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Monotone Classes of Dendrites

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Abstract. Continua X and Y are monotone equivalent if there exist monotone onto maps $f: X \to Y$ and $g: Y \to X$. A continuum X is isolated with respect to monotone maps if every continuum that is monotone equivalent to X must also be homeomorphic to X. In this paper we show that a dendrite X is isolated with respect to monotone maps if and only if the set of ramification points of X is finite. In this way we fully characterize the classes of dendrites that are monotone isolated.

1 Introduction

A *dendrite* is a locally connected continuum without simple closed curves. A map $f: X \to Y$ is said to be monotone if $f^{-1}(y)$ is connected for all $y \in f(X)$. Let $\mathcal{M}$ be the class of monotone mappings. Two continua X and Y are said to be *equivalent with respect to* $\mathcal{M}$ (or just *monotone equivalent*) if there are mappings in $\mathcal{M}$ from X onto Y and from Y onto X. The class $\mathcal{M}$ is said to be *neat*, since all homeomorphisms are in $\mathcal{M}$ and the composition of any two mappings in $\mathcal{M}$ is also in $\mathcal{M}$. Therefore, a family of continua is decomposed into disjoint equivalence classes in the sense that two continua belong to the same class provided that they are equivalent with respect to $\mathcal{M}$. A continuum is said to be *isolated with respect to* $\mathcal{M}$ provided the above-mentioned class to which X belongs consists only of X.

In [2], Theorems 6.7 and 6.14 show that universal dendrites are not isolated with respect to $\mathcal{M}$. The problem we solve was posed in 1991 in [2][Problem 6.1] and was also considered in [5, Problem 16 (717?), Conjecture 3.7 (718?)] The purpose of this paper is to prove the following theorem.

Theorem 1.1 *A dendrite X is isolated with respect to monotone maps if and only if the set of ramification points of X is finite.*

In [1], Camerlo, Darji, and Marcone present similar results on quasi-homeomorphisms. Two dendrites, X and Y are *quasi-homeomorphic* if for every $\epsilon > 0$ there exist ϵ-onto maps $f_\epsilon: X \to Y$ and $g_\epsilon: Y \to X$. It follows from [1, Theorem 3.2] in their paper that if two dendrites are monotone equivalent, then they are quasi-homeomorphic. However, the converse is not true. For example, if X is the simple harmonic comb and Y is two simple harmonic combs identified at one end-point of their respective

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spines, then it can be easily shown that X and Y are quasi-homeomorphic but not monotone equivalent. Hence, monotone equivalence is a finer equivalence relation.

In this paper, we will only show sufficiency of the main theorem, since necessity follows directly from [1, Theorems 3.2 and 3.5].

Theorem 1.2 *If D is a dendrite with a finite number of ramification points, then D is monotonically isolated.*

2 Preliminaries

2.1 Definitions, Notation, and Results on Dendrites

We will let $\mathcal{D}$ be the set of all dendrites. If D is a dendrite, let $R(D)$ denote the set of ramification points of D. If $A \subset D$, let $R_D(A) = A \cap R(D)$ or just $R(A)$ when there is no confusion. If $r \in R(D)$, then let $\operatorname{ord}(r)$ denote the order of r. Next let $\operatorname{Com}(K)$ be the set of components of K. Suppose that $B \subset D$ and $q \in R(B)$. Then define

$$\mathcal{C}(q, B, D) = \{\overline{A} \mid A \in \operatorname{Com}(D - B) \text{ such that } q \in \overline{A}\}.$$

When there is no confusion, we write $\mathcal{C}(q, B)$ for $\mathcal{C}(q, B, D)$ and $\mathcal{C}(q)$ for $\mathcal{C}(q, \{q\})$. Let $\mathcal{B} \subset \mathcal{D}$ be some class of dendrites. Then let $\mathcal{C}_{\mathcal{B}}$ be the subset of $\mathcal{C}$ whose elements belong to $\mathcal{B}$. For example, we let $\top$ be the class of dendrites that contain a triod, *i.e.*, dendrites that are not homeomorphic to an arc. Then

$$\mathcal{C}_{\top}(q, B, D) = \{A \in \mathcal{C}(q, B, D) \mid A \in \top\}.$$

Also, if $\mathcal{A}$ is a set, then for ease of notation, we let $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} A$ and $\sigma(\mathcal{A}) = \{\langle A_i \rangle_{i \in N} \mid A_i \in \mathcal{A} \text{ and } N \subset \mathbb{N}\}$.

Notice that $\mathcal{C}^*(q, B, D)$ is a dendrite whose intersection with B is $\{q\}$. Similarly, we will let superscripts of R be subsets of R with certain properties. For example, we define

$$R^{\top}(D) = \{r \in R(D) \mid |\mathcal{C}_{\top}(r, D)| = \infty\}.$$

Often, it will be useful to fix a point $r_D \in D$ and call that point a *root*. Then we say that (D, r_D) is a rooted dendrite and $\mathcal{D}_r = \{(D, r_D) \mid D \in \mathcal{D} \text{ and } r_D \text{ is a root of } D\}$. If D is a dendrite such that $|R_D([x, y])| < \infty$ for every arc $[x, y] \subset D$, then D is a *tree*. Note that the set of ramification points of a tree still may be infinite, although countable. Then define $\mathcal{T}_r$ to be the collection of rooted trees.

Suppose that there exists an arc $A \subset D$ such that $R(A)$ is infinite. Then D is called a *comb* and A is called a *spine* of D. Suppose that there exists an arc $A \subset X$ such that $\overline{R(A)}$ is homeomorphic to $\overline{\{1/n\}_{n=1}^{\infty}}$. Then D is called a *harmonic comb*. Furthermore, if the closure of each component of $D - A$ is an arc, then D is called a *simple harmonic comb*. A comb D is a *countable comb* if $\overline{R(A)}$ is countable for every arc $A \subset D$. On the other hand, if there exists a spine A such that $\overline{R(A)}$ is uncountable, then A is called a *wild spine* and D is called a *wild comb*. A wild spine $[x, y]$ is *archimedian* if $[x, y]$ is a maximal arc in D and if for every $p, q \in R([x, y])$ such that $p < q$ (in the natural ordering on $[x, y]$), there exists $r \in R([x, y])$ such that $p < r < q$. If B is a spine of D, then $\mathcal{C}^*(q, B, D)$ is a *tooth* of B and q is a *root* of $\mathcal{C}^*(q, B, D)$.

A subdendrite $D' \subset D$ is *free* if for every component C of $D - D'$, $\overline{C} \cap D'$ is an endpoint of D'. Notice that this implies that $D' = \overline{\operatorname{int}_D(D')}$. Let $[a, b]$ be an arc in dendrite D and let

$$D' = [a, b] \cup \bigcup \{ Y \mid Y \text{ is a component of } D - [a, b] \text{ such that } \overline{Y} \cap (a, b) \neq \varnothing \}.$$

Then we say that D' is a subdendrite of D *strung* by $[a, b]$. Note that in this case, D' is a free subdendrite.

The following well-known theorem will be very useful.

Theorem 2.1 *If D_1, D_2 are dendrites such that there exist a one-to-one map $h: D_1 \to D_2$, then there exists a monotone map $m: D_2 \to D_1$ such that $m|_{h^{-1}(D_1)} = h^{-1}$.*

We let D_m be the standard universal dendrite of order $m \in \{3, 4, \dots\} \cup \{\omega\}$ (see Ważewski [9], Menger [6], Charatonik [2]). Note that if D is a dendrite whose ramification points have order not greater than m, then D can be embedded in D_m. Furthermore, every dendrite can be embedded in D_ω. The following is [2, Corollary 6.4].

Theorem 2.2 ([2]) *For each $m, n \in \{3, 4, \dots\} \cup \{\omega\}$, there exists a monotone mapping of D_m onto D_n.*

To prove the main theorem, we break down the dendrites with an infinite number of ramification points into 4 classes:

(i) trees
(ii) countable combs
(iii) wild combs with perfect spines
(iv) dendrites that are monotone equivalent to D_ω.

3 Quasi-Orderings, Well-Quasi-Orderings, and Better-Quasi-Orderings

A relation is *quasi-ordered* if it is reflexive and transitive. Since monotone onto maps are preserved under composition, the existence of a monotone map between two continua induces a natural partial order. If $\mathcal{D}$ is the set of dendrites, then $\leq$ will be used to define a quasi-order on $\mathcal{D}$ in the following way:

$$D_1 \leq D_2 \text{ if and only if there exists a monotone onto map } m: D_2 \longrightarrow D_1.$$

The following variations of $\leq$ will be used:

- If $(D_1, r_1), (D_2, r_2) \in \mathcal{D}_r$, then we define the quasi-order, $\leq_r$, on $\mathcal{D}_r$ by $(D_1, r_1) \leq_r (D_2, r_2)$ if and only if there exists a monotone onto map $m: D_2 \to D_1$ such that $m(r_2) = r_1$. Note that $(D_1, r_1) \leq_r (D_2, r_2)$ implies $D_1 \leq D_2$, but the converse is not true.
- We define $(D_1, r_1) \leq_r^e (D_2, r_2)$ if there exists a 1-1 map $e: D_1 \to D_2$ such that $e(r_2) = r_1$.
- We define $\mathcal{T}_r^+ = \{0\} \cup \mathcal{T}_r \cup \{r_\infty\}$ and let $\leq_r^+$ be a quasi-order on $\mathcal{T}_r^+$ that is an extension of $\leq_r$ such that $0 \leq_r^+ r_\infty$ and $0 \leq_r^+ (T, r_T)$ for every $(T, r_T) \in \mathcal{T}_r$.

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- Suppose $N, M \subset \mathbb{N}$. If $\langle (A_i, a_i)\rangle_{i\in N}, \langle (B_i, b_i)\rangle_{i\in M} \in \sigma(\mathcal{D}_r)$, then we define $\langle (A_i, a_i)\rangle_{i\in N} \preceq_{\sigma_r} \langle (B_i, b_i)\rangle_{i\in M}$ if and only if there exists a strictly increasing onto function $f\colon N \to M$ such that $(A_i, a_i) \preceq_r (B_{f(i)}, b_{f(i)})$ for each $i \in N$.

We say that a collection of rooted dendrites $\{(D_i, r_i)\}_{i=1}^{\infty}$ is *weakly monotonically ordered* if for every i there exists $j_i > i$ such that $(D_i, r_i) \preceq_r (D_{j_i}, r_{j_i})$. We say that $\{(D_i, r_i)\}_{i=1}^{\infty}$ is *monotonically ordered* if $(D_i, r_i) \preceq_r (D_{i+1}, r_{i+1})$ for each i. Note that every weakly monotonically ordered sequence contains a monotonically ordered subsequence.

The symbol $\leqq$ will be used to define a quasi-ordering on generic sets. Often, we will need to vary these symbols to extend these quasi-orderings or to avoid confusion:

- We will let $\leqq_{\mathcal{A}}$ be the quasi-order defined on the set $\mathcal{A}$ when necessary to avoid confusion. The definition of the partial order will, of course, depend on the definition of the set.
- If $\leqq$ is a quasi-order defined on a set Q, then $\leqq_1$ will be defined on the power set of Q and $\leqq_{\omega_1}$ will be defined inductively on successive power sets. The precise definitions will be given later in this section.
- Suppose $N, M \subset \mathbb{N}$. If $\langle A_i\rangle_{i\in N}, \langle B_i\rangle_{i\in M} \in \sigma(\mathcal{A})$, then we define $\langle A_i\rangle_{i\in N} \leqq_{\sigma(\mathcal{A})} \langle B_i\rangle_{i\in M}$ if and only if there exists a strictly increasing onto function $f\colon N \to M$ such that $A_i \leqq_{\mathcal{A}} B_{f(i)}$ for each $i \in N$.

We will always use $\le$ to denote the usual order on $\mathbb{R}$. This includes the order on arcs.

A quasi-ordered set Q is *well-quasi-ordered (wqo)* if every strictly descending sequence is finite and every antichain (collection of pairwise incomparable elements) is finite. Let Q be quasi-ordered under $\leqq$ and define the following quasi-ordering, $\leqq_1$, on the power set $\mathcal{P}(Q)$ by $X \leqq_1 Y$ if and only if there exists a function $f\colon X \to Y$ such that $x \leqq f(x)$ for each $x \in X$, where $X, Y \in \mathcal{P}(Q)$. Rado [8] constructed a quasi-ordered set Q such that Q was wqo but $\mathcal{P}(Q)$ was not. So a stronger notion of well-quasi-ordering called *better-quasi-ordered (bqo)* was constructed by Nash–Williams that preserved the property under the power set. In general, only the notion of wqo is required in this paper. However, in order for the all required relations to be wqo, they must pass through intermediate steps as bqo using previous results in the literature. The definition of bqo we give is due to Laver [4] and is equivalent, to but less technical than Nash–Williams [7]: Q is bqo if $\mathcal{P}^{\omega_1}(Q)$ is wqo. Here, $\mathcal{P}^{\omega_1}(Q)$ is defined inductively by:

(a) $\mathcal{P}^0(Q) = Q$;
(b) if α is a successor ordinal, then $\mathcal{P}^{\alpha+1}(Q) = \mathcal{P}(\mathcal{P}^{\alpha}(Q))$;
(c) if β is a limit ordinal, then define $\mathcal{P}^{\beta} = \bigcup_{\alpha<\beta} \mathcal{P}^{\alpha}(Q)$.

Also, $\mathcal{P}^{\omega_1}(Q)$ is quasi-ordered by $\leqq_{\omega_1}$, which is a natural extension of both $\leqq$ and $\leqq_1$, and is defined inductively on $\alpha, \beta < \omega_1$ in the following way: Suppose that $X \in \mathcal{P}^{\alpha}(Q)$, $Y \in \mathcal{P}^{\beta}(Q)$. Then $X \leqq_{\omega_1} Y$ if and only if the following hold:

(a) If $\alpha = 0, \beta = 0$, then $X \leqq Y$, since $X, Y \in Q$.
(b) If $\alpha = 0, \beta > 0$, then there exists $Y' \in Y$ such that $X \leqq_{\omega_1} Y'$.
(c) If $\alpha > 0, \beta > 0$, then then for every $X' \in X$ there exists $Y' \in Y$ such that $X' \leqq_{\omega_1} Y'$.

Then the following homomorphism property should be clear.

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Proposition 3.1 *If Q is bqo, $Q' \subset Q$ and there is an onto order preserving function $f: Q' \to R$, then R is bqo.*

The following theorem is a compilation of the work in Nash–Williams.

Theorem 3.2 ([7])

(i) *If Q is bqo, then Q is wqo.*
(ii) *If Q_1 and Q_2 are bqo, then $Q_1 \cup Q_2$ and $Q_1 \times Q_2$ are bqo.*
(iii) *If Q is bqo, then $\mathcal{P}(Q)$ and $\sigma(Q)$ are both bqo.*

Suppose that A and B are linearly ordered sets. An *order embedding of A into B* is a one-to-one order preserving function from A into B. An ordered set S is *scattered* if an ordered set isomorphic to the rationals cannot be order embedded in S. Let $\mathcal{M}$ be the collection of all linearly ordered sets that can be expressed as the countable union of linearly ordered sets $\{L_i\}_{i=1}^{\infty}$ each of which is scattered. Let Q be a quasi-ordered set ordered by $\leqq_Q$. Next define

$$Q^{\mathcal{M}} = \{ (L, f) \mid f: L \longrightarrow Q \text{ such that } L \in \mathcal{M} \}$$

to be the *set of all labelings* of the elements of $\mathcal{M}$ by Q. Here, each f is called a *labeling*. Then there is a quasi-ordering, $\leqq_{Q^{\mathcal{M}}}$, of $Q^{\mathcal{M}}$ induced by the orderings on Q and $\mathcal{M}$ defined in the following way: $(L_1, f_1) \leqq_{Q^{\mathcal{M}}} (L_2, f_2)$ if and only if there exists an order embedding $e: L_1 \to L_2$ such that $f_1(x) \leqq_Q f_2(e(x))$ for all $x \in L_1$.

Finally, in Laver's dissertation the following important theorem was proved.

Theorem 3.3 ([3, 4]) (i) *$\mathcal{M}$ is better quasi-ordered under order embeddings.*
(ii) *If Q is better quasi-ordered, then $Q^{\mathcal{M}}$ is better quasi-ordered.*

The notation $[x, y]$ will denote an arc with endpoints x and y. Let

$$\mathcal{I} = \{ ([x, y], A) \mid A \subset [x, y] \text{ and } \overline{A} \text{ is countable} \}$$

and quasi-order $\mathcal{I}$ by $\leqq_{\mathcal{I}}$ in the following way: $([x_1, y_1], A_1) \leqq_{\mathcal{I}} ([x_2, y_2], A_2)$ if and only there exists a monotone map $m: [x_2, y_2] \to [x_1, y_1]$ such that

(a) $m(x_2) = x_1$,
(b) $m(y_2) = y_1$,
(c) $A_1 \subset m(A_2)$.

In a similar way, define *set of all labelings on* $\mathcal{I}$ by elements of Q:

$$Q^{\mathcal{I}} = \{ ([x, y], A, f) \mid ([x, y], A) \in \mathcal{I} \text{ and } f: A \longrightarrow Q \}.$$

We can quasi-order $Q^{\mathcal{I}}$ in a similar way: $([x_1, y_1], A_1, f_1) \leqq_{Q^{\mathcal{I}}} ([x_2, y_2], A_2, f_2)$ if and only if $([x_1, y_1], A_1) \leqq_{\mathcal{I}} ([x_2, y_2], A_2)$ and for each $y \in A_1$ there exists $x \in A_2$ such that $m(x) = y$ and $f_1(y) \leqq_Q f_2(x)$, where m is the monotone map described previously. Such a map is called a *label preserving monotone* or *lpm* map.

Proposition 3.4 *Let $A_1 \subset [x_1, y_1]$ and $A_2 \subset [x_2, y_2]$ such that*

(i) *A_1 and A_2 are both closed;*
(ii) *there exists an order embedding $e: A_1 \to A_2$.*

Then there exists a monotone map $m: [x_2, y_2] \to [x_1, y_1]$ *such that*

(a) $m(x_2) = x_1$,
(b) $m(y_2) = y_1$,
(c) $A_1 \subset m(A_2)$.

Proof Notice that e might not be continuous in the induced topology. However, $\overline{e(A_1)} \subset A_2$. If $x \notin \overline{e(A_1)}$, then define

$$a(x) = \max\{ r \in \overline{e(A_1)} \cup \{x_2\} \mid r < x \} \text{ and } b(x) = \min\{ r \in \overline{e(A_1)} \cup \{y_2\} \mid r > x \}.$$

Now define $m: [x_2, y_2] \to [x_1, y_1]$ in the following way

(a) $m(x_2) = x_1$;
(b) $m(y_2) = y_1$;
(c) if $x \in e(A_1)$, then let $m(x) = e^{-1}(x)$
(d) if $x \in \overline{e(A_1)} - e(A_1)$, then there exists $\{x_n\}_{n=1}^{\infty} \subset e(A_1)$ that limits to x. Let $m(x) = \lim_{n\to\infty} e^{-1}(x_n)$.
(e) If $x \notin \overline{e(A_1)}$, then define

$$m(x) = \left(\frac{m(b(x)) - m(a(x))}{b(x) - a(x)} \right) (x - a(x)) + m(a(x)).$$

Then clearly m is monotone with the prescribed properties. ∎

Note that the proposition is false if A_1 is not closed. Now we have the following corollary to Laver's theorem.

Corollary 3.5

(i) *$\mathcal{I}$ is better-quasi-ordered.*
(ii) *If Q is better quasi-ordered, then $Q^{\mathcal{I}}$ is better-quasi-ordered.*

4 Results on Trees

Recall that $\mathcal{T}_r$ is the collection of rooted trees. Let $\le_r^e$ define a quasi-order on $\mathcal{D}_r$ by $(D_1, r_1) \le_r^e (D_2, r_2)$ if and only if there exists a one-to-one map $e: D_1 \to D_2$ such that $e(r_1) = r_2$. The following theorem is due to Nash–Williams.

Theorem 4.1 ([7]) *$\mathcal{T}_r$ is better-quasi-ordered under $\le_r^e$.*

Then the following corollaries follow from Proposition 3.1 and Theorem 4.1.

Corollary 4.2 *$\mathcal{T}_r$ is better-quasi-ordered under $\le_r$.*

Corollary 4.3 *If $\{(T_i, r_i)\}_{i=1}^{\infty}$ is a sequence in $\mathcal{T}_r$, then there exists an N such that $\{(T_i, r_i)\}_{i=N}^{\infty}$ is weakly monotonically ordered.*

Let $\mathcal{Q}_r \subset \mathcal{T}_r$ and define $F(\mathcal{Q}_r)$ to be the *collection of fans on* $\mathcal{Q}_r$ by $(D, r_1) \in F(\mathcal{Q}_r)$ if and only if the closure of each component of $D - \{r_1\}$ is an element of $\mathcal{Q}_r$, that is, $\mathcal{C}(r_1) \subset \mathcal{Q}_r$.

Proposition 4.4 *If $\mathcal{Q}_r$ is bqo under $\le_r$, then $F(\mathcal{Q}_r)$ is bqo under $\le_r$.*

Proof Let $(D_1, r_1), (D_2, r_2) \in F(\mathcal{Q}_r)$ and let $\langle A^i_{r_1} \rangle_{i \in \rho_1}$ and $\langle B^i_{r_2} \rangle_{i \in \rho_2}$ be enumerations of the elements of $\mathcal{C}(r_1, D_1)$ and $\mathcal{C}(r_2, D_2)$, respectively, where $\rho_1, \rho_2 \subset \mathbb{N}$. Then $\langle A^i_{r_1} \rangle_{i \in \rho_1}, \langle B^i_{r_2} \rangle_{i \in \rho_2} \in \sigma(\mathcal{Q}_r)$. Suppose that $\langle A^i_{r_1} \rangle_{i \in \rho_1} \le_\sigma \langle B^i_{r_2} \rangle_{i \in \rho_2}$. Then there exists a strictly increasing function $f : \rho_1 \to \rho_2$ such that for each $i \in \rho_1$ there exists a monotone map $m_i : B^{f(i)}_{r_2} \to A^i_{r_1}$ such that $m_i(r_2) = r_1$. Now define $m: D_2 \to D_1$ by $m(x) = m_i(x)$ if $x \in B_{f(i)}$ and $m(x) = r_1$ otherwise. Then clearly m is monotone and hence $(D_1, r_1) \le_r (D_2, r_2)$. Since $\mathcal{Q}_r$ is bqo, it follows that $\sigma(\mathcal{Q}_r)$ is bqo by Theorem 3.2 and that $F(\mathcal{Q}_r)$ is bqo by Proposition 3.1. ■

Proposition 4.5 *If D is a tree, then $R(D)$ is closed.*

Proof Let $x \in D - R(D)$. Let A be a maximal arc in D such that $x \in A$. Since D is locally connected and $R(A)$ is finite, there exists an open set $U \subset A$ such that $x \in U$ and $U \cap R(A) = \varnothing$. Hence $D - R(D)$ is open. ■

If $r \in B \subset D$ and $\top$ is the class of dendrites that contain a triod, then let

$$\mathcal{C}_\top(q, B, D) = \{ A \in \mathcal{C}(q, B, D) \mid A \in \top \}.$$

That is, $\mathcal{C}_\top(r, B, D)$ is the collection of elements of $\mathcal{C}(r, B, D)$ that are not arcs.

Proposition 4.6 *If D is a tree with an infinite number of ramification points, then there exists $r \in D$ such that $\mathcal{C}_\top(r, D)$ is infinite.*

Proof First note that if D' is a subdendrite of D and $r \in D'$, then $|\mathcal{C}_\top(r, B, D')| \le |\mathcal{C}_\top(r, B, D)|$. Suppose, on the contrary, that $\mathcal{C}_\top(r, D)$ is finite for each r. Pick any $r_1 \in R(D)$. Then since $\mathcal{C}_\top(r_1, D)$ is finite, there exists $C_1 \in \mathcal{C}_\top(r_1, D)$ such that $R(C_1)$ is infinite. Suppose that distinct ramification points $r_1, \dots, r_n$ and $C_1, \dots, C_n$ have been found such that the following hold:

(a) $[r_1, \dots, r_{n-1}] \subset [r_1, \dots r_n]$,
(b) $C_n \in \mathcal{C}_\top(r_n, [r_1, r_n], C_{n-1})$ such that $R(C_n)$ is infinite.

Then pick $r_{n+1} \in R(C_n) - \{r_n\}$. Then $[r_1, r_n] \cap [r_n, r_{n+1}] = \{r_n\}$, so $[r_1, r_n] \subset [r_1, r_{n+1}]$. Also, since $\mathcal{C}_\top(r_{n+1}, [r_1, r_{n+1}], C_n)$ is finite, there exists $C_{n+1} \in \mathcal{C}_\top(r_{n+1}, [r_1, r_{n+1}], C_n)$ such that $R(C_{n+1})$ is infinite.

Let $[r_1, p] = \overline{\bigcup_{n=2}^\infty [r_1, r_n]}$. Then $[r_1, p]$ is a subarc of D that contains an infinite number of ramification points. However, this contradicts the fact that D is a tree. ■

Recall that $R^\top(D) = \{ r \in R(D) \mid |\mathcal{C}_\top(r, D)| = \infty \}$.

Lemma 4.7 *If D is a tree, then $R^\top(D)$ is closed.*

Proof Notice that $R^\top(D) \subset R(D)$. Suppose, on the contrary, that there exists $x \in \overline{R^\top(D)} - R^\top(D)$. Then $\mathcal{C}_\top(x, D)$ is finite and

$$R^\top(D) = \bigcup_{C \in \mathcal{C}_\top(x, D)} R^\top(C).$$

So it follows that there exists a $C' \in \mathcal{C}_{\top}(x, D)$ such that x is a limit point of $R^{\top}(C')$ and hence a limit point of $R(C')$. However, since C' is a tree and x is an endpoint of C', this contradicts Proposition 4.5. ∎

Theorem 4.8 *Let D be a tree with an infinite number of ramification points. Then D is not monotonically isolated.*

Proof Let $R^{\top}(D) = V_0$. By Proposition 4.6 and Lemma 4.7, V_0 is nonempty and closed. If V_0 is finite, let $V = R^{\top}(D)$. Otherwise, let V_1 be the set of limit points of V_0. Continuing inductively, suppose that V_α has been found. If V_α is finite, let $V = V_\alpha$. Otherwise, let $V_{\alpha+1}$ be the set of limit points points of V_α. Suppose that for some limit ordinal β, V_α is infinite for all $\alpha < \beta$; then let $V_\beta = \bigcap_{\alpha<\beta} V_\alpha$. Then since $\{V_\alpha\}_{\alpha<\beta}$ is a decreasing sequence of nonempty compact sets, V_β is nonempty. Since $R(D)$ is countable, V_β is at most countable, and therefore there exists α' such that $V_{\alpha'}$ is nonempty but finite. So let $V = V_{\alpha'} \subset R^{\top}(D)$.

Pick some $v \in V$. If $\mathcal{C}(v, D) - \mathcal{C}_{\top}(v, D)$ is finite, then let $\widehat{D} = D \cup [v, w]$ where $[v, w]$ is an arc such that $D \cap [v, w] = \{v\}$. Since $\mathcal{C}_{\top}(v, D)$ is an infinite collection of trees with root v and $\mathcal{T}_r$ is bqo, there exists a monotonically ordered sequence $\{T_i\}_{i=1}^{\infty} \subset \mathcal{C}_{\top}(v, D)$ by Corollary 4.3. Hence, for each i there exists a monotone onto map $m_i\colon T_i \to T_{i-1}$ such that $m_i(v) = v$ and $T_0 = [v, w]$. Since $D \subset \widehat{D}$, $D \le \widehat{D}$. Let $m\colon D \to \widehat{D}$ be defined by

$$m(x) = \begin{cases} m_i(x) & \text{if } x \in \bigcup_{i=1}^{\infty} T_i, \\ x & \text{if } x \notin \bigcup_{i=1}^{\infty} T_i. \end{cases}$$

Since m is easily checked to be monotone and onto, D and $\widehat{D}$ are monotone equivalent.

On the other hand, if $\mathcal{C}(v, D) - \mathcal{C}_{\top}(v, D)$ is infinite, then let $\widehat{D} = \mathcal{C}^*_{\top}(v, D)$ and

$$\mathcal{C}(v, D) - \mathcal{C}_{\top}(v, D) = \{[v, w_i]\}_{i=1}^{\infty}.$$

Notice that here $\widehat{D} \le D$. Again, there exists a monotonically ordered sequence $\{T_i\}_{i=1}^{\infty} \subset \mathcal{C}_{\top}(v, D)$. Hence for each i there exist monotone onto maps $m_i\colon T_{2i} \to T_{i-1}$ and $p_i\colon T_{2i-1} \to [v, w_i]$ such that $m_i(v) = v$ and $p_i(v) = v$ for each i. Let $p\colon \widehat{D} \to D$ be defined by

$$p(x) = \begin{cases} m_i(x) & \text{if } x \in \bigcup_{i=1}^{\infty} T_{2i}, \\ p_i(x) & \text{if } x \in \bigcup_{i=1}^{\infty} T_{2i-1}, \\ x & \text{if } x \notin \bigcup_{i=1}^{\infty} T_i. \end{cases}$$

Then p is clearly monotone, and thus D and $\widehat{D}$ are monotone equivalent.

Notice that in both cases, $|\mathcal{C}(v, D) - \mathcal{C}_{\top}(v, D)| \neq |\mathcal{C}(v, \widehat{D}) - \mathcal{C}_{\top}(v, \widehat{D})|$. Furthermore, if there exist a homeomorphism $h\colon D \to \widehat{D}$, then

$$\big\{ |\mathcal{C}(v, D) - \mathcal{C}_{\top}(v, D)| \big\}_{v\in V} = \big\{ |\mathcal{C}(h(v), \widehat{D}) - \mathcal{C}_{\top}(h(v), \widehat{D})| \big\}_{v\in V}.$$

However, this impossible, since V is finite. Thus, D is not monotonically isolated. ∎

5 Countable Combs

In this section we show that countable combs are not monotonically isolated. The main technique to do this is Theorem 5.1 in Subsection 5.1. In Subsection 5.2 we show that every sequence of countable combs that have bounded "levels" is weakly monotonically ordered. In Sections 5.3 and 5.4 we extend this to more complicated dendrites that have infinite levels (R^∞ dendrites) and even still richer dendrites called R^∞- monotone fractals. Then in Section 5.5, we apply Theorem 5.1 to the previously described continua and show that no countable comb is monotonically isolated.

5.1 Harmonic Combs

If D is a dendrite and $[p,q] \subset D$, then $[p,q]$ is a *harmonic spine* of D if $\overline{R([p,q])}$ is homeomorphic to $\overline{1/n}_{n=1}^{\infty}$ and $p,q \in \overline{R([p,q])}$. Without loss of generality we can assume that p is the unique limit point of $R([p,q])$ and then $q \in R((p,q))$. The subdendrite strung by a harmonic spine, $[p,q]$, is called a *strung harmonic comb* and is denoted by $S([p,q])$. Note that neither p nor q are ramification points of $S([p,q])$ itself. Since there are at most a countable number of harmonic spines in a dendrite, there are at most a countable number of strung harmonic combs of a dendrite. Let $R([(p,q)) = \{r_i\}_{i=1}^{\infty}$, where $p, r_{i+1} < r_i < q$ for each i in the ordering of $[p,q]$ and $(S([p,q]), \{r_i\}_{i=1}^{\infty})$ will be used to denote the strung harmonic comb along with the ramification points of the harmonic spine. Then $(S([p,q]), \{r_i\}_{i=1}^{\infty})$ is said to be a *(weakly) monotonically ordered strung harmonic comb* if $\{\mathcal{C}^*(r_i,[p,q],S([p,q]))\}_{i=1}^{\infty}$ is (weakly) monotonically ordered.

Theorem 5.1 *Suppose that X is a dendrite with a weakly monotonically ordered strung harmonic comb. Then there exists a dendrite Y that is monotonically equivalent to X but not homeomorphic to X.*

Proof Let $(S([p,q]), \{r_i\}_{i=1}^{\infty})$ be a weakly monotonically ordered strung harmonic comb in dendrite X and let $T_i = \mathcal{C}^*(r_i,(p,q),X)$. Every dendrite has at most a countable number of strung harmonic combs. So let $\{(S([p_j,q_j]), \{r_i^j\}_{i=1}^{\infty})\}_{j=1}^{\infty}$ be an ordering of these combs. (Note: if a dendrite has a harmonic comb, then it has an infinite number of strung harmonic combs.) Let $\{x_i\}_{i=1}^{\infty}$ be a sequence in $[p,q]$ such that $r_{i+1} < x_i < r_i$. Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of arcs such that $\operatorname{diam}(A_i) < 1/i$ and let $A_i = [a_i, b_i]$.

Let $X^1 = X \cup A_1$ and continuing inductively, let $X^{i+1} = X^i \cup A_{i+1}$ be defined in the following way:

(a) $X^i \cap A_{i+1} = \{x_{i+1}\}$.
(b) If $\operatorname{ord}_X(r_{2i}^i) = 3$, then x_i is not an endpoint of A_i. That is, $\operatorname{ord}_{X^i}(x_i) = 4$.
(c) If $\operatorname{ord}_X(r_{2i}^i) \neq 3$, then $x_i = b_i$. That is, $\operatorname{ord}_{X^i}(x_i) = 3$.

Let $Y = \bigcup_{i=1}^{\infty} X^i$ and $(\widetilde{S}([p,q]), \{y_i\}_{i=1}^{\infty})$ be the strung harmonic comb of Y that corresponds to $\widetilde{S}([p,q]) = S([p,q]) \cup \bigcup_{i=1}^{\infty} A_i$, where $y_{2i-1} = r_i$ and $y_{2i} = x_i$. Let $\widetilde{T}_i = \mathcal{C}^*(y_i,(p,q),Y)$ be the teeth of $\widetilde{S}([p,q])$. Notice that if $\operatorname{ord}_Y(y_{2i}) = 3$, then $\widetilde{T}_{2i} = [y_{2i}, a_i]$. If $\operatorname{ord}_Y(y_{2i}) = 4$, then $\widetilde{T}_{2i} = [y_{2i}, a_i] \cup [y_{2i}, b_i]$.

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Claim 5.1.1 $\widetilde{S}([p,q])$ is not homeomorphic to $S([p_i,q_i])$ for all i.

Let $i \in \mathbb{N}$. Then the claim follows from the fact that $y_{2i} = x_i$ is the $2i$-th root of $\widetilde{S}([p,q])$ and $\operatorname{ord}_{\widetilde{S}([p,q])}(x_i) \neq \operatorname{ord}_{S([p_i,q_i])}(r^i_{2i})$.

Thus, it may be concluded that Y is not homeomorphic to X.

Claim 5.1.2 There exists a monotone map $g: S([p,q]) \to \widetilde{S}([p,q])$.

Let $n(1) = 1$, and for each i let $n(i+1) \geq n(i) + 4$ such that there is a monotone map $g_i: T_{n(i)} \to T_i$ with $g_i(r_{n(i)}) = r_i$. We know that such a $n(i+1)$ exists, since D is monotonically ordered. Let $M_1: T_1 \to \widetilde{T}_1$ be a homeomorphism such that $M_1(r_1) = y_1$ and for $i > 1$ let $M_i: T_{n(i)} \to \widetilde{T}_{2i-1}$ be a monotone map such that $M_i(r_{n(i)}) = y_{2i-1}$.

Let S_i be the subdendrite of $S([p,q])$ strung by $[r_{n(i)}, r_{n(i+1)}]$ and $\widetilde{S}_i$ be the subdendrite of $\widetilde{S}([p,q])$ strung by $[y_{2i-1}, y_{2i+1}]$. Then let $f_i: S_i \to \widetilde{S}_i$ be a map such that the following hold:

(a) $[r_{n(i)}, r_{n(i)+1}]$ is mapped homeomorphically onto $[y_{2i-1}, y_{2i}]$;
(b) $[r_{n(i+1)-1}, r_{n(i+1)}]$ is mapped homeomorphically onto $[y_{2i}, y_{2i+1}]$;
(c) $T_{n(i)+1}$ is mapped monotonically onto $[y_{2i}, a_i]$ such that $f_i(r_{n(i)+1}) = y_{2i}$;
(d) if $\operatorname{ord}(y_{2i}) = 4$, then $T_{n(i)+2}$ is mapped monotonically onto $[y_{2i}, b_i]$ such that $f_i(r_{n(i)+2}) = y_{2i}$;
(e) if $\operatorname{ord}(y_{2i}) = 3$, then $T_{n(i)+2}$ is mapped to y_{2i};
(f) $[r_{n(i)+1}, r_{n(i+1)-1}]$ is mapped to y_{2i};
(g) T_j is mapped to y_{2i} for all $j \in \{n(i)+3, \ldots, n(i+1)-1\}$.

Then f_i is monotone. Next define $g: S([p,q]) \to \widetilde{S}([p,q])$ by

$$g(x) = \begin{cases} M_i(x) & \text{if } x \in T_{n(i)}, \\ f_i(x) & \text{if } x \in S_i, \\ p & x = p. \end{cases}$$

Then it can be checked that g is monotone.

Claim 5.1.3 X and Y are monotonically equivalent.

Let $G: X \to Y$ be defined by $G(x) = x$ if $x \notin S([p,q])$ and $G(x) = g(x)$ if $x \in S([p,q])$. Then G is clearly monotone. Since $X \subset Y$, it follows from Theorem 2.1 that there exists a monotone map from Y onto X. ∎

5.2 Countable Combs with Bounded Levels are bqo

An arc is said to be a *level 0 dendrite*. A dendrite D with root r is said to be a *level 1 dendrite* if it is not an arc and there exists an endpoint e such that the closure of the components of $D - [r,e]$ are arcs. A dendrite D with root r is a *level n dendrite* if it is not a level k dendrite for any k in $\{0,\ldots,n-1\}$, and there exists an endpoint e such that the closure of each component of $D - [r,e]$ has level less than n. Note that the root of each component of $D = [r,e]$ is where that component meets $[r,e]$.

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Let $\mathcal{L}_r^n$ be the collection of rooted n-level countable combs. For a dendrite $(D, r_1) \in \mathcal{L}_r^n$, let e_1 be some endpoint of D such that the closure of each component of $D - [r_1, e_1]$ has level of at most $n-1$. Let F be the collection of fans as described in Section 4 and for ease of notation let $\mathcal{LT}_r^n = F(\mathcal{T}_r \cup \bigcup_{i=0}^n \mathcal{L}_r^i) \cup \{0\}$. So define $f\colon \overline{R_D([r_1, e_1])} \to \mathcal{LT}_r^n$ by $f(q) = \mathcal{C}^*(q, [r_1, e_1], D)$ if $q \in R_D([r_1, e_1])$ and $f(q) = 0$ if $q \in \overline{R_D([r_1, e_1])} - R_D([r_1, e_1])$. Recall that $\mathcal{I} = \{([x, y], A) \mid A \subset [x, y] \text{ and } \overline{A} \text{ is countable}\}$. Then $([r_1, e_1], \overline{R_D([r_1, e_1])}, f) \in (\mathcal{LT}_r^n)^{\mathcal{I}}$.

Theorem 5.2 *$\mathcal{L}_r^n$ is bqo under $\le_r$.*

Proof Proof is by induction on n. Since $\mathcal{L}_r^0 \subset \mathcal{T}_r$, $\mathcal{L}_r^0$ is bqo. Suppose that $\mathcal{L}_r^0, \dots, \mathcal{L}_r^{n-1}$ are all bqo. Then $\mathcal{LT}_r^{n-1}$ is bqo by Theorems 3.2 and 4.1 and Proposition 4.4. Suppose that $(D_1, r_1), (D_2, r_2) \in \mathcal{L}_r^n$ such that there exist endpoints e_1, e_2 of D_1, D_2 respectively that have the following properties:

(a) The closure of each component of $D_1 - [r_1, e_1]$ and $D_2 - [r_2, e_2]$ has levels of at most $n-1$.
(b) There exist labellings f_1, f_2 such that

$$\Big([r_1, e_1], \overline{R_{D_1}\big([r_1, e_1]\big)}, f_1\Big) \le_{(\mathcal{LT}_r^n)^{\mathcal{I}}} \Big([r_2, e_2], \overline{R_{D_2}\big([r_2, e_2]\big)}, f_2\Big).$$

Then there exists a monotone map $m\colon [r_2, e_2] \to [r_1, e_1]$ such that

(a) $m(r_2) = r_1$,
(b) $m(e_2) = e_1$,
(c) $\overline{R_{D_1}([r_1, e_1])} \subset m(\overline{R_{D_2}([r_2, e_2])})$
(d) for each $q \in \overline{R_{D_1}([r_1, e_1])}$ there exists $x_q \in \overline{R_{D_2}([r_2, e_2])}$ such that $m(x_q) = q$ and $f_1(q) \le_r f_2(x_q)$ (note $0 \le_r S_{r'}$ for every $S_{r'} \in \mathcal{D}_r$).

Notice that (d) implies that there exists a monotone onto map

$$m_q\colon \mathcal{C}^*\big(x_q, [r_2, e_2], D_2\big) \longrightarrow \mathcal{C}^*\big(q, [r_1, e_1], D_1\big).$$

Now define $M\colon D_2 \to D_1$ by

$$M(x) = \begin{cases} m(x) & \text{if } x \in [r_2, e_2], \\ m_q(x) & \text{if } x \in \mathcal{C}^*(x_q, [r_2, e_2], D_2), \\ m(y) & \text{if } x \in \mathcal{C}^*(y, [r_2, e_2], D_2) \\ & \text{where } y \in \overline{R_{D_2}([r_2, e_2])} - \{x_q\}_{q \in \overline{R_{D_1}([r_1, e_1])}}. \end{cases}$$

Then M is clearly monotone and onto. Hence, $(D_1, r_1) \le_r (D_2, r_2)$. Since $(\mathcal{LT}_r^{n-1})^{\mathcal{I}}$ is bqo by Corollary 3.5, it follows that $\mathcal{L}_r^n$ is bqo by Proposition 3.1. ∎

5.3 Monotone Maps of R^∞ Combs

Now suppose that $[x, y] \subset D$ and $|R([x, y])| = \infty$. Let

$$R^1([x, y]) = \Big\{ q \in R([x, y]) \mid \text{ there exists } C \in \mathcal{C}(q, [x, y], D) \text{ and } p \in C \text{ such that } |R([q, p])| = \infty \Big\}.$$

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Continuing inductively, suppose that $R^n([x,y])$ has been defined. Then define

$$R^{n+1}([x,y]) = \Big\{ q \in R^n([x,y]) \mid \text{ there exists } C \in \mathcal{C}(q,[x,y],D) \text{ and } p \in C \text{ such that } |R^n([q,p])| = \infty \Big\}.$$

Let $R^\infty([x,y]) = \bigcap_{n=1}^\infty R^n([x,y])$. Define $R^n((x,y])$ and $R^\infty((x,y])$ similarly. Suppose that $B \subset D$. Then we can define

$$R^\infty(B) = \Big\{ q \in R(B) \mid \text{ there exists an endpoint } e \in \mathcal{C}^*(q,B,D) \text{ such that } |R^\infty([q,e])| = \infty \Big\}.$$

If $q \in R^\infty(B)$, then define

$$\mathcal{C}_\infty(q,B,D) = \{ \overline{A} \mid A \in \operatorname{Com}(D-B) \text{ such that } q \in \overline{A} \text{ and there exists an endpoint } e \in A \text{ such that } |R^\infty([q,e])| = \infty \}.$$

Note that if $q \in R^\infty([x,y])$, then $\mathcal{C}_\infty(q,[x,y],D) \neq \varnothing$. Being consistent with the $*$-notation, we define $\mathcal{C}^*_\infty(q,B,D) = \bigcup_{C \in \mathcal{C}_\infty(q,B,D)} C$.

We say that a comb has the R^∞ property (or is a R^∞ *comb*) if for every arc with the property that if $|R([x,y])| = \infty$ it is the case that $|R^\infty([x,y])| = \infty$ and there exists $x_1, y_1 \in D$ such that $|R([x_1,y_1])| = \infty$.

Proposition 5.3 *If there exists distinct $x, y \in D$ such that $R^1([x,y]) \neq R^\infty([x,y])$ then D is not a R^∞ comb.*

Proof If $q \in R^1([x,y]) - R^\infty([x,y])$, then there exists an m such that $q \in R^m([x,y]) - R^{m+1}([x,y])$. Thus there exists an endpoint e of $\mathcal{C}^*(q,[q,e],D)$ such that $|R([q,e])| = \infty$. However, since $q \notin R^{m+1}([x,y])$, it follows that $|R^\infty([q,e])| \leq |R^m([q,e])| < \infty$. Hence, D is not a R^∞ comb. ∎

So if D is a R^∞ comb, then let $R^F([x,y]) = R([x,y]) - R^\infty([x,y])$. It follows from Proposition 5.3 that if $q \in R^F([x,y])$, then $(\mathcal{C}^*(q,[x,y],D), q) \in \mathcal{T}_r$.

Let $\mathcal{T}_r^+ = \{0\} \cup \mathcal{T}_r \cup \{r_\infty\}$, where r_∞ will be the image of a ramification point in $R^\infty([x,y])$ under the following labeling, and extend the ordering $\leq_r$ on $\mathcal{T}_r$ to be such that $0 \leq_r^+ T$ for every $T \in \mathcal{T}_r \cup \{r_\infty\}$. Then $\mathcal{T}_r^+$ is bqo by Theorems 3.2 and 4.1.

Let D be a countable comb and $[x,y] \subset D$. Then let $f_{x,y}: \overline{R([x,y])} \to \mathcal{T}_r^+$ be a labeling of $\overline{R([x,y])}$ defined in the following way

$$f_{x,y}(q) = \begin{cases} 0 & \text{if } q \in \overline{R([x,y])} - R([x,y]), \\ \mathcal{C}^*(q,[x,y],D) & \text{if } q \in R^F([x,y]), \\ r_\infty & \text{if } q \in R^\infty([x,y]). \end{cases}$$

Hence, it follows that $([x,y], \overline{R_D([x,y])}, f_{x,y}) \in (\mathcal{T}_r^+)^{\mathcal{I}}$, which is ordered by $\leqq_{(\mathcal{T}_r^+)^{\mathcal{I}}}$ (see Section 3).

Let D_1 be a dendrite with root r_1 and let D_2 be a dendrite with root r_2 such that $|R^\infty([r_2,x])| = \infty$ for some $x \in D_2$. We say that D_2 *overshadows* D_1 if for every endpoint e_1 of D_1 and endpoint e_2 of D_2 such that $|R^\infty([r_2,e_2])| = \infty$, and if $r' \in$

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$R^\infty([r_2, e_2])$ and $T_{r'} \in \mathcal{C}_\infty(r', [r_2, e_2], D_2)$, then there exists an endpoint $e_{r'}$ of $T_{r'}$ and labels $f_{r_1,e_1}, f_{r',e_{r'}}$ such that

$$\Big([r_1, e_1], \overline{R_{D_1}\big([r_1, e_1]\big)}, f_{r_1,e_1}\Big) \leqq_{(\mathcal{T}_r^+)^{\mathcal{I}}} \Big([r_1, e_1], \overline{R_{D_2}\big([r', e_{r'}]\big)}, f_{r',e_{r'}}\Big).$$

Theorem 5.4 *If D_1 is a dendrite with root r_1 and D_2 is a dendrite with root r_2 such that D_2 overshadows D_1, then there exists a monotone map $m: D_2 \to D_1$ such that $m(r_2) = r_1$.*

Proof Let $D_0^1 = D_0^2 = \widehat{D}_0^1 = \widehat{D}_0^2 = \varnothing$. Let e_1 be any endpoint of D_1; then there exists an endpoint e_2 of D_2 such that

$$\Big([r_1, e_1]\overline{R_{D_1}\big([r_1, e_1]\big)}, f_{r_1,e_1}\Big) \leqq_{(\mathcal{T}_r^+)^{\mathcal{I}}} \Big([r_1, e_1]\overline{R_{D_2}\big([r_2, e_2]\big)}, f_{r_2,e_2}\Big).$$

Let $m_1: [r_2, e_2] \to [r_1, e_1]$ be an associated lpm map (see Section 3). Then for each $q \in R_{D_1}^F([r_1, e_1])$ there exists $\widehat{q} \in R_{D_2}^F([r_2, e_2])$ such that $m_1(\widehat{q}) = q$ and $f_{r_1,e_1}(q) \preceq_r^+ f_{r_2,e_2}(\widehat{q})$. It follows that there exists a monotone onto map $m_q^1: \mathcal{C}^*(\widehat{q}, [r_2, e_2], D_2) \to \mathcal{C}^*(q, [r_1, e_1], D_1)$ such that $m_q^1(\widehat{q}) = q$. Let

$$\begin{aligned} D_1^1 &= [r_1, e_1], \quad D_1^2 = [r_2, e_2],\\ \widehat{D}_1^1 &= D_1^1 \cup \bigcup_{q \in R_{D_1}^F([r_1, e_1])} \mathcal{C}^*(q, [r_1, e_1], D_1) \quad \text{and}\\ \widehat{D}_1^2 &= D_1^2 \cup \bigcup_{p \in R_{D_2}^F([r_2, e_2])} \mathcal{C}^*(p, [r_2, e_2], D_2). \end{aligned}$$

Now let $\widehat{m}_1: \widehat{D}_1^2 \to \widehat{D}_1^1$ be defined by

$$\widehat{m}_1(x) = \begin{cases} m_1(x) & \text{if } x \in D_1^2,\\ m_q^1(x) & \text{if } x \in \mathcal{C}^*(\widehat{q}, [r_2, e_2], D_2), \text{ where } m_1(\widehat{q}) = q \text{ and } q \in R_{D_1}^F([r_1, e_1]),\\ m_1(p) & \text{if } x \in \mathcal{C}^*(p, [r_2, e_2], D_2) \text{ and } p \in R_{D_2}^F([r_2, e_2]) - \{\widehat{q}\}_{q \in R_{D_1}^F([r_1, e_1])}. \end{cases}$$

Continuing inductively, suppose that dendrites $\widehat{D}_{n-1}^1$, D_n^1, $\widehat{D}_{n-1}^2$ and D_n^2 and monotone onto map $m_n: D_n^2 \to D_n^1$ have been found such that:

(a) $\widehat{D}_{n-1}^1 \subset D_n^1 \subset D_1$;
(b) $\widehat{D}_{n-1}^2 \subset D_n^2 \subset D_2$;
(c) each of the components of $D_1 - D_{n-1}^1$ has diameter less than $1/n$;
(d) the closure of the components of $D_n^1 - \widehat{D}_{n-1}^1$ and $D_n^2 - \widehat{D}_{n-1}^2$ are arcs;
(e) if $(t, e]$ is a component of $D_n^1 - \widehat{D}_{n-1}^1$, then there exists a component $(\widehat{t}, \widehat{e}]$ of $D_n^2 - \widehat{D}_{n-1}^2$ such that $m_n|_{[\widehat{t}, \widehat{e}]}$ is a lpm map onto $[t, e]$.

Thus, for each $q \in R_{D_1}^F([t, e])$ there exists $\widehat{q} \in R_{D_2}^F([\widehat{t}, \widehat{e}])$ such that $m_n(\widehat{q}) = q$ and $f_{t,e}(q) \preceq_r^+ f_{\widehat{t},\widehat{e}}(\widehat{q})$. It follows that there exists a monotone onto map

$$m_{q,[t,e]}^n: \mathcal{C}^*(\widehat{q}, [\widehat{t}, \widehat{e}], D_2) \to \mathcal{C}^*(q, [t, e], D_2)$$

such that $m_{q,[t,e]}^n(\widehat{q}) = q$. Let

$$\widehat{D}_n^1 = D_n^1 \cup \bigcup_{(t,e] \in \mathrm{Com}(D_n^1 - \widehat{D}_{n-1}^1)} \bigcup_{q \in R_{D_1}^F((t,e])} \mathcal{C}^*(q, [t, e] \cup D_n^1, D_1)$$

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and with the assignment $z \to \widehat{z}$ made previously, let

$$\widehat{D}_n^2 = D_n^2 \cup \bigcup_{(t,e]\in \mathrm{Com}(D_n^2 - \widehat{D}_{n-1}^2)} \ \bigcup_{p\in R_{D_1}^F((t,e])} \mathcal{C}^*\big(p, [t,e] \cup D_n^2, D_2\big).$$

Now let $\widehat{m}_n\colon \widehat{D}_n^2 \to \widehat{D}_n^1$ be defined by

$$\widehat{m}_n(x) = \begin{cases} m_n(x) & \text{if } x \in D_n^2, \\ m^1_{q,(t,e]}(x) & \text{if } x \in \mathcal{C}^*(\widehat{q}, [\widehat{t}, \widehat{e}], D_2), \\ & \quad \text{where } \widehat{q} = m_n(q) \text{ and } q \in R_{D_1}^F([t,e]), \\ m_n(p) & \text{if } x \in \mathcal{C}^*(p, [\widehat{t}, \widehat{e}], D_2) \\ & \quad \text{and } p \in R_{D_2}^F([\widehat{t}, \widehat{e}]) - \{\widehat{q}\}_{q\in R_{D_1}^F([t,e])}, \end{cases}$$

where $(t,e]$ is a component of $D_n^1 - \widehat{D}_{n-1}^1$. Notice that $\widehat{m}_n$ is a monotone onto map. Continuing with $(t,e]$ and $(\widehat{t}, \widehat{e}]$ as defined in (e), let $q \in R_{D_1}^\infty((t,e])$. For each $C \in \mathcal{C}(q, D_n^1, D_1)$ pick an endpoint $e = e(C)$ of C. There exists $\widehat{q} \in R_{D_2}^\infty((\widehat{t}, \widehat{e}])$ such that $\widehat{m}_n(\widehat{q}) = q$. Pick any $\widehat{C} \in \mathcal{C}_\infty(\widehat{q}, D_n^2, D_2)$. Then there exists an endpoint $\check{e} = \check{e}(\widehat{C})$ of $\widehat{C}$ such that $|R_{D_2}^\infty([\widehat{q}, \check{e}])| = \infty$. Let $r = r(\widehat{C}) \in R_{D_2}^\infty((\widehat{q}, \check{e}))$. Pick any $\widetilde{C} = \widetilde{C}(r) \in \mathcal{C}_\infty(r, [\widehat{q}, \check{e}], D_2)$. Then there exists an endpoint $\widetilde{e} = \widetilde{e}(C)$ of $\widetilde{C}$ such that

$$\Big([q,e], \overline{R_{D_1}\big([q,e]\big)}, f_{q,e}\Big) \leqq_{(\mathcal{T}_r^+)^{\mathcal{I}}} \Big([r,\widetilde{e}], \overline{R_{D_2}\big([r,\widetilde{e}]\big)}, f_{r,\widetilde{e}}\Big).$$

Note that since e depends on C, $\widetilde{e}$ depends on the same C to obtain the above relation. Let

$$m(q,C)\colon [r, \widetilde{e}(C)] \to [q, e(C)]$$

be the associated lpm map. Let

$$D_{n+1}^1 = \widehat{D}_n^1 \cup \bigcup_{q\in R^\infty(D_n^1 - D_{n-1}^1)} \ \bigcup_{C\in\mathcal{C}(q, D_n^1, D_1)} [q, e(C)],$$

$$D_{n+1}^2 = \widehat{D}_n^2 \cup \Big(\bigcup_{q\in R_\infty(D_n^1 - D_{n-1}^1)} [\widehat{q}, \check{e}(\widehat{q})]\Big) \cup \Big(\bigcup_{q\in R_\infty(D_n^1 - D_{n-1}^1)} \ \bigcup_{C\in\mathcal{C}(q,D_n^1,D_1)} [r, \widetilde{e}(C)]\Big),$$

where $R^\infty(D_n^1 - D_{n-1}^1) = \bigcup_{(t,e]\in\mathrm{Com}(D_n^1 - D_{n-1}^1)} R^\infty((t,e])$. Note that we may assume that the diameter of each component of $D_1 - D_{n+1}^1$ is less than $1/(n+1)$. Define $m_{n+1}\colon D_{n+1}^2 \to D_{n+1}^1$ by

$$m_{n+1}(x) = \begin{cases} m_n(x) & \text{if } x \in D_n^2, \\ \widehat{q} & \text{if } x \in [\widehat{q}, \check{e}(\widehat{q})] \\ m(q,C)(x) & \text{if } x \in [r, \widetilde{e}(C)]. \end{cases}$$

Notice that m_{n+1} is monotone and $m_{n+1}|_{D_n^2} = m_n$. Notice that $D_1 = \overline{\bigcup_{n=1}^\infty D_n^1}$ and let $\widehat{D}_2 = \overline{\bigcup_{n=1}^\infty D_n^2}$. Then let $\widehat{m}\colon \widehat{D}_2 \to D_1$ be defined by $\widehat{m}(x) = m_n(x)$ if $x \in D_n^2$ for some n. If $x \in \widehat{D}_2 - \bigcup_{n=1}^\infty D_n^2$, then there exists $x_n \in D_n^2$ for each n such that $x_n \to x$. Next, define $\widehat{m}(x) = \lim_{n\to\infty} m_n(x_n)$. It follows that $\widehat{m}$ is monotone. Finally, since $\widehat{D}_2 \subset D_2$, there exists a monotone onto map $m\colon D_2 \to D_1$ such that $m|_{\widehat{D}_2} = \widehat{m}$. ∎

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5.4 R^∞ Monotone Fractals

Let D be a R^∞ comb with root r_1; then D is R^∞ *self-similar with respect to monotone maps* ($R^\infty m$ self-similar) if for every endpoint e of D and $q \in R^\infty([r_1, e])$, there exists a monotone onto map $m: \mathcal{C}^*(q, [r_1, e]) \to D$ such that $m(q) = r_1$.

Theorem 5.5 *If D is a R^∞ comb, then D contains a free $R^\infty m$ self-similar subcomb.*

Proof We will use the result from Theorem 5.4 that if $D \nleq_r D'$, then D' does not overshadow D. Suppose that D_0 is an R^∞ comb with root r_0 that contain no free $R^\infty m$ self-similar subcomb. Then there exists an endpoint $\widehat{e}_0$ and $\widehat{r}_1 \in R^\infty_{D_0}([r_0, \widehat{e}_0])$ such that $\widehat{D}_1 = \mathcal{C}^*(\widehat{r}_1, [r_0, \widehat{e}_0], D_0)$ does not overshadow D_0. Therefore, there exists an endpoint e_0 of D_0, an endpoint $\widetilde{e}_1$ of $\widehat{D}_1$, and $r_1 \in R^\infty_{\widehat{D}_1}([\widehat{r}_1, \widetilde{e}_1])$ such that

$$\Big([r_0, e_0], \overline{R^\infty_{D_0}\big([r_0, e_0]\big)}, f_{r_0, e_0}\Big) \nleq_{(\mathcal{T}^+_r)^{\mathcal{I}}} \Big([r_1, e], \overline{R^\infty_{D_1}\big([r_1, e]\big)}, f_{r_1, e}\Big)$$

for any endpoint e of $D_1 = \mathcal{C}^*(r_1, [\widehat{r}_1, \widetilde{e}_1], \widehat{D}_1) \subset \widehat{D}_1$.

Continuing inductively, suppose that $\{[r_i, e_i]\}_{i=0}^{n-1}$ and $\{D_i\}_{i=0}^{n}$ have been found such that

(a) D_i is a R^∞ comb with root r_i,
(b) $D_i \subset D_{i-1}$,
(c) $[r_i, e_i] \subset D_i$,
(d) $([r_{i-1}, e_{i-1}], \overline{R^\infty_{D_{i-1}}([r_{i-1}, e_{i-1}])}, f_{r_{i-1}, e_{i-1}}) \nleq_{(\mathcal{T}^+_r)^{\mathcal{I}}} ([r_i, e], \overline{R^\infty_{D_i}([r_i, e])}, f_{r_i, e})$ for every endpoint $e \in D_i$.

It follows that D_n contains no free $R^\infty m$ self-similar subcomb. Then there exists an endpoint $\widehat{e}_n$ and $\widehat{r}_{n+1} \in R^\infty_{D_n}([r_n, \widehat{e}_n])$ such that $\widehat{D}_{n+1} = \mathcal{C}^*(\widehat{r}_{n+1}, [r_n, \widehat{e}_n], D_n)$ does not overshadow D_n. Therefore, there exists an endpoint e_n of D_n, an endpoint $\widetilde{e}_{n+1}$ of $\widehat{D}_{n+1}$, and $r_{n+1} \in R^\infty_{\widehat{D}_{n+1}}([\widehat{r}_{n+1}, \widetilde{e}_{n+1}])$ such that

$$\Big([r_n, e_n], \overline{R^\infty_{D_n}\big([r_n, e_n]\big)}, f_{r_n, e_n}\Big) \nleq_{(\mathcal{T}^+_r)^{\mathcal{I}}} \Big([r_{n+1}, e], \overline{R^\infty_{D_{n+1}}\big([r_{n+1}, e]\big)}, f_{r_{n+1}, e}\Big)$$

for any endpoint e of R^∞ comb

$$D_{n+1} = \mathcal{C}^*\big(r_{n+1}, [\widehat{r}_{n+1}, \widetilde{e}_{n+1}], \widehat{D}_{n+1}\big) \subset \widehat{D}_{n+1}.$$

Notice that if $i < j$, then $e_j \in D_j \subset D_{i+1}$. Thus, $[r_j, e_j] \subset [r_{i+1}, e_j]$. Since

$$\Big([r_{i+1}, e_j], \overline{R^\infty_{D_{i+1}}\big([r_{i+1}, e_j]\big)}, f_{r_{i+1}, e_j}\Big) \ngeq_{(\mathcal{T}^+_r)^{\mathcal{I}}} \Big([r_i, e_i], \overline{R^\infty_{D_i}\big([r_i, e_i]\big)}, f_{r_i, e_i}\Big)$$

by (d), it follows that

$$\Big([r_j, e_j], \overline{R^\infty_{D_j}\big([r_j, e_j]\big)}, f_{r_j, e_j}\Big) \ngeq_{(\mathcal{T}^+_r)^{\mathcal{I}}} \Big([r_i, e_i], \overline{R^\infty_{D_i}\big([r_i, e_i]\big)}, f_{r_i, e_i}\Big).$$

Thus, $\{([r_i, e_i], \overline{R^\infty_{D_i}([r_i, e_i])}, f_{r_i, e_i})\}_{i=1}^\infty$ must contain either an infinite antichain or a strictly decreasing infinite sequence. Either way, this contradicts the fact that $(\mathcal{T}^+_r)^{\mathcal{I}}$ is bqo and hence wqo. Hence, D_0 must have a free $R^\infty m$ self similar subcomb. ∎

5.5 Countable Combs are not Monotone Isolated

The following proposition simply follows from the fact that countable sets are not perfect.

Proposition 5.6 *Suppose that $[x,y]$ is an arc in a dendrite D such that $\overline{R([x,y])}$ is countable. Then there exists a subarc $[x',y']$ of $[x,y]$ such that $|R([x',y'])| = \infty$ and the the set of limit points of $R([x',y'])$ is $\{x'\}$.*

Theorem 5.7 *Countable combs are not monotonically isolated.*

Proof There are two important cases:

Case 1 Suppose that D is not an R^∞ comb.

Then there exists an arc $[x,y]$ such that $|R([x,y])| = \infty$ but $|R^\infty([x,y])| < \infty$. Hence, there exists a subarc $[x',y']$ such that $|R([x',y'])| = \infty$ but $R^\infty((x',y')) = \varnothing$.

Claim There exists an arc $[q,p]$ and an integer n such that

(a) $|R([p,q])| = \infty$
(b) $\mathcal{C}^*(r,[q,p],D) \in \mathcal{LT}^n_r$ for each $r \in R((p,q))$.

If $R^1((x',y')) = \varnothing$, then $\mathcal{C}^*(r,[x',y'],D) \in \mathcal{T}_r \subset \mathcal{LT}^1_r$ for each $r \in R((x',y'))$. So let $q = x'$ and $p = y'$. On the other hand, suppose that there exist $q \in R^1((x',y'))$. Then since $q \notin R^\infty((x',y'))$, there exists an n such that

$$q \in R^n\big((x',y')\big) - R^{n+1}\big((x',y')\big).$$

Then there exists $p \in \mathcal{C}^*(q,[x',y'],D)$ such that $|R([q,p])| = \infty$. It follows that $\mathcal{C}^*(r,[q,p],D) \in \mathcal{LT}^n_r$ for each $r \in R((p,q))$, and the claim is shown.

Next, by Proposition 5.6 there exists a subarc $[q',p']$ such that $|R([q',p'])| = \infty$, the the set of limit points of $R([q',p'])$ is $\{q'\}$ (or similarly $\{p'\}$) and $R^\infty((q',p']) = \varnothing$. Order $R((q',p'])$ by $\{q_i\}_{i=1}^\infty$ where $q' < q_{i+1} < q_i \le p'$ in the natural ordering of $[q',p']$. Since $\mathcal{LT}^n_r$ is bqo, there exists an N such that $\{\mathcal{C}^*(q_i,[q',p']),D\}_{i=N}^\infty$ is weakly monotonically ordered. Hence the subdendrite strung by $[q',q_N]$ is a free, weakly monotonically ordered harmonic comb. Hence, D is not monotonically isolated, by Theorem 5.1, and Case 1 is completed.

Case 2 Suppose that D is a R^∞ comb.

Then by Theorem 5.5, D contains a free $R^\infty m$ self similar subcomb D' with root r'. Then there exists an endpoint e of D' such that $|R^\infty([r',e]| = \infty$. Again by Proposition 5.6 there exists a subarc $[q',p']$ such that $|R([q',p'])| = \infty$, the the set of limit points of $R([q',p'])$ is $\{q'\}$ (or similarly $\{p'\}$) and $R^\infty((q',p']) = \varnothing$. Notice if $q,p \in R^\infty((q',p'])$, then there exist monotone maps

$$m_p\colon \mathcal{C}^*(p,[q',p'],D') \to D' \quad \text{and} \quad m_q\colon \mathcal{C}^*(q,[q',p'],D') \to D'.$$

But since $\mathcal{C}^*(p,[q',p'],D'), \mathcal{C}^*(q,[q',p'],D') \subset D'$, we have that $\mathcal{C}^*(p,[q',p'],D')$ and $\mathcal{C}^*(q,[q',p'],D')$ are monotonically equivalent. Hence,

$$\left\{ \mathcal{C}^*\left(q,[q',p'],D'\right)\right\}_{q\in R^\infty((q',p'])}$$

is bqo. Also, if $q \in R((q',p']) - R^\infty((q',p']) = R^F((q',p'])$, then, by Proposition 5.3, $\mathcal{C}^*(q,[q',p'],D') \in \mathcal{T}_r$.

Order $R((q',p'])$ by $\{q_i\}_{i=1}^\infty$, where $q' < q_{i+1} < q_i \le p'$ in the natural ordering of $[q',p']$. Since $\mathcal{T}_r \cup \{D'\}$ is bqo, there exists an N such that $\{\mathcal{C}^*(q_i,[q',p'],D')\}_{i=N}^\infty$ is weakly monotonically ordered. Hence, the subdendrite strung by $[q',q_N]$ is a free, monotonically ordered harmonic comb. Hence, D is not monotonically isolated, by Theorem 5.1. ■

Corollary 5.8 *If X is a dendrite with a free countable comb, then X is not monotonically isolated.*

Proof Notice that in the proof of Theorem 5.7, we concluded that every countable comb has a free, weakly monotonically ordered harmonic comb. Hence, D is not monotonically isolated by Theorem 5.1. ■

6 Wild Combs

Let X be a wild comb with wild spine A. For each $p \in R(A)$, define $T_p = \mathcal{C}^*(p,A,X)$ and $\mathcal{T}_A^X = \{T_p \mid p \in R(A)\}$. If $p \in A - R(A)$, then define $T_p = \{p\}$. Suppose that X and Y are wild combs with respective spines A_X and A_Y. Then define $\mathcal{T}_{A_Y}^Y \lhd \mathcal{T}_{A_X}^X$ if for every $T_y \in \mathcal{T}_{A_Y}^Y$ and subarc $B \subset A_X$ such that $\overline{R(B)}$ is uncountable, there exists $T_x \in \mathcal{T}_{A_X}^X$ such that $T_y \le_r T_x$.

In this section we show that wild combs are not monotonically isolated by first showing that if $\mathcal{T}_{A_Y}^Y \lhd \mathcal{T}_{A_X}^X$, then there exists an onto monotone map $m\colon X \to Y$. Then the following cases are shown:

(a) If X is a wild comb with a perfect spine that contains a free harmonic comb, then X is not monotonically isolated by Theorem 5.1.
(b) If X is a wild comb with a perfect spine such that no perfect spine contains a free arc, then X is not monotonically isolated.
(c) If X is a wild comb with a perfect spine such that contains a free arc, then X is not monotonically isolated.
(d) It will be shown in the next section that if X is a wild comb that contains no perfect spine, then X is monotonically equivalent to D_3.

Proposition 6.1 *If $\mathcal{T}_{A_Y}^Y \lhd \mathcal{T}_{A_X}^X$ and $[p,q] \subset A_X$ such that $\overline{R([p,q])}$ is uncountable, then $\mathcal{T}_{A_Y}^Y \lhd \mathcal{T}_{[p,q]}^X$.*

Proof This follows directly from the definition of $\lhd$. ■

Lemma 6.2 *Let X be a wild comb. Then there exists a wild comb Y with a wild spine A_Y such that $\overline{R(A_Y)} = A_Y$ and a monotone map $m\colon X \to Y$.*

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Proof Let $A_X = [a,b]$ be a wild spine of X and let

$$\mathcal{A} = \big\{ [x,y] \subset A_X \mid \overline{R(A_X)} \cap (x,y) = \varnothing \text{ and if } [w,r] \subset A_X \text{ such that } [x,y] \text{is a proper subset of } [w,r], \text{ then } \overline{R(A_X)} \cap (w,r) \neq \varnothing \big\}.$$

Let $Y = X/\mathcal{A}$ be the dendrite such that each $[x,y] \in \mathcal{A}$ is identified with a point and let $m: X \to Y$ be the natural quotient map. Need to show that $A_Y = A_X/\mathcal{A}$ is an arc. Let $\mathcal{E}$ be the collection of endpoints of the elements of $\mathcal{A}$. Since $\mathcal{A}$ is countable, $\mathcal{E}$ must be countable. Thus, $\overline{R(A_X)} \cap A - \mathcal{E}$ is uncountable. So $A - \bigcup_{B \in \mathcal{A}}$ is uncountable and therefore A_Y is an arc. Since every open interval of A_Y must contain a ramification point of Y, $\overline{R(A_Y)} = A_Y$. ∎

Proposition 6.3 *Let I_X and I_Y be arcs, $\{x_i\}_{i=1}^{\infty} \subset I_X$ and $\{y_i\}_{i=1}^{\infty} \subset I_Y$ such that $x_i < x_j < x_k$ if and only if $y_i < y_j < y_k$. Suppose that $\{x_{i_j}\}_{j=1}^{\infty}$ is a subsequence such that*

(i) $x = \lim_{j\to\infty} x_{i_j}$,
(ii) *either $x_{i_j} < x$ for all j or $x_{i_j} > x$ for all j.*

Then $\lim_{j\to\infty} y_{i_j}$ exists.

Proof Without loss of generality, assume $x_{i_j} < x$. Let $y = \sup\{y_{i_j}\}_{j=1}^{\infty}$. Suppose that t is a limit point of $\{y_{i_j}\}_{j=1}^{\infty}$ less that y. Let $\epsilon = (1/3)(y-t)$. Then there exists j' and an increasing sequence $\{j(n)\}_{n=1}^{\infty}$ such that

(a) $y_{i_{j'}} \in (y-\epsilon, y]$,
(b) $y_{i_{j(n)}} \in (t-\epsilon, t+\epsilon)$ for all n.

Hence, $y_{i_{j(n)}} < y_{i_{j'}}$ for all n. It follows that $x_{i_{j(n)}} < x_{i_{j'}} < x$. Hence, $\{x_{i_{j(n)}}\}_{n=1}^{\infty}$ is a subsequence of $\{x_{i_j}\}_{j=1}^{\infty}$ that does not converge to x. This is a contradiction. Hence, $y = \lim_{j\to\infty} y_{i_j}$. ∎

Lemma 6.4 *Let I_X and I_Y be arcs, $\{x_i\}_{i=1}^{\infty} \subset I_X$ and $\{y_i\}_{i=1}^{\infty} \subset I_Y$ such that*

(i) $x_1 = \min \overline{\{x_i\}_{i=1}^{\infty}}$ *and* $y_1 = \min \overline{\{y_i\}_{i=1}^{\infty}}$,
(ii) $x_2 = \max \overline{\{x_i\}_{i=1}^{\infty}}$ *and* $y_2 = \max \overline{\{y_i\}_{i=1}^{\infty}}$,
(iii) *x_j is an isolated point of $\overline{\{x_i\}_{i=1}^{\infty}}$ for each j,*
(iv) *if $s < t$ are limit points of $\{y_i\}_{i=1}^{\infty}$, then $[s,t] \cap \{y_i\}_{i=1}^{\infty} \neq \varnothing$,*
(v) *$x_i < x_j < x_k$ if and only if $y_i < y_j < y_k$.*

Then there exists a monotone onto map $m: [x_1, x_2] \to [y_1, y_2]$ such that $m(x_i) = y_i$ for each i.

Proof First we must prove the following claim.

Claim If $\lim_{j\to\infty} x_{i_j}$ exists, then $\lim_{j\to\infty} y_{i_j}$ exists.

Let $x = \lim_{j\to\infty} x_{i_j}$ and note by (iii) that $x \notin \{x_i\}_{i=1}^{\infty}$. By Proposition 6.3 we may assume that there exists increasing sequences of natural numbers $\{\sigma(n)\}_{n=1}^{\infty}$ and $\{\tau(n)\}_{n=1}^{\infty}$ such that

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(a) $\{\sigma(n)\}_{n=1}^{\infty} \cup \{\tau(n)\}_{n=1}^{\infty} = \{i_j\}_{j=1}^{\infty}$
(b) $x_{\sigma(n)} < x < x_{\tau(n)}$ for all n.

By Proposition 6.3 there exists $s \le t$ such that $s = \lim_{n\to\infty} y_{\sigma(n)}$ and $t = \lim_{n\to\infty} y_{\tau(n)}$. Suppose that there exists j' such that $y_{i_{j'}} \in [s,t]$. Then $y_{\sigma(n)} \le y_{i_{j'}} \le y_{\tau(n)}$ for all n. It follows from (v) that $x_{\sigma(n)} \le x_{i_{j'}} \le x_{\tau(n)}$. Hence, $x_{i_{j'}} = x$, which is impossible. Thus, it follows from (iv) that $s = t = \lim_{j\to\infty} y_{i_j}$.

Let

$$\Phi = \{y_j \mid (y_j - \epsilon, y_j) \cap \{y_i\}_{i=1}^{\infty} \neq \varnothing\},$$
$$\Lambda = \{y_j \mid (y_j, y_j + \epsilon) \cap \{y_i\}_{i=1}^{\infty} \neq \varnothing\}$$

be the elements of $\{y_i\}_{i=1}^{\infty}$ that are also respectively right-hand and left-hand limit points of $\{y_i\}_{i=1}^{\infty}$. Notice that it follows from (iv) that each component of $[y_1, y_2] - \overline{\{y_i\}_{i=1}^{\infty}}$ must be of the form (s_k, y_k), (y_i, t_i) or (y_i, y_k) for some i, k where s_k and t_i are limit points of $\{y_i\}_{i=1}^{\infty}$. Hence, it follows that each component of $[x_1, x_2] - \overline{\{x_i\}_{i=1}^{\infty}}$ must be one of the following forms:

(a) (x_i, x_k) if (y_i, y_k) is a component of $[y_1, y_2] - \overline{\{y_i\}_{i=1}^{\infty}}$ for the same i, k. Here $m(x)$ will map $[x_i, x_k]$ linearly onto $[y_i, y_k]$ such that $m(x_i) = y_i$ and $m(x_k) = y_k$.
(b) (s'_k, x_k), where s'_k is a limit point of $\{x_i\}_{i=1}^{\infty}$ corresponding to the component (s_k, y_k) of $[y_1, y_2] - \overline{\{y_i\}_{i=1}^{\infty}}$. Here $m(x)$ will map $[s'_k, x_k]$ linearly onto $[s_k, y_k]$ such that $m(s'_k) = s_k$ and $m(x_k) = y_k$.
(c) (x_i, t'_i), where t'_i is a limit point of $\{x_i\}_{i=1}^{\infty}$ corresponding to the component (t_i, y_i) of $[y_1, y_2] - \overline{\{y_i\}_{i=1}^{\infty}}$. Here $m(x)$ will map $[x_i, t'_i]$ linearly onto $[y_i, t_i]$ such that $m(x_i) = y_i$ and $m(t'_i) = t_i$.
(d) (α_j, x_j), where α_j is a limit point of $\{x_i\}_{i=1}^{\infty}$ and $y_j \in \Phi$. Here $m([\alpha_j, x_j]) = y_j$.
(e) (x_j, β_j), where β_j is a limit point of $\{x_i\}_{i=1}^{\infty}$ and $y_j \in \Lambda$. Here $m([x_j, \beta_j]) = y_j$.

Then it is easy to check that $m\colon [x_1, x_2] \to [y_1, y_2]$ is monotone. ∎

Lemma 6.5 *Let X and Y be wild combs with respective spines A_X and A_Y such that $\mathcal{T}^Y_{A_Y} \lhd \mathcal{T}^X_{A_X}$. Then there exists a monotone onto map $m\colon X \to Y$.*

Proof Let $A_X = [a, b]$ and $A_Y = [c, d]$. Also define

$$\mathcal{L} = \Big\{ s \in A_Y \mid (s,t) \in \mathrm{Com}\big(A_Y - \overline{R(A_Y)}\big)\Big\},$$
$$\mathcal{R} = \Big\{ t \in A_Y \mid (s,t) \in \mathrm{Com}\big(A_Y - \overline{R(A_Y)}\big)\Big\}.$$

Notice that $\mathcal{L} \cup \mathcal{R}$ is countable. So let

$$\{y_i\}_{i=1}^{\infty} = R(A_Y) \cup \mathcal{L} \cup \mathcal{R} \cup \Big(\overline{R(A_Y)} \cap \{c, d\}\Big) = Q_Y$$

such that $y_1 = \min Q_Y$ and $y_2 = \max Q_Y$. Note that if $y_i \notin R(A_Y)$, then $T^Y_{y_i} = \{y_i\} \preceq_r T^X_x$ for all $x \in R(A_X)$.

Since $\mathcal{T}^Y_{A_Y} \lhd \mathcal{T}^X_{A_X}$, there exists $x_1, x_2 \in R((a,b))$ and $\epsilon_1, \epsilon_2 > 0$ such that

(a) $a < x_1 < x_1 + \epsilon_1 < x_2 - \epsilon_2 < x_2 < b$,
(b) $T^Y_{y_1} \preceq_r T^X_{x_1}$ and $T^Y_{y_2} \preceq_r T^X_{x_2}$,
(c) $\overline{R((x_1 + \epsilon_1, x_2 - \epsilon_2))}$ is uncountable.

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Continuing inductively, suppose that for each $i \in \{1, \dots, N\}$, $x_i \in R([x_1, x_2])$ and $\epsilon_i > 0$ have been chosen such that

(a) $(x_i - \epsilon_i, x_i + \epsilon_i) \cap (x_k - \epsilon_k, x_k + \epsilon_k) = \varnothing$ when $i \neq k$,
(b) if $x_i + \epsilon_i < x_k - \epsilon_k$ then $\overline{R([x_i + \epsilon_i, x_k - \epsilon_k])}$ is uncountable,
(c) $y_i < y_j < y_k$ if and only if $x_i < x_j < x_k$,
(d) $T^Y_{y_i} \leq_r T^X_{x_i}$.

Let $y_p = \max_{1 \le i \le N}\{y_i \mid y_i < y_{N+1}\}$ and $y_q = \min_{1 \le i \le N}\{y_i \mid y_i > y_{N+1}\}$. Then there exists $x_{N+1} \in R((x_p + \epsilon_p, x_q - \epsilon_q))$ and $\epsilon_{N+1} > 0$ such that

(a) $T^Y_{y_{N+1}} \leq_r T^X_{x_{N+1}}$,
(b) $x_p + \epsilon_p < x_{N+1} - \epsilon_{N+1} < x_{N+1} + \epsilon_{N+1} < x_q - \epsilon_q$,
(c) $\overline{R((x_p + \epsilon_p, x_{N+1} - \epsilon_{N+1}))}$, and $\overline{R((x_{N+1} + \epsilon_{N+1}, x_q - \epsilon_q))}$ are uncountable.

Notice that for every j, x_j is an isolated point of $\{x_i\}_{i=1}^{\infty}$ and that if $p < q$ are limit points of Q_Y, then $[p, q] \cap Q_Y \neq \varnothing$. Otherwise, (p, q) would be a component of $\overline{R(A_Y)}$ such that $p \notin \mathcal{L}$ and $q \notin \mathcal{R}$, which are both impossible. Therefore, by Lemma 6.4, there exists a monotone onto map $m: [x_1, x_2] \to [y_1, y_2]$ such that $m(x_i) = y_i$. Furthermore, m can be easily extended to a monotone onto map $\widehat{m}: [a, b] \to [c, d]$ such that $\widehat{m}(x) = m(x)$ whenever $x \in [x_1, x_2]$.

For each i let $m_i: T^X_{x_i} \to T^Y_{y_i}$ be an onto monotone map such that $m_i(x_i) = y_i$. Define $f: X \to Y$ by

$$f(z) = \begin{cases} m_i(z) & \text{if } z \in T^X_{x_i}, \\ \widehat{m}(x) & \text{if } z \in T^X_x \text{ where } x \in R([A_X]) - \{x_i\}_{i=1}^{\infty}, \\ \widehat{m}(z) & \text{if } z \in A_X = [a, b]. \end{cases}$$

Since $\widehat{m}$ and each m_i are monotone, f must be monotone. ∎

Let X be a wild comb with wild spine A. A is *perfect* if for every $y \in R(A)$ and arc $B \subset A$ such that $\overline{R(B)}$ is uncountable, there exists $x \in R(B)$ such that $T_y \leq_r T_x$.

Lemma 6.6 *Let X be a wild comb with spine A such that $\mathcal{T}^X_A$ is bqo. Then X has a perfect spine.*

Proof Suppose that X has no perfect spine. Then there exists a $x_1 \in X$ and an arc $A_1 \subset X$ such that $\overline{R(A_1)}$ is uncountable and $T_{x_1} \not\leq_r T_a$ for all $a \in A_1$. Since A_1 is not perfect there exists $x_2 \in A_1$ and an arc $A_2 \subset A_1$ such that $\overline{R(A_2)}$ is uncountable and $T_{x_i} \not\leq_r T_a$ for all $i \in \{1, 2\}$ and $a \in A_2$.

Continuing inductively, suppose that $x_1, \dots, x_{n-1}$ and A_n have be found such that $T_{x_i} \not\leq_r T_a$ for all $i \in \{1, \dots, n-1\}$ and $a \in A_n$, where $\overline{R(A_n)}$ is uncountable. Since A_n is not perfect, there exists $x_n \in A_n$ and an arc $A_{n+1} \subset A_n$ such that $\overline{R(A_{n+1})}$ is uncountable and $T_{x_i} \not\leq_r T_a$ for all $i \in \{1, \dots, n\}$ and $a \in A_{n+1}$. Thus, $\{T_{x_i}\}_{i=1}^{\infty}$ either contains an infinite anti-chain or an infinite strictly decreasing sequence. Either contradicts the fact that $\mathcal{T}^X_A$ is bqo and hence not wqo. ∎

Lemma 6.7 *Suppose that X is a wild comb that*

(i) *contains no harmonic comb,*

(ii) *has a perfect spine A_X that contains a free arc $[a,b]$.*

Then X is not monotonically isolated.

Proof Let H be a simple harmonic comb with spine A_H and $Y = X \cup H$, where $A_H = [a,b]$ and A_Y is the corresponding spine for Y. Since Y contains a free harmonic comb and X does not, then they cannot be homeomorphic. Since $X \subset Y$, there exists a monotone map from $m: Y \to X$. Also, if I is an arc and T is any dendrite, we have that $I \le T$. So it follows that $\mathcal{T}^Y_{A_Y} \lhd \mathcal{T}^X_{A_X}$. Thus, by Lemma 6.5, there exists a monotone map $m': X \to Y$. Hence X and Y are monotonically equivalent. ∎

Lemma 6.8 *Suppose that X is a wild comb with a perfect spine such that every perfect spine contains no free arc. Then X is not monotonically isolated.*

Proof Let $[a,b]$ be a perfect spine in X and note that if $[p,q] \subset [a,b]$, then $\mathcal{T}^X_{[a,b]} \lhd \mathcal{T}^X_{[p,q]}$ by Proposition 6.1. Let $[c,d] \subset (a,b)$. Define $Y \subset X$ such that for each $r \in \overline{R}([c,d])$ identify T_r with r. Clearly, this defines a monotone map $m: X \to Y$. Conversely, since $\mathcal{T}^X_{[a,b]} \lhd \mathcal{T}^X_{[d,b]} = \mathcal{T}^Y_{[d,b]}$, it follows from Lemma 6.5 there is a monotone map $m': Y \to X$. So X and Y are monotonically equivalent. Since Y contains a perfect spine with a free arc and X does not, they cannot be homeomorphic. ∎

Let

$$R^W\big([x,y]\big) = \big\{ q \in R\big([x,y]\big) \mid \mathcal{C}^*\big(q,[x,y]\big) \text{ is a wild comb with root } q \big\}.$$

Theorem 6.9 *If D is a wild comb with a perfect spine, then D is not monotonically isolated.*

Proof If D has a free harmonic comb, then D is not monotonically isolated by Theorem 5.1. If D contains a free arc but no harmonic comb, then D is not monotonically isolated by Lemma 6.7. If D contains no free arc, then it is not monotonically isolated by Lemma 6.8. ∎

7 Dendrites that are Monotonically Equivalent to D_ω.

Lemma 7.1 *Suppose that D is a wild comb that contains no perfect spine and no free countable comb. Then if $[x,y]$ is an arc such that $\overline{R([x,y])}$ is uncountable, it follows that $\overline{R^W([x,y])}$ is uncountable.*

Proof For the purpose of a contradiction, suppose that $\overline{R([x,y])}$ is uncountable and $\overline{R^W([x,y])}$ is countable. Then there there exist a subarc $[x',y']$ such that $\overline{R([x',y'])}$ is uncountable and $\overline{R^W([x',y'])}$ is empty. Since $\mathcal{C}^*(q,[x',y'])$ cannot be either a countable comb or a wild comb for any $q \in R([x',y'])$, it follows that $\mathcal{C}^*(q,[x',y']) \in \mathcal{T}_r$ for each $q \in R([x',y'])$. Since $\mathcal{T}_r$ is bqo, $[x',y']$ contains a perfect spine, which is a contradiction. ∎

Recall that a wild spine $[x,y]$ is *archimedian* if $[x,y]$ is a maximal arc in D and if for every $p,q \in R([x,y])$ such that $p < q$ (in the natural ordering on $[x,y]$), there

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exists $r \in R([x,y])$ such that $p < r < q$. A comb is archimedian if it contains an archimedian wild spine.

Theorem 7.2 *Suppose that D is a wild comb with the property that if $\overline{R([x,y])}$ is uncountable, then $\overline{R^W([x,y])}$ is uncountable. Then D is monotonically equivalent to D_3.*

Proof First we need to show the following claim:

Claim If $[x,y]$ is an arc in D such that $\overline{R([x,y])}$ is uncountable, then there exists an archimedian comb $A \subset D$ with spine $[x,y]$ such that if $(p,q]$ is a component of $A - [x,y]$ then $\overline{R_D((p,q])}$ is uncountable.

Since $\overline{R^W([x,y])}$ is uncountable, there exists $a(x,y) \subset R^W([x,y])$ with the property that if $v,w \in a(x,y)$ such that $v < w$ (in the natural ordering on $[x,y]$), then there exists $r \in a(x,y)$ such that $v < r < w$. Since $\mathcal{C}^*(t,[x,y])$ is a wild comb, for each $t \in a(x,y)$ there exists an endpoint e_t of $\mathcal{C}^*(t,[x,y])$ such that $\overline{R((t,e_t])}$ is uncountable. Let $A = [x,y] \cup \bigcup_{t\in a(x,y)}[t,e_t]$, and the claim follows.

Now suppose that in fact $\overline{R([x,y])}$ is uncountable and let $A_1 \subset D$ be an archimedian comb with spine $[x,y] = A_0$ and such that $\overline{R_D((p,q])}$ uncountable for each component $(p,q]$ of $A_1 - [x,y]$. Continuing inductively, suppose that A_{n-1} and A_n have been found with the properties

(a) $A_{n-1} \subset A_n$,
(b) each component of $A_n - A_{n-1}$ is an arc,
(c) if $(p,q]$ is a component of $A_n - A_{n-1}$, then $\overline{R|_D((p,q])}$ is uncountable.

It follows that if $(p,q]$ is a component of $A_n - A_{n-1}$, then there exists an archimedian comb $A_{p,q} \subset D$ with spine $[p,q]$ and such that if $(s,t]$ is a component of $A_{p,q} - [p,q]$, then $\overline{R_D((s,t])}$ is uncountable. Let $A = \overline{\bigcup_{n=1}^{\infty} A_n}$. If we shrink each free arc of A to a point, we have a monotone map onto D_3. Since $A \subset D$, it follows that there is a monotone map from D onto D_3 and hence $D \le D_3$. ∎

8 Main Theorem

In this section we combine our results to prove the main theorem.

Theorem 8.1 *If D is a dendrite with an infinite number of ramification points, then D is not monotonically isolated.*

Proof If D has an infinite number of ramification points, then D falls into one of the following categories:

(a) D contains no arc with an infinite number of ramification points.
Then D is an infinite tree and is not monotonically isolated by Theorem 4.8.
(b) D contains some arc with an infinite number of ramification points.
  (b.1) D contains a free countable comb.
  Then D is not monotonically isolated by Corollary 5.8.
  (b.2) D does not contain a free countable comb.

Then D is a wild comb.

(b.2.1) D contains a perfect spine.
Then D is not monotonically isolated by Theorem 6.9.

(b.2.2) D contains no perfect spine.
Then D is a wild comb with the property that if $\overline{R([x,y])}$ is uncountable, then $\overline{R^W([x,y])}$ is uncountable by Lemma 7.1. It follows from Theorems 7.2 and 2.2 that D is not monotonically isolated. ∎

Theorem 1.1 now follows from Theorems 1.2 and 8.1.

References

[1] R. Camerlo, U. B. Darji, and A. Marcone, *Classification problems in continuum theory.* Trans. Amer. Math. Soc. **357**(2005), no. 11, 4301–4328. http://dx.doi.org/10.1090/S0002-9947-05-03956-5

[2] J. J. Charatonik, *Monotone mappings of universal dendrites.* Topology Appl. **38**(1991), no. 2, 163–187. http://dx.doi.org/10.1016/0166-8641(91)90083-X

[3] R. Laver, *Order types and well-quasi-orderings.* Thesis (Ph.D.), University of California, Berkeley, 1969.

[4] ———, *On Fraïssé's order type conjecture.* Ann. of Math. (2) **93**(1971), 89–111. http://dx.doi.org/10.2307/1970754

[5] V. Martínez de la Vega and J. Martínez Montejano, *Open problems on dendroids.* In: Open Problems in Topology. II, Elsevier, 2007, pp. 319–334.

[6] K. Menger, *Kurventheorie*, Teubner, Leipzig, 1932.

[7] C. St. J. A. Nash-Williams, *On well-quasi-ordering infinite trees.* Proc. Cambridge Philos. Soc. **61**(1965), 697–720. http://dx.doi.org/10.1017/S0305004100039062

[8] R. Rado, *Partial well-ordering of sets of vectors.* Mathematika **1**(1954), 89–95. http://dx.doi.org/10.1112/S0025579300000565

[9] T. Ważewski, *Sur les courbes de Jordan ne renferment aucune courbe simple fermée de Jordan*. Ann. Soc. Pol. Math **2**(1923), 49–170.

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