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THE DIRECT INTEGRAL OF SOME WEIGHTED BERGMAN SPACES

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Abstract Let G be the abelian Lie group $\mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$, acting on the complex space $X = \mathbb{R}^{n+k} \times \mathrm{i}G$. Let F be a strictly convex function on $\mathbb{R}^{n+k}$. Let H be the Bergman space of holomorphic functions on X which are square-integrable with respect to the weight e^{-F}. The G-action on X leads to a unitary G-representation on the Hilbert space H. We study the irreducible representations which occur in H by means of their direct integral. This problem is motivated by geometric quantization, which associates unitary representations with invariant Kähler forms. As an application, we construct a model in the sense that every irreducible G-representation occurs exactly once in H.

Keywords: Bergman space; direct integral; unitary representation; model

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1. Introduction

Let G be a connected abelian Lie group and X be a complex space given by

$$G = \mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k, \qquad X = \mathbb{R}^{n+k} + \mathrm{i}G.$$

The complex structure of X is given by $\mathbb{C}^n \times \mathbb{C}^k/\mathbb{Z}^k$ (quotient on the imaginary part of $\mathbb{C}^k$). In this paper, we consider the G-action on X, and study the resulting unitary G-representation on some weighted Bergman spaces of holomorphic functions on X. We also explain the significance of this result from the viewpoint of Kähler geometry and geometric quantization.

We will adopt the following convention on the coordinates. Write

$$\left.\begin{aligned} z = (z_1, z_2), \quad z_1 = (z^1, \dots, z^n), \quad z_2 = (z^{n+1}, \dots, z^{n+k}), \\ z = x + \mathrm{i}y, \quad z_j = x_j + \mathrm{i}y_j, \quad z^j = x^j + \mathrm{i}y^j, \end{aligned}\right\} \tag{1.1}$$

on $X = \mathbb{C}^n \times \mathbb{C}^k/\mathbb{Z}^k$. Here z_1 and z_2 are respectively the coordinates on $\mathbb{C}^n$ and $\mathbb{C}^k/\mathbb{Z}^k$. Also, $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are respectively the coordinates on $\mathbb{R}^n \times \mathbb{R}^k$ and $\mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$. For instance a 1-form on X is given by $\sum_j f_j \,\mathrm{d}x^j + h_j \,\mathrm{d}y^j$, where j sums over $1, \dots, n+k$. A complex invariant 1-form $c_1 \,\mathrm{d}y^{n+1} + \cdots + c_k \,\mathrm{d}y^{n+k}$ on $\mathbb{R}^k/\mathbb{Z}^k$ can simply be written as $c\,\mathrm{d}y_2$ with $c \in \mathbb{C}^k$. We also normalize $2\pi \sim 1$ to simplify notation, so that, for all $r = (r^1, \dots, r^{n+k}) \in \mathbb{R}^n \times \mathbb{Z}^k$ and $z = (z^1, \dots, z^{n+k}) \in X$, we can write

$$e^{rz} = \exp(r^1 z^1 + \cdots + r^{n+k} z^{n+k}). \tag{1.2}$$

Observe that for $a = n+1, \dots, n+k$, each y^a is a variable on $\mathbb{R}/\mathbb{Z}$, so it is only defined modulo $\mathbb{Z}$. But since $r^a \in \mathbb{Z}$, and the exponential of the imaginary number is $\mathbb{Z}$ periodic, it follows that (1.2) is a well-defined complex number.

Write the group operation of $\mathbb{R}^k/\mathbb{Z}^k$ additively. The action of $G = \mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$ on $X = \mathbb{R}^{n+k} + \mathrm{i}G$ is defined by

$$g(x + \mathrm{i}y) = x + \mathrm{i}(g + y), \quad \text{where } x \in \mathbb{R}^{n+k} \text{ and } g, y \in G. \tag{1.3}$$

Let $F : \mathbb{R}^{n+k} \to \mathbb{R}$ be a strictly convex function, namely a smooth function whose Hessian matrix is positive definite everywhere. It is extended to X by setting $\tilde{F}(x+\mathrm{i}y) = F(x)$. Consider the Bergman space H with weight e^{-F}:

$$H = \left\{ h : X \to \mathbb{C} \text{ holomorphic; } \int_X |h(z)|^2 e^{-F(x)}\,\mathrm{d}x\,\mathrm{d}y < \infty \right\}. \tag{1.4}$$

The G-action on X leads to a unitary G-representation on the Hilbert space H by

$$G \to \operatorname{Aut}(H), \quad (g \cdot h)(z) = h(g(z))$$

for all $g \in G$, $h \in H$ and $z \in X$. In this paper, we study the irreducible G-representations that occur in H.

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary G-representations. Since G is abelian, each representation space in $\hat{G}$ is one dimensional. In fact

$$\hat{G} = \mathbb{R}^n \times \mathbb{Z}^k,$$

where each $t \in \mathbb{R}^n \times \mathbb{Z}^k$ defines a unitary representation of G on $\mathbb{C}$ in the following manner. Write $t = (t^1, \dots, t^{n+k}) \in \mathbb{R}^n \times \mathbb{Z}^k$ and $g = (g^1, \dots, g^{n+k}) \in \mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$ in the spirit of (1.1). Let $e^{\mathrm{i}tg} = \exp \mathrm{i}(t^1 g^1 + \cdots + t^{n+k} g^{n+k})$. As explained in (1.2), this is well defined. Now $t \in \hat{G}$ defines a G-representation on $\mathbb{C}$ by

$$G \to \operatorname{Aut}(\mathbb{C}), \quad g(z) = e^{\mathrm{i}tg} z. \tag{1.5}$$

Since $e^{\mathrm{i}tg}$ has norm 1, we clearly have $g(z)\overline{g(z)} = z\bar{z}$, so (1.5) is unitary with respect to the usual Hermitian inner product on $\mathbb{C}$.

We intend to define the occurrence of $t \in \hat{G}$ in the weighted Bergman space H. If a holomorphic function $f : X \to \mathbb{C}$ spans a one-dimensional G-representation which is equivalent to (1.5), then $f(x + \mathrm{i}(g + y)) = e^{\mathrm{i}tg} f(x + \mathrm{i}y)$ for all $g \in G$ and $x + \mathrm{i}y \in X$, which implies that f is a constant multiple of the function $z \mapsto e^{tz}$. Therefore, given $h \in H$, one may attempt to write h as a 'span' of the functions e^{rz} over $r \in \hat{G}$, namely

$$h = \sum_{r \in \hat{G}} \phi(r) e^{rz}. \tag{1.6}$$

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This works well in the case where $n = 0$ and $\hat{G} = \mathbb{Z}^k$ is discrete, as carried out in [**2**]. But with $n > 0$, the first component of the parameter $r = (r_1, r_2)$ in (1.6) is no longer discrete under the natural measure of $\mathbb{R}^n$. Each $t_1 \in \mathbb{R}^n$ has no point mass, so the value of a coefficient $\phi(t_1, t_2)$ in (1.6) is not as important as the values of $\phi(r, t_2)$ for all r near t_1. The $\mathbb{Z}^k$ component has discrete measure, so we can ignore the neighbourhood of t_2. The natural measure of $\hat{G} = \mathbb{R}^n \times \mathbb{Z}^k$ is called the Plancherel measure. Write $\mathrm{d}r$ for the Plancherel measure of the $\mathbb{R}^n$ component. With these facts in mind, we replace (1.6) by the following direct integral [**5**] expression. Let $C_0^\infty(\mathbb{R}^n)$ denote the compactly supported smooth functions on $\mathbb{R}^n$.

Definition 1.1. We say that $t = (t_1, t_2) \in \hat{G}$ occurs in H if and only if there exists some $h \in H$ of the form

$$h(z) = \int_{\mathbb{R}^n} \phi(r) e^{r z_1} \, \mathrm{d}r \, e^{t_2 z_2}, \tag{1.7}$$

where $\phi \in C_0^\infty(\mathbb{R}^n)$ and $\phi(r) \neq 0$ for all r sufficiently near t_1.

Here $\phi \in C_0^\infty(\mathbb{R}^n)$ is just a convenient condition for the construction of h. As we will see in (2.5) and (2.8), h may still be in H even if ϕ belongs to some wider class of weighted L^2-functions on $\mathbb{R}^n$. But since the definition concerns the occurrence of t in H, only the non-vanishing property of ϕ near t_1 matters.

Let $F' : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ denote the gradient mapping, and $\mathrm{Im}(F')$ its image. The following is the main result of this paper.

Theorem 1.2. *The unitary irreducible representation $t \in \hat{G}$ occurs in H if and only if $t \in \mathrm{Im}(\frac{1}{2} F')$.*

The special case $G = \mathbb{R}^k / \mathbb{Z}^k$ has been proved in [**2**]. The main argument here is therefore to deal with the $\mathbb{R}^n$-action via the direct integral expression.

In § 2, we prove Theorem 1.2. In § 3, we explain the geometric motivation for this problem. Namely, let $\omega = \mathrm{i}\partial\bar{\partial}F$ be a Kähler form on X, where F is G-invariant. Then F is a strictly convex function on $\mathbb{R}^{n+k}$. The standard machinery of geometric quantization [**4**] leads to a holomorphic Hermitian line bundle L on X. The space $H(L)$ of square-integrable holomorphic sections of L is a unitary G-representation. We will show that $H(L)$ is G-equivariant to the H of (1.4). Also, the condition $t \in \mathrm{Im}(\frac{1}{2} F')$ in Theorem 1.2 is equivalent to t being in the image of the moment map of ω.

Finally, we will construct a model [**3**] in the sense that every irreducible G-representation occurs exactly once in H.

2. Weighted Bergman space

In this section, we study the weighted Bergman space H (see (1.4)) and prove Theorem 1.2. We first gather some basic facts about the strictly convex function. It has at most one critical point and, if the critical point exists, then it is the global minimum.

Proposition 2.1. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a strictly convex function. Then the image $\mathrm{Im}(F')$ of the gradient mapping is convex. Also,*

$$\int_{\mathbb{R}^n} e^{-F(x)+rx}\,\mathrm{d}x < \infty$$

if and only if $r \in \mathrm{Im}(F')$.

Proof. For any $r \in \mathbb{R}^n$, define the strictly convex function F_r by

$$F_r(x) = F(x) - rx. \tag{2.1}$$

Observe that $r \in \mathrm{Im}(F')$ if and only if F_r has a global minimum.

Let $r, s \in \mathrm{Im}(F')$, and let $q = ar + (1-a)s$ for some $0 < a < 1$. We want to show that $q \in \mathrm{Im}(F')$. Define the strictly convex functions F_q, F_r and F_s as in (2.1). Since $r, s \in \mathrm{Im}(F')$, the functions F_r and F_s have global minimum, and so do the strictly convex functions aF_r and $(1-a)F_s$. Therefore, for $x \in \mathbb{R}^n$,

$$|x| \to \infty \implies aF_r(x), (1-a)F_s(x) \to \infty. \tag{2.2}$$

It is easy to check that

$$F_q = aF_r + (1-a)F_s. \tag{2.3}$$

Then (2.2) and (2.3) imply that $F_q(x) \to \infty$ as $|x| \to \infty$. So F_q has a global minimum, which implies that $q \in \mathrm{Im}(F')$. This proves the first part of the proposition.

Next we have

$$\begin{aligned} r \in \mathrm{Im}(F') &\iff 0 \in \mathrm{Im}(F'_r) \\ &\iff F'_r \text{ has a global minimum} \\ &\iff \int_{\mathbb{R}^n} e^{-F_r(x)}\,\mathrm{d}x < \infty \qquad \text{by [\mathbf{2}, Proposition 3.3]} \\ &\iff \int_{\mathbb{R}^n} e^{-F(x)+rx}\,\mathrm{d}x < \infty. \end{aligned}$$

This completes the proof. □

We now prove Theorem 1.2. The toric part $\mathbb{R}^k/\mathbb{Z}^k$ of G has been handled in [**2**], so our main focus is on the Euclidean part $\mathbb{R}^n$. For clarity of explanation, we start with the special case $G = \mathbb{R}^n$. The general case $G = \mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$ then follows from an easy modification.

Proof of Theorem 1.2 (special case $G = \mathbb{R}^n$). Let $G = \hat{G} = \mathbb{R}^n$ and $X = \mathbb{C}^n$. Use the usual coordinates $z = x + \mathrm{i}y$ on $\mathbb{C}^n$, where $G = \mathbb{R}^n$ acts along the y-variables as in (1.3). Let $\phi \in C_0^\infty(\mathbb{R}^n)$, and consider

$$h(z) = \int_{\mathbb{R}^n} \phi(r) e^{rz}\,\mathrm{d}r. \tag{2.4}$$

For each fixed x, the function $y \mapsto h(x+\mathrm{i}y)$ is the Fourier transform of $\phi(r)e^{rx}$. So by the Plancherel theorem (see, for example, [**7**]),

$$\int_{\mathbb{R}^n} |h(x+\mathrm{i}y)|^2 \, \mathrm{d}y = \int_{\mathbb{R}^n} |\phi(r)|^2 e^{2rx} \, \mathrm{d}r.$$

Applying $\int_{\mathbb{R}^n} (\cdots) e^{-F(x)} \, \mathrm{d}x$ to both sides, we get

$$\begin{aligned} \int_{\mathbb{C}^n} |h(x+\mathrm{i}y)|^2 e^{-F(x)} \, \mathrm{d}y \, \mathrm{d}x &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi(r)|^2 e^{2rx-F(x)} \, \mathrm{d}r \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} |\phi(r)|^2 \left(\int_{\mathbb{R}^n} e^{2rx-F(x)} \, \mathrm{d}x \right) \mathrm{d}r. \end{aligned} \tag{2.5}$$

We want to show that

$$t \in \operatorname{Im}(\tfrac{1}{2}F') \iff t \text{ occurs in } H. \tag{2.6}$$

Suppose that $t \notin \operatorname{Im}(\frac{1}{2}F')$, and so we want to show that t does not occur in H. Assume otherwise, namely there exist $h \in H$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ satisfying (2.4), such that ϕ does not vanish near t. By Proposition 2.1, $\operatorname{Im}(\frac{1}{2}F')$ is convex. So there exists a region $U \subset \mathbb{R}^n$ of positive measure such that ϕ does not vanish on U, and U does not intersect $\operatorname{Im}(\frac{1}{2}F')$. For all $r \in U$, $\phi(r) \neq 0$, and

$$\int_{\mathbb{R}^n} e^{2rx-F(x)} \, \mathrm{d}x$$

diverges by Proposition 2.1. Therefore, the last expression of (2.5) diverges. However, the first expression of (2.5) is just the square of the Hilbert space norm of $h \in H$, and so it converges. This is a contradiction, so t does not occur in H.

To complete the proof of (2.6), suppose that, conversely, $t \in \operatorname{Im}(\frac{1}{2}F')$. We want to show that t occurs in H. Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a smooth function with compact support K, such that

$$t \in U \subset K \subset \operatorname{Im}(\tfrac{1}{2}F'), \tag{2.7}$$

where $U = \phi^{-1}(0, \infty)$, and $K = \bar{U}$ is the support of ϕ. Let

$$h(z) = \int_{\mathbb{R}^n} \phi(r) e^{rz} \, \mathrm{d}r.$$

Let $m_1 = \max_{r \in K} |\phi(r)|^2$. Since $K \subset \operatorname{Im}(\frac{1}{2}F')$, by Proposition 2.1,

$$r \mapsto \int_{\mathbb{R}^n} e^{2rx-F(x)} \, \mathrm{d}x$$

is a smooth function on K. Let m_2 be its maximum value over K. Then the last expression of (2.5) is bounded above by $\int_K m_1 m_2 \, \mathrm{d}r < \infty$. So the first expression of (2.5) converges, which implies that $h \in H$. Since $t \in U$, it follows that ϕ does not vanish near t in (2.4), so t occurs in H. This completes the proof of (2.6), and Theorem 1.2 follows. $\square$

The following argument for the general case is achieved by repeating the above proof, with only minor modifications. We therefore omit the details where appropriate.

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Proof of Theorem 1.2 (general case $G = \mathbb{R}^n \times \mathbb{R}^k/\mathbb{Z}^k$). We use the coordinates $z = (z_1, z_2)$ introduced in (1.1). Write $h(z) = f(z_1)e^{t_2 z_2}$ as in (1.7), where

$$f(z_1) = \int_{\mathbb{R}^n} \phi(r) e^{r z_1} \, \mathrm{d}r$$

for some $\phi \in C_0^\infty(\mathbb{R}^n)$. Analogously to the method in (2.5), it leads to

$$\begin{aligned}\int_{\mathbb{R}^k} \int_{\mathbb{C}^n} |f(z_1)|^2 e^{2t_2 x_2 - F(x)} \, \mathrm{d}y_1 \, \mathrm{d}x_1 \, \mathrm{d}x_2 &= \int_{\mathbb{R}^k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\phi(r)|^2 e^{2r x_1 + 2t_2 x_2 - F(x)} \, \mathrm{d}r \, \mathrm{d}x_1 \, \mathrm{d}x_2 \\ &= \int_{\mathbb{R}^n} |\phi(r)|^2 \left(\int_{\mathbb{R}^{n+k}} e^{2r x_1 + 2t_2 x_2 - F(x)} \, \mathrm{d}x \right) \mathrm{d}r.\end{aligned} \tag{2.8}$$

We want to prove (2.6). Suppose that $t \notin \mathrm{Im}(\frac{1}{2}F')$, and we want to show that t does not occur in H. Assume otherwise, namely there exist some $h \in H$ and $\phi \in C_0^\infty(\mathbb{R}^n)$ in (1.7), such that ϕ does not vanish near t_1. We repeat the earlier proof for $G = \mathbb{R}^n$, with h replaced by f, and U replaced by $U \times t_2$. Then again, the divergence of the last expression of (2.8) implies that $h(z) = f(z_1)e^{t_2 z_2}$ cannot be an element of H.

Conversely, suppose that $t \in \mathrm{Im}(\frac{1}{2}F')$. We again repeat the earlier proof, with (2.7) replaced by $t \in U \times t_2 \subset K \times t_2 \subset \mathrm{Im}(\frac{1}{2}F')$. Using the same m_1 and m_2, the last expression of (2.8) is bounded by $\int_K m_1 m_2 \, \mathrm{d}r$. Therefore (2.8) converges and $h \in H$. This completes the proof of (2.6), and Theorem 1.2 follows. □

3. Kähler structure

In this section, we describe the significance of the weighted Bergman space H (see (1.4)) as well as Theorem 1.2. We adopt the coordinate system introduced in (1.1).

Suppose that ω is a Kähler form on X with a G-invariant potential function F. Namely, $\omega = \mathrm{i}\partial\bar{\partial}F$ and $F(z) = F(x)$. Then

$$\begin{aligned}\omega &= \mathrm{i}\partial\bar{\partial}F \\ &= \tfrac{1}{4}\mathrm{i} \sum_{j,l} \left(\frac{\partial}{\partial x^j} - \mathrm{i}\frac{\partial}{\partial y^j} \right) \left(\frac{\partial}{\partial x^l} + \mathrm{i}\frac{\partial}{\partial y^l} \right) F (\mathrm{d}x^j + \mathrm{i}\,\mathrm{d}y^j) \wedge (\mathrm{d}x^l - \mathrm{i}\,\mathrm{d}y^l) \\ &= \frac{1}{2} \sum_{j,l} \frac{\partial^2 F}{\partial x^j \partial x^l} \, \mathrm{d}x^j \wedge \mathrm{d}y^l.\end{aligned} \tag{3.1}$$

Since ω is Kähler, the matrix

$$\left(\frac{\partial^2 F}{\partial x^j \partial x^l} \right)$$

is positive definite. This implies that $F : \mathbb{R}^{n+k} \to \mathbb{R}$ is strictly convex.

Note that a general real $(1,1)$-form is

$$\sum_{j,l} a_{jl} (\mathrm{d}x^j \wedge \mathrm{d}x^l + \mathrm{d}y^j \wedge \mathrm{d}y^l) + b_{jl} \, \mathrm{d}x^j \wedge \mathrm{d}y^l.$$

It therefore has no potential function unless all $a_{jl} = 0$.

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From now on, fix a Kähler form $\omega = \mathrm{i}\partial\bar{\partial}F$ on X. By (3.1), write

$$\omega = \mathrm{d}\sum_l \left(\frac{1}{2}\frac{\partial F}{\partial x^l}\,\mathrm{d}y^l\right).$$

The 1-form

$$\frac{1}{2}\sum_l \frac{\partial F}{\partial x^l}\,\mathrm{d}y^l$$

is G-invariant, so the moment map μ of ω is simply the gradient mapping [**1**, Theorem 4.2.10],

$$\mu : X \to \mathbb{R}^{n+k}, \quad \mu(x+\mathrm{i}y) = \tfrac{1}{2}F'(x). \tag{3.2}$$

Note that, given any $q \in \mathbb{R}^{n+k}$ in the image of μ, the corresponding symplectic quotient [**6**] is always a point. Since F is strictly convex, it implies that F' is injective, so there exists a unique $p \in \mathbb{R}^{n+k}$ such that $\frac{1}{2}F'(p) = q$. Hence, $\mu^{-1}\{q\} = \{p+\mathrm{i}y \in X;\ y \in G\}$, and the symplectic quotient $(\mu^{-1}\{q\})/G$ is a point.

The usual machinery of geometric quantization [**4**] says that ω leads to a holomorphic line bundle L on X with Chern class $[\omega] = 0$, and it has a connection, ∇, whose curvature is ω. Furthermore, L has an invariant Hermitian structure $\langle\cdot\,,\cdot\rangle$. A section s on L is said to be holomorphic if $\nabla_v s = 0$ for all anti-holomorphic vector fields v. Define the square-integrable holomorphic sections by

$$H(L) = \left\{ s \text{ holomorphic section on } L;\ \int_X \langle s, s\rangle \,\mathrm{d}x\,\mathrm{d}y < \infty \right\}. \tag{3.3}$$

The G-action on X leads to a unitary G-representation in $H(L)$. We will show that it is G-equivariant to the weighted Bergman space H (see (1.4)).

Lemma 3.1. *Let α be a complex closed 1-form on X, invariant under the $\mathbb{R}^k/\mathbb{Z}^k$-action. There then exist some $c \in \mathbb{C}^k$ and function f on X such that $\alpha = c\,\mathrm{d}y_2 + \mathrm{d}f$.*

Proof. Let $\pi : X \to \mathbb{R}^k/\mathbb{Z}^k$ be the projection $(z_1, z_2) \mapsto y_2$. The fibre of π is contractible, so α is cohomologous to a closed 1-form in the image of π^*. Namely, $\alpha = \pi^*\beta + \mathrm{d}f$ for some function f. Averaging over the compact group $\mathbb{R}^k/\mathbb{Z}^k$ if necessary, we may assume that β is $\mathbb{R}^k/\mathbb{Z}^k$-invariant. Hence, there exists some $c \in \mathbb{C}^k$ such that $\beta = c\,\mathrm{d}y_2$. This proves the lemma. □

Proposition 3.2. *There is a non-vanishing G-invariant holomorphic section s_0 on L such that $\langle s_0, s_0\rangle = e^{-F}$.*

Proof. The arguments are similar to those presented in [**2**, § 3], so we merely sketch the idea here. Pick a non-vanishing smooth section s of L. Then $\gamma = \mathrm{i}\nabla s/s$ is a complex 1-form on X, and satisfies $\mathrm{d}\gamma = \omega$. Let $\mu = -\mathrm{i}\partial F$. Then $\mu - \gamma$ is a closed 1-form on X. Averaging over the compact group $\mathbb{R}^k/\mathbb{Z}^k$ if necessary, we may assume that $\mu - \gamma$ is $\mathbb{R}^k/\mathbb{Z}^k$-invariant. By Lemma 3.1, $\mu - \gamma = c\,\mathrm{d}y_2 + \mathrm{d}f$ for some $c \in \mathbb{C}^k$ and function f.

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Using the section $e^{cx_2-\mathrm{i}f}s$ of L and the arguments leading to [**2**, Equation (3.11)], it follows that $c \in \mathbb{Z}^k$. Therefore, as explained in (1.2), the expression $e^{-\mathrm{i}(cy_2+f)}$ is well defined.

Let $s_0 = e^{-\mathrm{i}(cy_2+f)}s$. Since s is non-vanishing, s_0 is also non-vanishing. Similar arguments to those after [**2**, Equation (3.11)] show that we have $\xi \cdot s_0 = 0$ for all $\xi \in \mathbb{R}^{n+k}$. Thus, s_0 is G-invariant. Direct computation shows that $\nabla s_0 = -\partial F \otimes s_0$. Since ∂F is a $(1,0)$-form, it follows that $\nabla_v s_0 = 0$ for all anti-holomorphic vector fields v and, hence, that s_0 is a holomorphic section. Analogously to the arguments leading to [**2**, (3.13)], we have $\langle s_0, s_0 \rangle = e^{-F}$. This proves the proposition. □

By Proposition 3.2, it follows that H of (1.4) and $H(L)$ of (3.3) are G-equivariant via the trivialization $h \mapsto hs_0$. We can rephrase Theorem 1.2, together with (3.2), by saying that $t \in \hat{G}$ occurs in $H(L)$ if and only if t lies in the image of the moment map of ω.

An application of Theorem 1.2 is the construction of the model of G. The concept of a model is first introduced in [**3**] for classical groups. It refers to a unitary G-representation in which every member of $\hat{G}$ occurs exactly once. In our case, we see that the occurrence of $t \in \hat{G}$ is controlled by the image set $\mathrm{Im}(\frac{1}{2}F')$.

Corollary 3.3. *The unitary G-representation H is a model if and only if F' is surjective. Equivalently, $H(L)$ is a model if and only if the moment map of ω is surjective.*

Proof. By Theorem 1.2, H is a model if and only if $\hat{G} = \mathbb{R}^n \times \mathbb{Z}^k$ is contained in $\mathrm{Im}(\frac{1}{2}F')$. By Proposition 2.1, $\mathrm{Im}(\frac{1}{2}F')$ is convex, so this is equivalent to $\mathrm{Im}(\frac{1}{2}F') = \mathbb{R}^{n+k}$. The argument for the second statement of the corollary is similar. The proof follows. □

For instance, define the quadratic function

$$F(x) = F(x^1, \dots, x^{n+k}) = (x^1)^2 + \cdots + (x^{n+k})^2.$$

Then $F'(x) = 2x$ is surjective, so H is a model of G.

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