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Large deviation principle for piecewise monotonic maps with density of periodic measures

YONG MOO CHUNG† and KENICHIRO YAMAMOTO‡

† *Department of Applied Mathematics, Hiroshima University, Higashi-Hiroshima 739-8527, Japan*
(e-mail: chung@amath.hiroshima-u.ac.jp)
‡ *Department of General Education, Nagaoka University of Technology, Nagaoka 940-2188, Japan*
(e-mail: k_yamamoto@vos.nagaokaut.ac.jp)

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Abstract. We show that a piecewise monotonic map with positive topological entropy satisfies the level-2 large deviation principle with respect to the unique measure of maximal entropy under the conditions that the corresponding Markov diagram is irreducible and that the periodic measures of the map are dense in the set of ergodic measures. This result can apply to a broad class of piecewise monotonic maps, such as monotonic mod one transformations and piecewise monotonic maps with two monotonic pieces.

Key words: large deviation, piecewise monotonic map, periodic measure
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1. *Introduction*

Let X be a metrizable space and $T: X \to X$ be a Borel measurable map. We denote by $\mathcal{M}(X)$ the set of all Borel probability measures on X endowed with weak-* topology, by $\mathcal{M}_T(X) \subset \mathcal{M}(X)$ the set of all T-invariant ones, and by $\mathcal{M}^e_T(X) \subset \mathcal{M}_T(X)$ the set of ergodic ones. We say that (X, T) satisfies the *(level-2) large deviation principle* with a *reference measure* $m \in \mathcal{M}(X)$ if there exists a lower semi-continuous function $\mathcal{J}: \mathcal{M}(X) \to [0, \infty]$, called a *rate function*, such that

$$\limsup_{n\to\infty} \frac{1}{n} \log m(\{x \in X : \delta^T_n(x) \in \mathcal{K}\}) \leq -\inf_{\mathcal{K}} \mathcal{J}$$

holds for any closed set $\mathcal{K} \subset \mathcal{M}(X)$ and

$$\liminf_{n\to\infty} \frac{1}{n} \log m(\{x \in X : \delta_n^T(x) \in \mathcal{U}\}) \geq -\inf_{\mathcal{U}} \mathcal{J}$$

holds for any open set $\mathcal{U} \subset \mathcal{M}(X)$. Here, $\delta_n^T : X \to \mathcal{M}(X)$ is defined by $\delta_n^T(x) := (1/n)\sum_{j=0}^{n-1} \delta_{T^j(x)}$, where δ_y signifies the Dirac mass at a point $y \in X$. We say that $\mu \in \mathcal{M}(X)$ is a *periodic measure* if there exist $x \in X$ and $n > 0$ such that $T^n(x) = x$ and $\mu = \delta_n^T(x)$ hold. Then it is clear that $\mu \in \mathcal{M}_T^e(X)$. We denote by $\mathcal{M}_T^p(X) \subset \mathcal{M}_T^e(X)$ the set of all periodic measures on X.

Henceforth, let $X = [0, 1]$ be the unit interval, and $T : X \to X$ be a *piecewise monotonic map*; that is, there exist an integer $k > 1$, and $0 = a_0 < a_1 < \cdots < a_k = 1$, which we call *critical points*, such that $T|_{(a_{j-1},a_j)}$ is continuous and strictly monotone for each $1 \leq j \leq k$. If $T|_{(a_{j-1},a_j)}$ is increasing (respectively decreasing) for $1 \leq j \leq k$, then we call T a *piecewise increasing* (respectively *decreasing*) map. Throughout this paper, we further assume the following conditions for a piecewise monotonic map T.

- $\bigcup_{n\geq 0} T^{-n}\{a_0, a_1, \ldots, a_k\}$ is dense in X. In other words, the partition $\{(a_{j-1}, a_j) : 1 \leq j \leq k\}$ by the monotone intervals is a generator for the dynamical system (X, T).
- The topological entropy $h_{\text{top}}(X, T)$ of T is positive (see [2, Ch. 9] for the definition and basic properties of topological entropy for piecewise monotonic maps).

It is proved in [11] that there exists a measure $m = m_T \in \mathcal{M}_T^e(X)$ of maximal entropy for T; that is, the metric entropy of m coincides with $h_{\text{top}}(X, T)$. In this study, we investigate whether the large deviation principle holds for a piecewise monotonic map with a measure of maximal entropy as reference. In such a situation, it was shown in [24, 27] that the large deviation principle holds if the map has the specification property (see [3, §1] for the definition of the specification property). Moreover, Pfister and Sullivan have proved in [20] that all β-transformations satisfy the large deviation principle, while the specification property holds only for a set of $\beta > 1$ of Lebesgue measure zero.

The purpose of this paper is to provide a criterion for satisfying the large deviation principle using common concepts for piecewise monotonic maps. More precisely, we assume the following natural conditions for piecewise monotonic maps $T : X \to X$.

- *Irreducibility (IR).* The Markov diagram $(\mathcal{D}_T, \to)$ of T is irreducible (see §2 for the definitions).
- *Density of periodic measures (DP).* The set $\mathcal{M}_T^p(X)$ of periodic measures is dense in $\mathcal{M}_T^e(X)$.

Condition (IR) implies transitivity of the map, and that the measure of maximal entropy is unique (see [11]). Moreover, condition (DP) holds for any map with the specification property (see [22]). Our main result is the following theorem.

THEOREM A. *Let $T : X \to X$ be a piecewise monotonic map satisfying (IR) and (DP), and let m be the unique measure of maximal entropy. Then (X, T) satisfies the large deviation principle with m as reference, and the rate function $\mathcal{J} : \mathcal{M}(X) \to [0, \infty]$ is expressed by (3.5). Moreover, if we assume that T is piecewise increasing and either left* *or right continuous, then*

$$\mathcal{J}(\mu) = \begin{cases} h_{\mathrm{top}}(X, T) - h_T(\mu), & \mu \in \mathcal{M}_T(X), \\ \infty, & \textit{otherwise}. \end{cases} \tag{1.1}$$

Here, $h_T(\mu)$ denotes the metric entropy of $\mu \in \mathcal{M}_T(X)$.

We remark that Theorem A can be generalized for the case where the reference measure m is the equilibrium state of a potential discussed in [14]. It will be treated in a forthcoming paper.

The contraction principle [7] gives the following formula for fluctuations of time averages of continuous observables.

COROLLARY B. *Let $T : X \to X$ be as in Theorem A. Then, for any continuous function $\varphi : X \to \mathbb{R}$ and a closed interval $J \subset \mathbb{R}$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \log m\bigg(\bigg\{x \in X : \frac{1}{n}\sum_{j=0}^{n-1} \varphi(T^j(x)) \in J\bigg\}\bigg) = -\inf_J \mathcal{J}_\varphi$$

where $\mathcal{J}_\varphi : \mathbb{R} \to [0, \infty]$ denotes the (level-1) rate function for φ given by $\mathcal{J}_\varphi(\alpha) = \inf\{\mathcal{J}(\mu) : \int \varphi \, d\mu = \alpha\}$.

Remark 1.1. If T is continuous, then condition (IR) implies that (some iteration of) the map has the specification property (see, for example, [3, Theorem 1.1]), and then the large deviation principle holds (see [25, 27]). By contrast, in the case of discontinuity, we emphasize that there are many examples of piecewise monotonic maps that satisfy (IR) and (DP) but do not have the specification property (see applications below).

Remark 1.2. Condition (IR) is slightly stronger than transitivity. It is highly likely that Theorem A remains true if we assume transitivity instead of (IR). However, (IR) simplifies the argument used in this study for lifting measures from a subshift with a finite alphabet to a Markov shift with a countable alphabet (see the proof of Proposition 3.1). Hence, we avoided this generalization in this study. We provide a sufficient condition for (IR) in §4.

Remark 1.3. The density of periodic measures (DP) assumed in Theorem A has been studied by many researchers independently of the theory of large deviations. The condition holds for a broad class of piecewise monotonic maps (see [13, 15, 23], for example). However, as of this writing, there is no known transitive piecewise monotonic map without the density of periodic measures (this is an open problem posed by Hofbauer and Raith in [15]). Therefore, we hope that the result of our study will contribute to a complete description of the large deviation principle for transitive piecewise monotonic maps.

Theorem A can be applied to demonstrate the large deviation principle for the following classes of piecewise monotonic maps.

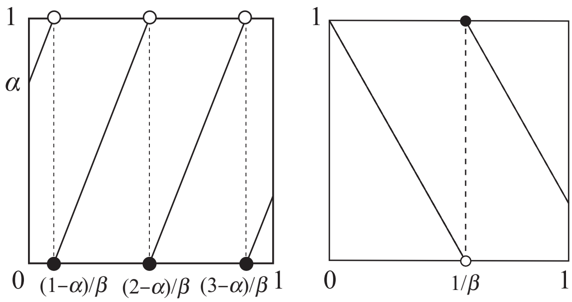

FIGURE 1. Graphs of $T_{\alpha,\beta}$ (left) and $T_{-\beta}$ (right).

- *β-transformations (piecewise increasing).* The β-transformation $T_\beta\colon X \to X$ with $\beta > 1$ was introduced by Rényi [21] and defined by

$$T_\beta(x) = \begin{cases} \beta x \pmod 1, & x \neq 1, \\ \lim\limits_{y \to 1-0} (\beta y \pmod 1), & x = 1. \end{cases}$$

  Then T_β satisfies (IR) and (DP) for any $\beta > 1$ (see [9, 23]).
- *Linear mod 1 transformations (piecewise increasing).* The linear mod one transformation $T_{\alpha,\beta}\colon X \to X$ with $\beta > 1$ and $0 \leq \alpha < 1$ was introduced by Parry [19] and defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \beta x + \alpha \pmod 1, & x \neq 1, \\ \lim\limits_{y \to 1-0} (\beta y + \alpha \pmod 1), & x = 1. \end{cases} \tag{1.2}$$

  In Example 4.1, we prove that $T_{\alpha,\beta}$ satisfies (IR) and (DP) for any $0 \leq \alpha < 1$ and $\beta > 2$.
- *$(-\beta)$-transformations (piecewise decreasing).* The $(-\beta)$-transformation $T_{-\beta}\colon X \to X$ with $\beta > 1$ was introduced by Ito and Sadahiro [16] and defined by

$$T_{-\beta}(x) = -\beta x + \lfloor \beta x \rfloor + 1, \tag{1.3}$$

  where $\lfloor \xi \rfloor$ denotes the largest integer that is no more than ξ. We show that $T_{-\beta}$ satisfies (IR) and (DP) for any $(1 + \sqrt{5})/2 < \beta < 2$ in Example 4.2.
- *Maps with two monotonic pieces (may be neither piecewise increasing nor decreasing).* A piecewise monotonic map $T\colon X \to X$ has *two monotonic pieces* if there is $0 < a < 1$ such that both $T|_{(0,a)}$ and $T|_{(a,1)}$ are strictly monotonic and continuous. In this case, condition (DP) was demonstrated in [15, Theorem 2] for when the map satisfies condition (IR). A typical example which has two monotonic pieces is the class of one-dimensional Lorenz maps. For condition (IR) for Lorenz maps, we refer to [1, Ch. 3] and [18].

The graphs of linear mod one transformations $T_{\alpha,\beta}$ and $(-\beta)$-transformations $T_{-\beta}$ are plotted in Figure 1. These transformations do not satisfy the specification property for Lebesgue almost every parameter (see [3]), and it was not known whether the

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large deviation principle holds. As mentioned before, we apply Theorem A to these transformations.

The remainder of this paper is organized as follows. In §2 we establish our definitions and prepare several lemmas. Subsequently, we present a proof of Theorem A in §3 and apply it to concrete examples in §4.

2. *Preliminaries*

2.1. *Symbolic dynamics.* For a finite or countable set A, we denote by $A^{\mathbb{N}}$ the one-sided infinite product of A equipped with the product topology of the discrete topology of A. Let σ be the shift map on $A^{\mathbb{N}}$ (that is, $(\sigma(\omega))_i = \omega_{i+1}$ for each $i \in \mathbb{N}$ and $\omega = (\omega_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}}$). When a subset Σ of $A^{\mathbb{N}}$ is σ-invariant and closed, we call it a *subshift* and call A the *alphabet* of Σ. When Σ is of the form

$$\Sigma = \{(\omega_i)_{i\in\mathbb{N}} \in A^{\mathbb{N}} : M_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in \mathbb{N}\}$$

with a matrix $M = (M_{ij})_{(i,j)\in A^2}$, each entry of which is 0 or 1, we call Σ a *Markov shift*. To emphasize the dependence of Σ on M, it is denoted by Σ_M, and M is called the *adjacency matrix* of Σ_M. In a similar manner, we define the shift map, subshift, and Markov shift for the two-sided infinite product $A^{\mathbb{Z}}$ of A.

For a subshift Σ on an alphabet A, let $[u] = \{(\omega_i) \in \Sigma : u = \omega_1 \cdots \omega_n\}$ for each $u \in A^n$, $n \geq 1$, and set $\mathcal{L}(\Sigma) = \{u \in \bigcup_{n\geq 1} A^n : [u] \neq \emptyset\}$. We also denote $\mathcal{L}_n(\Sigma) := \{u \in \mathcal{L}(\Sigma) : |u| = n\}$ for $n \geq 1$, where $|u|$ denotes the length of u that is, $|u| = n$ if $u = u_1 \cdots u_n \in A^n$. For $u = u_1 \cdots u_m$ and $v = v_1 \cdots v_n$ in $\mathcal{L}(\Sigma)$, we denote $uv = u_1 \cdots u_m v_1 \cdots v_n$. Finally, we say that Σ is *transitive* if, for any $u, v \in \mathcal{L}(\Sigma)$, we can find $w \in \mathcal{L}(\Sigma)$ such that $uwv \in \mathcal{L}(\Sigma)$ holds. For the rest of this paper, we denote by $h_{\mathrm{top}}(\Sigma, \sigma)$ the *topological entropy* of the restriction $\sigma : \Sigma \to \Sigma$ of the shift map σ to a subshift Σ, and by $h_\sigma(\mu)$ the *metric entropy* of $\mu \in \mathcal{M}_\sigma(\Sigma)$.

2.2. *Markov diagram.* Let $X = [0, 1]$ and $T : X \to X$ be a piecewise monotonic map with critical points $0 = a_0 < a_1 < \cdots < a_k = 1$. Let $X_T := \bigcap_{n=0}^{\infty} T^{-n}(\bigcup_{j=1}^{k}(a_{j-1}, a_j))$, and define the *coding map* $\mathcal{I} : X_T \to \{1, \ldots, k\}^{\mathbb{N}}$ by

$$(\mathcal{I}(x))_i = j \quad \text{if and only if } T^{i-1}(x) \in (a_{j-1}, a_j),$$

which is injective since the partition $\{(a_{j-1}, a_j) : 1 \leq j \leq k\}$ is a generator. We denote the closure of $\mathcal{I}(X_T)$ in $\{1, \ldots, k\}^{\mathbb{N}}$ by Σ_T^+. Then Σ_T^+ is a subshift, and (Σ_T^+, σ) is called the *coding space* of (X, T).

In what follows, we define the Markov diagram, introduced by Hofbauer [11], which is a countable oriented graph with subsets of Σ_T^+ as vertices. Let $C \subset \Sigma_T^+$ be a closed subset with $C \subset [j]$ for some $j \in \{1, \ldots, k\}$. We say that a non-empty closed subset $D \subset \Sigma_T^+$ is a *successor* of C if $D = [l] \cap \sigma(C)$ for some $l \in \{1, \ldots, k\}$. The expression $C \to D$ denotes that D is a successor of C. We now define a set $\mathcal{D}_T$ of vertices by induction. First, we set $\mathcal{D}_0 := \{[1], \ldots, [k]\}$. If $\mathcal{D}_n$ is defined for $n \geq 0$, then we set

$$\mathcal{D}_{n+1} := \mathcal{D}_n \cup \{D : D \text{ is a successor for some } C \in \mathcal{D}_n\}.$$

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We note that $\mathcal{D}_n$ is a finite set for each $n \geq 0$ since the number of successors of any closed subset of Σ_T^+ is at most k by definition. Finally, we set

$$\mathcal{D}_T := \bigcup_{n \geq 0} \mathcal{D}_n.$$

We call the oriented graph $(\mathcal{D}_T, \to)$ the *Markov diagram* of T. The Markov diagram $(\mathcal{D}_T, \to)$ is *irreducible* if, for any $C, D \in \mathcal{D}_T$, there exist $C_1, \ldots, C_n \in \mathcal{D}_T$ such that $C = C_1 \to \cdots \to C_n = D$. For notational simplicity, we use the expression $\mathcal{D}$ instead of $\mathcal{D}_T$ if no confusion arises. We remark that

$$\mathcal{D} = \{\sigma^{|u|-1}[u] : u \in \mathcal{L}(\Sigma_T^+)\}$$

holds, and that $\sigma^{|u|}[uj]$ is a successor of $\sigma^{|u|-1}[u]$ for each pair $u \in \mathcal{L}(\Sigma_T^+)$ and $j \in \{1, \ldots, k\}$ with $uj \in \mathcal{L}(\Sigma_T^+)$.

For a subset $\mathcal{C} \subset \mathcal{D}$, we define a matrix $M(\mathcal{C}) = (M(\mathcal{C})_{C,D})_{(C,D)\in\mathcal{C}^2}$ by

$$M(\mathcal{C})_{C,D} = \begin{cases} 1, & \text{C} \to \text{D}, \\ 0, & \text{otherwise}. \end{cases}$$

Then $\Sigma_{M(\mathcal{C})} = \{(C_i)_{i\in\mathbb{Z}} \in \mathcal{C}^{\mathbb{Z}} : C_i \to C_{i+1}, i \in \mathbb{Z}\}$ is a two-sided Markov shift with a countable alphabet $\mathcal{C}$ and an adjacency matrix $M(\mathcal{C})$. For notational simplicity, we write $\Sigma_{\mathcal{C}}$ instead of $\Sigma_{M(\mathcal{C})}$ for the remainder of this paper. It is clear that $\Sigma_{\mathcal{D}}$ is transitive if and only if $(\mathcal{D}_T, \to)$ is irreducible, and it is known that the equality

$$h_{\text{top}}(\Sigma_{\mathcal{D}}, \sigma) = h_{\text{top}}(\Sigma_T^+, \sigma) = h_{\text{top}}(X, T)$$

holds (see [8]). Moreover, there is a relationship between the countable Markov shift $\Sigma_{\mathcal{D}}$ and a natural extension of the coding shift Σ_T^+. To be more precise, let us define a natural extension Σ_T of Σ_T^+ by

$$\Sigma_T := \{(\omega_i)_{i\in\mathbb{Z}} \in \{1, \ldots, k\}^{\mathbb{Z}} : \omega_i\omega_{i+1} \cdots \in \Sigma_T^+, i \in \mathbb{Z}\},$$

and a map $\Psi\colon \Sigma_{\mathcal{D}} \to \{1, \ldots, k\}^{\mathbb{Z}}$ by

$$\Psi((C_i)_{i\in\mathbb{Z}}) := (\omega_i)_{i\in\mathbb{Z}} \quad \text{for } (C_i)_{i\in\mathbb{Z}} \in \Sigma_{\mathcal{D}},$$

where $\omega_i \in \{1, \ldots, k\}$ is a unique integer such that $C_i \subset [\omega_i]$ holds for each $i \in \mathbb{Z}$. The following lemma, proved in [11], states that $(\Sigma_{\mathcal{D}}, \sigma)$ is topologically conjugate to (Σ_T, σ), except for 'small' sets (see also [4, Appendix]).

LEMMA 2.1. [11, Lemmas 2 and 3] *The map $\Psi\colon \Sigma_{\mathcal{D}} \to \{1, \ldots, k\}^{\mathbb{Z}}$ is continuous and satisfies $\sigma \circ \Psi = \Psi \circ \sigma$. Moreover, there exist two shift-invariant subsets $\overline{N} \subset \Sigma_{\mathcal{D}}$ and $N \subset \Sigma_T$ satisfying the following properties.*

- *We have $\Psi(\Sigma_{\mathcal{D}} \setminus \overline{N}) = \Sigma_T \setminus N$.*
- *The restriction map $\Psi\colon \Sigma_{\mathcal{D}} \setminus \overline{N} \to \Sigma_T \setminus N$ is bijective and bi-measurable.*
- *There is no periodic point in N.*
- *For any invariant measure $\mu \in \mathcal{M}_\sigma(\Sigma_T)$ with $\mu(N) = 1$, we have $h_\sigma(\mu) = 0$.*
- *For any invariant measure $\overline{\mu} \in \mathcal{M}_\sigma(\Sigma_{\mathcal{D}})$ with $\overline{\mu}(\overline{N}) = 1$, we have $h_\sigma(\overline{\mu}) = 0$.*

For the rest of this section, we assume that $(\mathcal{D}_T, \to)$ is irreducible. Then the following lemma holds.

LEMMA 2.2. ([**11**, Theorem 2(iii)] and [**12**, p. 377, Corollary 1(ii)]) *There is a unique pair of* $\overline{m} \in \mathcal{M}_\sigma^e(\Sigma_\mathcal{D})$ *and* $m^+ \in \mathcal{M}_\sigma^e(\Sigma_T^+)$ *of maximal entropy; that is,*

$$h_\sigma(\overline{m}) = h_\sigma(m^+) = h_{\mathrm{top}}(\Sigma_T^+, \sigma).$$

Moreover, the following properties hold for $\overline{m}$ *and* m^+.

(1) $m^+ = \overline{m} \circ (\Psi^+)^{-1}$, *where* $\Psi^+ := \pi \circ \Psi$ *and* $\pi : \Sigma_T \to \Sigma_T^+$ *is a natural projection;* $(\omega_i)_{i\in\mathbb{Z}} \mapsto (\omega_i)_{i\in\mathbb{N}}$.

(2) *There are two families* $\{L(C)\}_{C\in\mathcal{D}}$ *and* $\{R(C)\}_{C\in\mathcal{D}}$ *of real positive numbers satisfying the following properties:*
- $\overline{m}[C_1 \cdots C_n] = L(C_1)R(C_n)\exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\}$ *for any* $n \ge 1$ *and* $C_1 \cdots C_n \in \mathcal{L}_n(\Sigma_\mathcal{D})$;
- *both* $\sum_{C\in\mathcal{D}} L(C)$ *and* $\sup_{C\in\mathcal{D}} R(C)$ *are finite.*

Finally, we prove a weak Gibbs property for the unique measure of maximal entropy on Σ_T^+, which has a key role in obtaining the large deviation bound in §3.

LEMMA 2.3. *Let* $m^+ \in \mathcal{M}_\sigma^e(\Sigma_T^+)$ *be as in Lemma 2.2. Then, for any finite set* $\mathcal{F} \subset \mathcal{D}$, *we can find* $K = K_\mathcal{F} > 1$ *such that*

$$m^+[u] \le K\,\exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\} \quad (n \ge 1, u \in \mathcal{L}_n(\Sigma_T^+)),$$

$$m^+[u] \ge K^{-1}\,\exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\} \quad (n \ge 1, u \in \mathcal{L}_n(\Psi^+(\Sigma_\mathcal{F}))).$$

Proof. Let $\overline{m}$, $\{L(C)\}_{C\in\mathcal{D}}$ and $\{R(C)\}_{C\in\mathcal{D}}$ be as in Lemma 2.2. For a finite set $\mathcal{F} \subset \mathcal{D}$, we put

$$K = K_\mathcal{F} := \max\bigg\{2, LR, \Big(\min_{C,D\in\mathcal{F}} L(C)R(D)\Big)^{-1}\bigg\},$$

where we set $L := \sum_{C\in\mathcal{D}} L(C)$ and $R := \sup_{C\in\mathcal{D}} R(C)$. Note that $K < \infty$ by Lemma 2.2. Let $n \ge 1$ and $u = u_1 \cdots u_n \in \mathcal{L}_n(\Psi^+(\Sigma_\mathcal{F}))$. Then we can find a word $C_1 \cdots C_n \in \mathcal{L}_n(\Sigma_\mathcal{F})$ such that $[C_1 \cdots C_n] \subset (\Psi^+)^{-1}[u]$. Hence, it follows from Lemma 2.2 that

$$\begin{aligned} m^+[u] &\ge \overline{m}[C_1 \cdots C_n] = L(C_1)R(C_n)\exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\} \\ &\ge K^{-1}\exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\}. \end{aligned}$$

We note that $\Psi^+(\Sigma_\mathcal{D}) = \Sigma_T^+$ (see [**12**], for instance). For $n \ge 1$ and $u \in \mathcal{L}_n(\Sigma_T^+)$, we set

$$\mathcal{C} = \mathcal{C}(u) := \{C \in \mathcal{D} : [C] \cap (\Psi^+)^{-1}[u] \ne \emptyset\}.$$

Then, for any $C \in \mathcal{C}$, we can find a unique word $P(C) = C_1 \cdots C_n \in \mathcal{L}_n(\Sigma_\mathcal{D})$ such that $C_1 = C$ and $[C] \cap (\Psi^+)^{-1}[u] = [P(C)]$. It is easy to see that $\bigcup_{C\in\mathcal{D}}[P(C)] = (\Psi^+)^{-1}[u]$. Therefore, we have

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$$\begin{aligned} m[u] &\le \sum_{C\in C} \overline{m}[P(C)] \\ &\le \sum_{C\in\mathcal{D}} L(C)R\exp\{-nh_{\rm top}(\Sigma_T^+,\sigma)\} \\ &\le K\exp\{-nh_{\rm top}(\Sigma_T^+,\sigma)\}, \end{aligned}$$

which proves the lemma. □

3. *Proof of Theorem A*

In this section we present a proof of Theorem A. First, we show the following analogous result of Theorem A on coding spaces.

THEOREM C. *Let $T: X \to X$ be as in Theorem A, (Σ_T^+,σ) be a coding space of (X,T), and m^+ be the unique measure of maximal entropy on Σ_T^+. Then (Σ_T^+,σ) satisfies the large deviation principle with m^+ as reference, and the rate function $\mathcal{J}^+: \mathcal{M}(\Sigma_T^+) \to [0,\infty]$ is expressed by*

$$\mathcal{J}^+(\mu^+) = \begin{cases} h_{\rm top}(\Sigma_T^+,\sigma) - h_\sigma(\mu^+), & \mu^+ \in \mathcal{M}_\sigma(\Sigma_T^+), \\ \infty, & \textit{otherwise}. \end{cases} \tag{3.1}$$

Proof. To prove the theorem, it is sufficient to show

$$\limsup_{n\to\infty} \frac{1}{n}\log m^+((\delta_n^\sigma)^{-1}(\mathcal{K})) \le - \inf_{\mu^+\in\mathcal{K}\cap\mathcal{M}_\sigma(\Sigma_T^+)} (h_{\rm top}(\Sigma_T^+,\sigma) - h_\sigma(\mu^+)) \tag{3.2}$$

for any closed set $\mathcal{K} \subset \mathcal{M}(\Sigma_T^+)$, and

$$\liminf_{n\to\infty} \frac{1}{n}\log m^+((\delta_n^\sigma)^{-1}(\mathcal{U})) \ge - \inf_{\mu^+\in\mathcal{U}\cap\mathcal{M}_\sigma(\Sigma_T^+)} (h_{\rm top}(\Sigma_T^+,\sigma) - h_\sigma(\mu^+)) \tag{3.3}$$

for any open set $\mathcal{U} \subset \mathcal{M}(\Sigma_T^+)$. The upper bound (3.2) follows from the first inequality in Lemma 2.3 and [20, Theorem 3.2] (see also [6, §4]).

In what follows, we will show the lower bound (3.3). Let $\epsilon > 0$, $\mu^+ \in \mathcal{M}_\sigma(\Sigma_T^+)$ and $\mathcal{U}$ be an open neighborhood of μ^+ in $\mathcal{M}(\Sigma_T^+)$. To show (3.3), it is enough to show that

$$\liminf_{n\to\infty} \frac{1}{n}\log m^+((\delta_n^\sigma)^{-1}(\mathcal{U})) \ge h_\sigma(\mu^+) - h_{\rm top}(\Sigma_T^+,\sigma) - 2\epsilon.$$

In the following proposition, we use condition (DP) to show the entropy-density of ergodic measures, that is, any invariant measure can be approximated by ergodic ones with similar entropies. The entropy-density plays a key role in estimating the lower bounds on large deviations (see, for example, [20, §1]).

PROPOSITION 3.1. *There exist a finite set $\mathcal{F} \subset \mathcal{D}$ and $\rho^+ \in \mathcal{M}_\sigma^e(\Psi^+(\Sigma_{\mathcal{F}}))$ such that $\rho^+ \in \mathcal{U}$ and $h_\sigma(\rho^+) \ge h_\sigma(\mu^+) - \epsilon$.*

Remark 3.2. Recently, Takahasi [25] proved the entropy-density of ergodic measures with compact supports for countable Markov shifts. From this result, Proposition 3.1 follows

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if every invariant measure on Σ_T^+ can be lifted to $\Sigma_{\mathcal{D}}$. Unfortunately, there are unliftable measures in general (see, for example, [10, 17]). In the proof, to overcome this difficulty, we show that the set of liftable measures is entropy-dense in the set of invariant measures by using (DP).

Proof of Proposition 3.1. It follows from condition (DP) and [26, Theorem A] that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. If $h_\sigma(\mu^+) - \epsilon \le 0$, then we choose $\rho^+ \in \mathcal{M}_\sigma^p(\Sigma_T^+) \cap \mathcal{U}$ and take a periodic point $\omega \in \Sigma_T^+$ in the support of ρ^+. It follows from [12, Theorem 8] that there are finite vertices $C_1, \ldots, C_n \in \mathcal{D}$ such that $\Psi^+(\cdots C_1 \cdots C_n C_1 \cdots C_n \cdots) = \omega$ holds. We set $\mathcal{F} := \{C_1, \ldots, C_n\}$. Then it is clear that $\rho^+ \in \mathcal{M}_\sigma^e(\Psi^+(\Sigma_{\mathcal{F}}))$ and $h_\sigma(\rho^+) = 0 \ge h_\sigma(\mu^+) - \epsilon$. This suffices for the case $h_\sigma(\mu^+) - \epsilon \le 0$.

Henceforth, assume that $h_\sigma(\mu^+) - \epsilon > 0$. First, we define two pushforward maps $\pi_*\colon \mathcal{M}_\sigma(\Sigma_T) \to \mathcal{M}_\sigma(\Sigma_T^+)$ and $\Psi_*\colon \mathcal{M}(\Sigma_{\mathcal{D}}) \to \mathcal{M}(\Sigma_T)$ by $\pi_*(\vartheta) := \vartheta \circ \pi^{-1}$ for $\vartheta \in \mathcal{M}_\sigma(\Sigma_T)$, and $\Psi_*(\overline{\vartheta}) := \overline{\vartheta} \circ \Psi^{-1}$ for $\overline{\vartheta} \in \mathcal{M}(\Sigma_{\mathcal{D}})$. Then π_* is a homeomorphism and $h_\sigma(\vartheta) = h_\sigma(\pi_*(\vartheta))$ holds for any $\vartheta \in \mathcal{M}_\sigma(\Sigma_T)$ (see [8, Proposition 5.8]). Moreover, it follows from Lemma 2.1 that Ψ_* is continuous, the restriction map $\Psi_*\colon \mathcal{M}_\sigma(\Sigma_{\mathcal{D}} \setminus \overline{N}) \to \mathcal{M}_\sigma(\Sigma_T \setminus N)$ is well defined and bi-measurable, and $h_\sigma(\overline{\vartheta}) = h_\sigma(\Psi_*(\overline{\vartheta}))$ holds for any $\overline{\vartheta} \in \mathcal{M}_\sigma(\Sigma_{\mathcal{D}} \setminus \overline{N})$.

We set $\mu := \pi_*^{-1}(\mu^+) \in \pi_*^{-1}(\mathcal{U})$. Then we can find $0 \le c \le 1$ and $\mu_1, \mu_2 \in \mathcal{M}_\sigma(\Sigma_T)$ such that $\mu = c\mu_1 + (1-c)\mu_2$, $\mu_1(\Sigma_T \setminus N) = 1$ and $\mu_2(N) = 1$ hold. We note that $h_\sigma(\mu_2) = 0$ by Lemma 2.1. Recall that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$. By the definition of π_*, we have $\mathcal{M}_\sigma^p(\Sigma_T^+) = \pi_*(\mathcal{M}_\sigma^p(\Sigma_T))$, and hence, $\mathcal{M}_\sigma^p(\Sigma_T)$ is also dense in $\mathcal{M}_\sigma(\Sigma_T)$. Therefore, we can find $\nu_2 \in \mathcal{M}_\sigma^p(\Sigma_T)$ such that $\nu := c\mu_1 + (1-c)\nu_2 \in \pi_*^{-1}(\mathcal{U})$ and $h_\sigma(\nu) = h_\sigma(\mu)$ hold. Since N has no periodic point, $\nu_2(\Sigma_T \setminus N) = 1$, which also implies that $\nu(\Sigma_T \setminus N) = 1$. Hence, we can define $\overline{\nu} := \Psi_*^{-1}(\nu) \in (\pi_* \circ \Psi_*)^{-1}(\mathcal{U})$. Note that $\overline{\nu}$ is supported on a transitive countable Markov shift $\Sigma_{\mathcal{D}}$, and $(\pi_* \circ \Psi_*)^{-1}(\mathcal{U})$ is open. Therefore, it follows from [25, Main Theorem] that there exist a finite set $\mathcal{F} \subset \mathcal{D}$ and an ergodic measure $\overline{\rho} \in (\pi_* \circ \Psi_*)^{-1}(\mathcal{U})$ supported on $\Sigma_{\mathcal{F}}$ such that $h_\sigma(\overline{\rho}) \ge h_\sigma(\overline{\nu}) - \epsilon > 0$. Since $h_\sigma(\overline{\rho}) > 0$, we have $\overline{\rho}(\overline{N}) = 0$ by Lemma 2.1. We define an ergodic measure ρ^+ on Σ_T^+ by $\rho^+ := \pi_*(\Phi_*(\overline{\rho}))$. Clearly, we have $\rho^+ \in \mathcal{M}_\sigma^e(\Psi^+(\Sigma_{\mathcal{F}})) \cap \mathcal{U}$. Moreover, since $\overline{\rho} \in \mathcal{M}_\sigma(\Sigma_{\mathcal{D}} \setminus \overline{N})$, we have $h_\sigma(\overline{\rho}) = h_\sigma(\rho^+)$, which implies that $h_\sigma(\rho^+) \ge h_\sigma(\mu^+) - \epsilon$. □

We continue the proof of Theorem C. Note that $\Psi^+(\Sigma_{\mathcal{F}}) \subset \Sigma_T^+$ is a subshift, and $\rho^+(\Psi^+(\Sigma_{\mathcal{F}})) = 1$. Hence, from the estimates in [20], we have the following lemma.

LEMMA 3.3. [20, Propositions 2.1 and 4.1] *There exists $n_0 \in \mathbb{N}$ such that, for any $n \ge n_0$,*

$$\#\{u \in \mathcal{L}_n(\Psi^+(\Sigma_{\mathcal{F}})) : [u] \subset (\delta_n^\sigma)^{-1}(\mathcal{U})\} \ge \exp\{n(h_\sigma(\rho^+) - \epsilon)\}.$$

For notational simplicity, for each $n \ge n_0$, we set $\mathcal{L}_{n,\rho^+} := \{u \in \mathcal{L}_n(\Psi^+(\Sigma_{\mathcal{F}})) : [u] \subset (\delta_n^\sigma)^{-1}(\mathcal{U})\}$. Since $\mathcal{F}$ is finite, it follows from Lemma 2.3 that there is $K > 1$ such that $m^+[u] \ge K^{-1} \exp\{-nh_{\mathrm{top}}(\Sigma_T^+, \sigma)\}$ for any $u \in \mathcal{L}_{n,\rho^+}$ and $n \ge n_0$. Therefore, by $\bigcup_{u \in \mathcal{L}_{n,\rho^+}} [u] \subset (\delta_n^\sigma)^{-1}(\mathcal{U})$, we have

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$$
\begin{aligned}
&\liminf_{n\to\infty} \frac{1}{n} \log m^+((\delta_n^\sigma)^{-1}(\mathcal{U})) \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n} \log m^+\bigg(\bigcup_{u\in\mathcal{L}_{n,\rho^+}} [u] \bigg) \\
&\quad = \liminf_{n\to\infty} \frac{1}{n} \log \sum_{u\in\mathcal{L}_{n,\rho^+}} m^+[u] \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n} \log(\#\mathcal{L}_{n,\rho^+} K^{-1} \exp\{-n h_{\mathrm{top}}(\Sigma_T^+, \sigma)\}) \\
&\quad \geq \liminf_{n\to\infty} \frac{1}{n} \log(K^{-1} \exp\{n(h_\sigma(\rho^+) - h_{\mathrm{top}}(\Sigma_T^+, \sigma) - \epsilon)\}) \\
&\quad \geq h_\sigma(\mu^+) - h_{\mathrm{top}}(\Sigma_T^+, \sigma) - 2\epsilon,
\end{aligned}
$$

which proves Theorem C. □

Proof of Theorem A. Let $T\colon X \to X$ and m be as in Theorem A with critical points $0 = a_0 < a_1 < \cdots < a_k = 1$. Then it follows from Theorem C that (Σ_T^+, σ) satisfies the large deviation principle with the unique measure of maximal entropy m^+ on Σ_T^+, and the rate function $\mathcal{J}^+\colon \mathcal{M}(\Sigma_T^+) \to [0, \infty]$ is expressed by equation (3.1).

Since the coding map $\mathcal{I}\colon X_T \to \Sigma_T^+$ is injective (see §2), the set

$$
\bigcap_{n\geq 0} \mathrm{cl}(T^{-n}(a_{\omega_{n+1}-1}, a_{\omega_{n+1}})) \tag{3.4}
$$

consists of a unique element for any $(\omega_i)_{i\in\mathbb{N}} \in \Sigma_T^+$. Here, $\mathrm{cl}(A)$ denotes the closure of a set A. We define a map $\Phi\colon \Sigma_T^+ \to X$ by $\Phi((\omega_i)_{i\in\mathbb{N}}) = y$, where y is a unique element of the set (3.4), and let $\Phi_*\colon \mathcal{M}(\Sigma_T^+) \to \mathcal{M}(X)$ be a pushforward map; $\mu^+ \mapsto \mu^+ \circ \Phi^{-1}$. Then the following properties hold (see [8], for instance):

- Φ is a continuous surjection, $\Phi(\mathcal{I}(X_T)) = X_T$, and the restriction map $\Phi\colon \mathcal{I}(X_T) \to X_T$ is bijective;
- $\Phi \circ \sigma(\omega) = T \circ \Phi(\omega)$ for $\omega \in \mathcal{I}(X_T)$;
- $m = \Phi_*(m^+)$ and $m(X_T) = m^+(\mathcal{I}(X_T)) = 1$;
- $h_\sigma(\mu^+) = h_T(\Phi_*(\mu^+))$ if $\Phi_*(\mu^+) \in \mathcal{M}_T(X)$.

From the aforementioned properties, we have $m \circ (\delta_n^T)^{-1} = m^+ \circ (\delta_n^\sigma)^{-1} \circ \Phi_*^{-1}$. Hence, it follows from the contraction principle [7, Theorem 4.2.1] that (X, T) satisfies the large deviation principle with m, and the rate function $\mathcal{J}\colon \mathcal{M}(X) \to [0, \infty]$ is expressed by

$$
\mathcal{J}(\mu) := \inf\{\mathcal{J}^+(\mu^+) : \mu^+ \in \mathcal{M}(\Sigma_T^+) \text{ with } \Phi_*(\mu^+) = \mu\}. \tag{3.5}
$$

Hereafter, assume that T is piecewise increasing and either right or left continuous. Since $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$, it follows from [26, Theorem B and Lemma 2.7] that $\Phi_*(\mathcal{M}_\sigma(\Sigma_T^+)) = \mathcal{M}_T(X)$. Hence, $\mathcal{J}(\mu) = \infty$ if $\mu \notin \mathcal{M}_T(X)$, and

$$
\begin{aligned}
\mathcal{J}(\mu) &= \inf\{h_{\mathrm{top}}(\Sigma_T^+, \sigma) - h_\sigma(\mu^+) : \mu^+ \in \mathcal{M}_\sigma(\Sigma_T^+), \mu = \Phi_*(\mu^+)\} \\
&= h_{\mathrm{top}}(X, T) - h_T(\mu)
\end{aligned}
$$

if $\mu \in \mathcal{M}_T(X)$. This completes the proof of Theorem A. □

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4. *Applications*

In this section, we apply Theorem A to demonstrate the large deviation principle for the concrete examples presented in §1. First, we provide a sufficient condition for (IR).

Let $T: X \to X$ be a piecewise monotonic map with critical points $0 = a_0 < a_1 < \cdots < a_k = 1$, and we consider the following condition.

Exactness (EX). For any open interval $I \subset X$, there exist a positive integer τ, integers $1 \leq i_0, \ldots, i_{\tau-1} \leq k$, and an open subinterval $L \subset I$ such that $T^j(L) \subset (a_{i_j-1}, a_{i_j})$ for all $0 \leq j \leq \tau - 1$ and $T^\tau(L) = (0, 1)$.

LEMMA 4.1. *Condition (EX) implies (IR).*

Proof. Let $C, D \in \mathcal{D}$. To prove the lemma, it is sufficient to show that there are $n \geq 1$ and $C_1, \ldots, C_n \in \mathcal{D}$ such that $C_1 = C$, $C_n = D$ and $C_i \to C_{i+1}$ for each $1 \leq i \leq n - 1$. Since $C, D \in \mathcal{D}$, there are $u = u_1 \cdots u_l$ and $v = v_1 \cdots v_m$ in $\mathcal{L}(\Sigma_T^+)$ such that $C = \sigma^{l-1}[u]$ and $D = \sigma^{m-1}[v]$. If we set $I := \bigcap_{j=1}^{l} T^{-(j-1)}((a_{u_j-1}, a_{u_j}))$, then I is a non-empty open interval since $u \in \mathcal{L}(\Sigma_T^+)$. By condition (EX), there exist an open subinterval $L \subset I$, $\tau \geq 1$, and $1 \leq i_0, \ldots, i_{\tau-1} \leq k$ such that $T^j(L) \subset (a_{i_j-1}, a_{i_j})$ for $0 \leq j \leq \tau - 1$ and $T^\tau(L) = (0, 1)$. Since $T^{l-1}(I) \subset (a_{u_{l-1}}, a_{u_l})$, we have $l \leq \tau$. We set

$$w = \begin{cases} i_l \cdots i_{\tau-1}, & \tau > l, \\ \lambda, & \tau = l, \end{cases}$$

where λ denotes the empty word. Then it is not difficult to see that $\sigma^\tau[uw] = \Sigma_T^+$. Hence, we have $\sigma^{\tau+m-1}[uwv] = D$, which proves the lemma. □

Example 4.1. Let $0 \leq \alpha < 1$, $\beta > 2$, and $T = T_{\alpha,\beta}: X \to X$ be as in (1.2). Then it follows from [**13**, Theorem 2] that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$, and therefore $\mathcal{M}_T^p(X)$ is also dense in $\mathcal{M}_T^e(X)$ (see also [**26**, Theorem A]). Moreover, it is proved in [**5**, Proposition 3.14] that T satisfies condition (EX). Hence, by Lemma 4.1 and Theorem A, (X, T) satisfies the large deviation principle with the unique measure of maximal entropy m. Since T is piecewise increasing and right continuous, the rate function $\mathcal{J}: \mathcal{M}(X) \to [0, \infty]$ is expressed by (1.1).

Example 4.2. Let $(1 + \sqrt{5})/2 < \beta < 2$ and $T = T_{-\beta}: X \to X$ be as in (1.3). Since $\beta > (1 + \sqrt{5})/2$, we can prove that T satisfies condition (EX) in a similar manner to the proof of [**9**, Proposition 8]. Hence, by Lemma 4.1, T satisfies (IR). Moreover, it is clear that T has two monotonic pieces and is transitive. Hence, it follows from [**15**, Theorem 2] that $\mathcal{M}_\sigma^p(\Sigma_T^+)$ is dense in $\mathcal{M}_\sigma(\Sigma_T^+)$, and hence, $\mathcal{M}_T^p(X)$ is also dense in $\mathcal{M}_T^e(X)$. Therefore, by Theorem A, (X, T) satisfies the large deviation principle with the unique measure of maximal entropy m.

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