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ON THE BOREL–CANTELLI PROBLEM

BY
JONATHAN SHUSTER[1]

1. **Introduction.** Let $(\Omega, \mathscr{F}, P)$ be a probability space, and $A_1, A_2 \dots$ be a sequence of members of $\mathscr{F}$. The classical Borel–Cantelli problem is to determine the probability that infinitely many events A_k occur. The classical results may be found in Feller [2, p. 188]; while related work may be found in Spitzer [3, p. 317], and Dawson and Sankoff [1]. The latter works are generalizations of the Borel–Cantelli lemmas, taken in different directions.

In this paper, necessary and sufficient conditions will be given for infinitely many events A_k to occur, with probability 1. A lower bound for the probability that only finitely many A_k occur, is developed. In addition, necessary and sufficient conditions that only finitely many A_k occur, with probability 1, are given.

2. **The main results.**

THEOREM 1. (a) *If for every set $A \in \mathscr{F}$, such that $P(A) > 0$,*

$$\sum_{k=1}^{\infty} P(A \cap A_k) = +\infty, \quad \text{then} \quad P\left(\bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} A_k\right) = 1.$$

(b) *If there exists a set $A \in \mathscr{F}$, such that*

$$\sum_{k=1}^{\infty} P(A \cap A_k) < +\infty, \quad \textit{then} \quad P\left(\bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} A_k\right) \leq 1 - P(A).$$

Proof of (a). Let $B_r = \bigcap_{k=r}^{\infty} A_k^c$, where $E^c = \Omega - E$, $r = 1, 2, \dots$ Since $B_r \subset A_k^c$, $k = r, r+1, \dots$,

$$\sum_{k=1}^{\infty} P(B_r \cap A_k) = \sum_{k=1}^{r-1} P(B_r \cap A_k) < r, \quad r = 1, 2, \dots$$

Hence, by the hypothesis of (a), $P(B_r) = 0$, $r = 1, 2, \dots$; that is,

$$P\left(\bigcup_{r=1}^{\infty} B_r\right) = P\left(\bigcup_{r=1}^{\infty} \bigcap_{k=r}^{\infty} A_k^c\right) = 0.$$

The conclusion of (a) follows from DeMorgan's rules.

Proof of (b). Let A be a set such that the sum (b) converges. Let $I_k(\omega)$ be the indicator function of the set $A \cap A_k$, $k = 1, 2, \dots$, and $T = \sum_{k=1}^{\infty} I_k$. Clearly, $E(T) = \sum_{k=1}^{\infty} P(A \cap A_k) < \infty$. Therefore, $\{\omega \in A : \omega \text{ is in infinitely many } A_k\}$ is an

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event of measure zero, since otherwise $E(T) = +\infty$. That is, for almost every $\omega \in A$, only finitely many events A_k occur.

This completes the proof.

The following notation is required in the balance of the paper. Let $\tilde{B}_0, \tilde{B}_1, \ldots$. be the events that exactly $0, 1, \ldots A_k$'s occur. Note that:

$$\tilde{B}_0 = \bigcap_{k=1}^{\infty} A_k^c, \quad \text{and} \quad \tilde{B}_r = \bigcup_{[j_1, \ldots, j_r]} \left[A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_r} \cap \left(\bigcap_{k \neq j_i} A_k^c\right)\right],$$

and are therefore measurable. (j_i's all different, $j_i = 1, 2, \ldots$).
Let $C_n = \bigcup_{r=0}^{n} \tilde{B}_r$, $n = 0, 1, \ldots$.

The following lemma will be used in Theorem 2.

LEMMA.

$$\sum_{k=1}^{\infty} P(C_n \cap A_k) \leq n, \quad n = 0, 1, \ldots$$

Proof. Let $J_{n,k}$ be the indicator function of $C_n \cap A_k$. Consider $\{J_{n,k}(\omega): k = 1, 2, \ldots\}$, for each $\omega \in \Omega$. From the definition of the set C_n, at most n of the $J_{n,k}$ can be 1 for any ω. Therefore, one obtains:

$$\sum_{k=1}^{\infty} J_{n,k}(\omega) \leq n.$$

Upon taking expectations on both sides, the required conclusion is reached.

THEOREM 2. *In order that only finitely many events A_k, occur with probability one, it is necessary and sufficient that for every $\epsilon > 0$, there exists a measurable set D_ϵ, such that*:

$$\text{(i)} \qquad P(D_\epsilon) > 1 - \epsilon \quad \text{and} \quad \text{(ii)} \sum_{k=1}^{\infty} P(D_\epsilon \cap A_k) < \infty.$$

Proof. If only finitely many A_k occur with probability one, the events C_n, defined above, satisfy:

$$C_n \nearrow E, \quad \text{with } P(E) = 1.$$

By virtue of the fact that the measure is continuous from below, the above lemma shows the conditions are necessary. The sufficiency is an immediate consequence of Theorem 1(b).

This completes the proof.

3. **Conclusions.** (i) If a set $A \in \mathscr{F}$, can be found such that $\sum P(A \cap A_k)$ converges, then only finitely many A_k occur with a probability not less than $P(A)$.

(ii) Combining the results of Theorems 1 and 2, applying the second to a conditional probability space, the following equation holds:

$$P\left(\bigcap_{r=1}^{\infty} \bigcup_{k=r}^{\infty} A_k\right) = 1 - \sup\left\{P(A): A \in \mathscr{F}, \quad \sum_{k=1}^{\infty} P(A \cap A_k) < \infty\right\}.$$

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(iii) The above theorems point out what can go wrong, if one tried to reverse the argument of the classical lemmas. In addition, the divergence argument does not depend on the independence of the A_k's.

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McGILL UNIVERSITY,
MONTREAL, QUEBEC
UNIVERSITY OF FLORIDA,
GAINSVILLE, FLORIDA

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