2$

and $H^*(SP^2(\mathbb{H}P^2))$ is given by

0	4	7	8	11	12	13	15	16
1	g	$u_{1,1}$	g^2, h	$u_{1,2}$	$gh/2$	$u_{2,2}$	$u_{3,2}$	$h^2/2$

,

with relations $g^3 = 3gh/2$ and $g^2h/2 = h^2/2$ in both cases.

Similarly, $H^*(SP^2(\mathbb{C}P^3))$ is given by

0	2	4	6	7	8	9	10	11	12
1	g	g^2, h	g^3, $gh/2$	$u_{1,2}$	$g^2h/2$, $h^2/2$	$u_{1,3}$	$gh^2/4$	$u_{2,3}$	$h^3/4$

and $H^*(SP^2(\mathbb{H}P^3))$ is given by

0	4	7	8	11	12	13	15	16	17
1	g	$u_{1,1}$	g^2, h	$u_{1,2}$	g^3, $gh/2$	$u_{2,2}$	$u_{3,2}$, $u_{1,3}$	$g^2h/2$, $h^2/2$	$u_{2,3}$

19	20	21	23	24
$u_{3,3}$	$gh^2/4$	$u_{4,3}$	$u_{5,3}$	$h^3/4$

,

with relations $g^4 = 4g^2h/2 - h^2/2$ and $g^3h/2 = 3gh^2/4$ in both cases.

Appendix B. This table records the first occurrence of key notation:

Symbol	ζ	θ	λ	τ	χ	ξ	ω	a	c	d	g	h	m
Page	707	708	709	708	707	707	707	713	713	707	721	721	714

x	y	z	$u_{i,j}$	r_k	F^d	P^d	RK^n	Γ	$Pin^\ddagger(d)$
713	713	713	720	723	708	707	711	709	708

.

For any ring R, the free module on basis elements $y_1,\ldots,y_n$ is denoted by $R\langle y_1,\ldots,y_n\rangle$, as first occurs on page 716.

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