BULLETIN 13 (1), 1970

INJECTIVE HULLS OF SEMILATTICES

BY
G. BRUNS AND H. LAKSER

A (meet-) semilattice is an algebra with one binary operation $\wedge$, which is associative, commutative and idempotent. Throughout this paper we are working in the category of semilattices. *All categorical or general algebraic notions are to be understood in this category.* In every semilattice S the relation

$$a \leq b \quad \text{if and only if} \quad a \wedge b = a$$

defines a partial ordering of S. The symbol "$\bigvee$" denotes least upper bounds under this partial ordering. If it is not clear from the context in which partially ordered set a least upper bound is taken, we add this set as an index to the symbol; for example, $\bigvee_A X$ denotes the least upper bound of X in the partially ordered set A.

In this paper we characterize injective semilattices (Theorem 1) and essential extensions of semilattices (Theorem 2) and we explicitly construct the injective hull of any semilattice (Corollary 2).

1. **Injective Semilattices.** We recall that in any category K of algebras an algebra C is said to be *injective* if and only if every homormorphism φ from a subalgebra A of an algebra B into C has an extension to all of B. An algebra A is said to be a *retract* of an extension B if and only if there exists a homomorphism φ: $B \to A$ which maps A identically. A homomorphism φ with this property is called a *retraction.*

LEMMA 1. *Every complete lattice C satisfying the identity*

$$(1) \qquad a \wedge \bigvee M = \bigvee (a \wedge x \mid x \in M)$$

for all $a \in C$, $M \subseteq C$, is an injective semilattice.

Proof. Let A be a subsemilattice of a semilattice B and let φ be a homomorphism of A into C. Define $\bar{\varphi}$: $B \to C$ by $\bar{\varphi}(b) = \bigvee(\varphi(a) \mid a \leq b, a \in A)$. Clearly $\bar{\varphi}$ extends φ. We show that $\bar{\varphi}$ is a homomorphism. Let $b_1, b_2 \in B$. Then

$$\begin{aligned} \bar{\varphi}(b_1) \wedge \bar{\varphi}(b_2) &= \bigvee(\varphi(a_1) \mid a_1 \leq b_1, \quad a_1 \in A) \wedge \bigvee(\varphi(a_2) \mid a_2 \leq b_2, \quad a_2 \in A) \\ &= \bigvee(\varphi(a_1 \wedge a_2) \mid a_1 \leq b_1, a_2 \leq b_2, \quad a_1, a_2 \in A) \\ &= \bigvee(\varphi(a) \mid a \leq b_1 \wedge b_2, \quad a \in A) \\ &= \bar{\varphi}(b_1 \wedge b_2). \end{aligned}$$

LEMMA 2. *Every retract of a complete lattice C satisfying* (1) *is complete and satisfies* (1).

Received by the editors May 2, 1969.

https://doi.org/10.4153/CMB-1970-023-6 Published online by Cambridge University Press

Proof. Let A be a subsemilattice of C and let $\varphi: C \to A$ be a retraction. It is well known that A is also a complete lattice. We show that A satisfies (1). Let $a \in A$, $M \subseteq A$. Clearly $a \wedge \bigvee_A M$ is an upper bound in A of $\{a \wedge x \mid x \in M\}$. If $u \in A$ is an arbitrary upper bound of this set then

$$\begin{aligned} u = \varphi(u) &\geq \varphi(\textstyle\bigvee_C (a \wedge x \mid x \in M)) = \varphi(a \wedge \bigvee_C M) \\ &= \varphi(a) \wedge \varphi(\textstyle\bigvee_C M) \geq a \wedge \bigvee_A M. \end{aligned}$$

This means that $a \wedge \bigvee_A M$ is the least upper bound in A of $\{a \wedge x \mid x \in M\}$, proving (1).

THEOREM 1. *A semilattice S is injective if an only if it is a complete lattice and satisfies* (1).

Proof. By Lemma 1 our conditions are sufficient for injectivity. Assume, conversely, that the semilattice S is injective. The mapping that assigns to each $a \in S$ the principal ideal $(a]$ generated by a is an embedding of S into C, the power set lattice of S. Clearly, C satisfies the conditions of our theorem. Since S is injective it is a retract of each extension. Thus, by Lemma 2, S is a complete lattice satisfying (1).

2. **Injective Hulls.** We recall that an extension C of an algebra A is said to be an *essential extension* if and only if, for every algebra B, any homomorphism $\varphi: C \to B$, whose restriction to A is one-to-one, is itself one-to-one. Clearly each essential extension of A can be embedded over A in any injective extension of A. (That is, the embedding restricted to A coincides with the embedding of A in the injective extension). An *injective hull* of an algebra A is an essential, injective extension of A. Since, as we have seen, any semilattice can be embedded in an injective one, it follows from general considerations (by a proof similar to [1], Proposition VII 2.8, page 261) that every semilattice has an injective hull. We give, however, an explicit description of injective hulls of semilattices.

We call a subset M of a semilattice S *admissible* if and only if the following two conditions hold:

(a) $\bigvee M$ exists,

(b) for each $a \in S$, $\bigvee(a \wedge x \mid x \in M)$ exists and $\bigvee(a \wedge x \mid x \in M) = a \wedge \bigvee M$.

A subset A of a semilattice S is said to be a *D-ideal* if and only if it satisfies the following two conditions:

(c) if $x \in A$ and $y \leq x$ then $y \in A$,

(d) if $M \subseteq A$ and M is admissible then $\bigvee M \in A$.

The set $I_D(S)$ of all D-ideals of S is clearly a complete lattice under set inclusion, and arbitrary meets coincide with set-theoretical intersections. Moreover, the mapping that assigns to each a in S the principal ideal $(a]$ is an embedding of S into $I_D(S)$, so that $I_D(S)$ can be considered to be an extension of S. We describe least upper bounds in $I_D(S)$.

https://doi.org/10.4153/CMB-1970-023-6 Published online by Cambridge University Press

LEMMA 3. *For each family* $(A_i \mid i \in I)$ *in* $I_D(S)$,
$$\bigvee(A_i \mid i \in I) = \{\bigvee M \mid M \subseteq \bigcup (A_i \mid i \in I), M \text{ admissible}\}$$

Proof. Let $A=\{\bigvee M \mid M \subseteq \bigcup(A_i \mid i \in I), M \text{ admissible}\}$. Clearly, $\bigcup(A_i \mid i \in I) \subseteq A \subseteq \bigvee(A_i \mid i \in I)$. Hence it suffices to show that A is a D-ideal. Assume $y \in A$ and $x \leq y$; thus $y=\bigvee M$ for some admissible $M \subseteq \bigcup(A_i \mid i \in I)$. Then $x=x \wedge \bigvee M = \bigvee(x \wedge z \mid z \in M) \in A$, since $\{x \wedge z \mid z \in M\}$ is also admissible. Assume next that $N \subseteq A$, N admissible. Then for each $x \in N$ there is an admissible set $M_x \subseteq \bigcup(A_i \mid i \in I)$ with $x=\bigvee M_x$. Put $M=\bigcup(M_x \mid x \in N)$. Clearly $\bigvee N=\bigvee M$ and $M \subseteq \bigcup(A_i \mid i \in I)$. For $u \in S$ we have

$$\begin{aligned} u \wedge \bigvee M = u \wedge \bigvee(\bigvee M_x \mid x \in N) &= \bigvee(u \wedge \bigvee M_x \mid x \in N) \\ &= \bigvee(u \wedge y) \mid y \in M_x, x \in N) = \bigvee(u \wedge y \mid y \in M), \end{aligned}$$

which shows that M is admissible. It follows that $\bigvee N \in A$, and hence that A is a D-ideal.

As a consequence of Lemma 3 we obtain:

COROLLARY 1. $I_D(S)$ *satisfies* (1) *and, hence, is an injective extension of* S.

An extension E of a semilattice S is said to be *join-dense* if and only if each element $a \in E$ is the join of elements in S. The extension E is said to *preserve distributive joins* if and only if, for each admissible subset M of S, $\bigvee_E M$ exists and $\bigvee_E M=\bigvee_S M$ holds. Clearly, the extension $I_D(S)$ of S has both of these properties. We now characterize essential extensions.

THEOREM 2. *An extension* E *of a semilattice* S *is essential if and only if it is join-dense and preserves distributive joins.*

Proof. If E is an essential extension then it can be embedded over S into $I_D(S)$, since $I_D(S)$ is injective. But $I_D(S)$ satisfies the conditions of our theorem; it follows that E satisfies these conditions.

Assume, conversely, that E is a join-dense extension of S preserving distributive joins. We show that E is an essential extension. Let φ be a homomorphism of E into a semilattice B, and assume that $\varphi \mid S$ is one-to-one. Assume further that φ is not one-to-one. Then there exist elements $a, b \in E$, $a<b$, with $\varphi(a)=\varphi(b)$. Since E is a join-dense extension there exists $u \in S$ satisfying $u \leq b$ and $u \nleq a$. It follows that

$$\varphi(a \wedge u) = \varphi(a) \wedge \varphi(u) = \varphi(b) \wedge \varphi(u) = \varphi(b \wedge u) = \varphi(u).$$

We claim that for each $c \in S$

$$u \wedge c = \bigvee_S(c \wedge u \wedge x \mid x \leq a, x \in S). \tag{2}$$

Clearly, $u \wedge c$ is an upper bound of $\{c \wedge u \wedge x \mid x \leq a, x \in S\}$. If it were not the least upper bound in S there would exist an upper bound $v \in S$ such that $v<u \wedge c$. Since E is a join-dense extension, $a \wedge u \wedge c=\bigvee_E(c \wedge u \wedge x \mid x \leq a, x \in S)$. It would thus

https://doi.org/10.4153/CMB-1970-023-6 Published online by Cambridge University Press

follow that $\varphi(a \wedge u \wedge c) = \varphi(\bigvee_E(c \wedge u \wedge x \mid x \leq a,\ x \in S) \leq \varphi(v) < \varphi(u \wedge c)$, contradicting $\varphi(a \wedge u) = \varphi(u)$. Hence (2) holds. But (2) implies that $M = \{u \wedge x \mid x \leq a, x \in S\}$ is an admissible subset of S and that $\bigvee_S M = u$. Since E preserves distributive joins it follows that $u = \bigvee_S M = \bigvee_E M \leq a$, a contradiction. Thus E is an essential extension of S.

Since any two injective hulls of an algebra A are isomorphic over A it follows that we have the following Corollary:

COROLLARY 2. *The following properties of an extension E of a semilattice S are equivalent*:

(i) *E is an injective hull of S*,

(ii) *E is isomorphic over S to* $I_D(S)$,

(iii) *E is a complete, join-dense extension of S preserving distributive joins and satisfying the distributive law* (1).

REFERENCE

1. P. M. Cohn, *Universal algebra*, Harper and Row, 1965.

MCMASTER UNIVERSITY,
HAMILTON, ONTARIO
UNIVERSITY OF MANITOBA,
WINNIPEG, MANITOBA

https://doi.org/10.4153/CMB-1970-023-6 Published online by Cambridge University Press