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ON SEMIREGULAR RINGS WHOSE FINITELY GENERATED MODULES EMBED IN FREE MODULES

Dedicated to the memory of Professor Maurice Auslander

JUAN RADA AND MANUEL SAORÍN

ABSTRACT. We consider rings as in the title and find the precise obstacle for them not to be Quasi-Frobenius, thus shedding new light on an old open question in Ring Theory. We also find several partial affirmative answers for that question.

It is well-known that a ring for which every left module embeds in a free module is Quasi-Frobenius (QF). However, the following is still an open question:

A) Given a ring R for which every finitely generated left R-module embeds in a free (or projective) module, is R QF?

Until the early eighties there were many partial affirmative answers to this question. Among them, when R is left perfect [10], left self-injective ([2] or [12]), left or right noetherian ([6] and [4]) or when the injective envelope of ${}_RR$ is a projective module [7] (see [4] for a good survey on these results). Menal [7] introduces a modified version of Question A:

B) Does there exist a cardinal c with the property that every ring all whose c-generated left R-modules embed in free modules is necessarily a QF ring?

From that time, as far as we know, both questions have not seen any new partial answer until very recently, when Gómez Pardo and Guil Asensio [5] proved that if the embedding in projective of Question A is required to be essential the answer is yes. This, as a byproduct, implied an affirmative answer in case R is supposed to be left CS (*i.e.* every left ideal is essential in a direct summand of ${}_RR$).

A natural generalization of both perfect rings and self-injective rings are the so-called semiregular rings (see definition below), a class of rings which strictly includes the semiperfect ones as well. In these notes, we try to get an insight in Questions A and B when the ring R is semiregular. We find that in case the Jacobson radical $J(R)$ is left T-nilpotent, the answer to A is yes (Theorem 2), while in case the transfinite powers of J become eventually zero or the intersection of any descending chain of cyclic right ideals is zero, the answer to Question B is affirmative by taking $c = \aleph_0$, the infinite countable cardinal (Theorem 3). In the general semiregular situation, we see that the answer to

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Question A could only be negative in case there existed a proper direct summand of ${}_RR$ which is the left annihilator of a finite subset of $J(R)$ (Corollary 5). As a result we get a list of previously unobserved properties that, added to the semiregularity of R, imply an affirmative answer for A (Corollary 6).

In the last part, we see that some strict generalizations of the left T-nilpotency of J still allow a lot of information on the structure of injective indecomposables (Theorem 7), from which we can obtain new partial positive answers to A and B (Corollaries 8 and 9).

In the sequel "ring" means "associative ring with identity". All modules are unital and we shall write ${}_RM$ or M_R when we want to stress that a module is left or right module. In particular, ${}_RR$ and R_R will denote the canonical structures of left and right R-module in R. If R is a ring, its Jacobson radical will be denoted by $J(R)$, or simply J if no confusion appears. The *left transfinite sequence* of powers of J is defined as follows: $J^1 = J$ and, in case J^β has been defined for every ordinal $\beta < \alpha$, we put $J^\alpha = \bigcap_{\beta<\alpha} J^\beta$, when α is limit, and $J^\alpha = JJ^{\alpha-1}$, when α is non-limit. There exists a least ordinal γ such that $J^\gamma = J^\alpha$, for all ordinals $\alpha \geq \gamma$ and we put $\overline{J}(R) = J^\gamma$. The Jacobson radical J is *left T-nilpotent* when, for every sequence $x_0, x_1, \ldots, x_n, \ldots$ of elements of J, there exists $n \epsilon \mathbb{N}$ such that $x_0 x_1 \cdots x_n = 0$.

A ring R is called *semiregular* [8] when R/J is regular (in the sense of von Neumann) and idempotents lift modulo J. That is equivalent to say that every finitely presented left (or right) R-module has a projective cover. Such a ring has the property that, for every finitely generated submodule M of a projective module P, P admits a decomposition $P = P_1 \oplus P_2$, where $P_1 \subseteq M$ and $P_2 \cap M$ is a submodule of JP (note that then $M = P_1 \oplus (P_2 \cap M)$).

A ring R is called *left FP-injective* when the dual functor $(-)^* = \mathrm{Hom}_R(-, {}_RR)$ preserves exact sequences $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ in which ${}_RM$ is a finitely presented module. More generally, R will be said *left (cyclic)* $\aleph_0$*-injective* (see [3]) when every homomorphism $f: I \longrightarrow {}_RR$, where I is a finitely generated (cyclic) left ideal of R, extends to a homomorphism $\hat{f}: {}_RR \longrightarrow {}_RR$.

In order to deal with Question A we shall say that a ring R is *left FGF* (see [4]) whenever every finitely generated left R-module embeds in a free module (or, equivalently, in a projective module). Question A can be hence reformulated as: Does left FGF imply QF?

For all ring-theoretical terminology not defined here, the reader is referred to [1] and [11].

From the explicit description of direct limits in ${}_R\,\mathrm{Mod}$ given in [11, p. 17-18] follows the next lemma which is crucial in the sequel.

LEMMA 1. *Let* $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \cdots \longrightarrow M_n \xrightarrow{f_n} M_{n+1} \cdots$ *be a sequence of homomorphisms of R-modules. If* $\varinjlim_n (M_n, \{f_n\}) = 0$ *then, for every* $x \epsilon M_0$*, there exists an integer* $k = k(x) \geq 0$ *such that* $(f_k \circ \cdots \circ f_0)(x) = 0$*. In particular, when* M_0 *is finitely generated there exists an integer* $k \geq 0$ *such that* $f_k \circ \cdots \circ f_0 = 0$*.*

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THEOREM 2. *Let R be a ring such that $R/J(R)$ is regular and $J(R)$ left T-nilpotent. If R is left FGF then R is QF.*

PROOF. We will prove that every finitely generated left R-module is essentially embeddable in a projective module. The result will follow then by [5, Corollary 3.5].

Let M be a finitely generated left R-module. Since R is semiregular left FGF, M has a decomposition $M = P \oplus M_0$ where P is projective and $\mu_0 \colon M_0 \longrightarrow P_0$ is a monomorphism such that P_0 is finitely generated projective and $\operatorname{Im}\mu_0 \subseteq JP_0$. By viewing μ_0 as an inclusion, we consider a pseudocomplement V_0 of M_0 in P_0 and hence $M_0 \xrightarrow{\mu_0} P_0 \xrightarrow{\pi} P_0/V_0$, where π is the canonical projection, is an essential monomorphism. If P_0/V_0 is projective we are done. If not, we have a decomposition $P_0/V_0 = P_0' \oplus V_0'$ where P_0' is projective and V_0' is embeddable in the radical of a finitely generated projective module. Now we lift this decomposition back to P_0, so that $P_0 = P_0' \oplus P_0''$ and V_0 may be viewed as a submodule of P_0''. Thus we have a diagram as follows:

$$\begin{array}{ccc}
 & & 0 \oplus V_0 \\
 & & \big\downarrow {\scriptstyle (0,\iota_0)} \\
M_0 & \xrightarrow{\mu_0} & P_0' \oplus P_0'' \\
 & & \big\downarrow {\scriptstyle (1,p_0)} \\
 & & P_0' \oplus (P_0''/V_0)
\end{array}$$

where $\mu_0 \colon M_0 \longrightarrow P_0' \oplus P_0''$ is a monomorphism such that $\operatorname{Im}\mu_0 \subseteq JP_0' \oplus JP_0''$, $p_0 \colon P_0'' \longrightarrow P_0''/V_0$ is the canonical projection, $0 \oplus V_0$ is a pseudocomplement of $\operatorname{Im}\mu_0$ in $P_0' \oplus P_0''$ with canonical inclusion $\iota_0 \colon V_0 \longrightarrow P_0''$ and $f_0 = (1, p_0) \circ \mu_0 \colon M_0 \longrightarrow M_1$, where $M_1 = P_0' \oplus (P_0''/V_0)$, is an essential monomorphism.

Proceeding in this way, since now P_0''/V_0 is embeddable in the radical of a finitely generated projective module, we complete the diagram as follows:

$$\begin{array}{ccccc}
 & & 0 \oplus V_0 & & \\
 & & \big\downarrow {\scriptstyle (0,\iota_0)} & & \\
M_0 & \xrightarrow{\mu_0} & P_0' \oplus P_0'' & & 0 \oplus 0 \oplus V_1 \\
 & & \big\downarrow {\scriptstyle (1,p_0)} & & \big\downarrow {\scriptstyle (0,0,\iota_1)} \\
 & & P_0' \oplus (P_0''/V_0) & \xrightarrow{(1,\mu_1)} & P_0' \oplus P_1' \oplus P_1'' \\
 & & & & \big\downarrow {\scriptstyle (1,1,p_1)} \\
 & & & & P_0' \oplus P_1' \oplus (P_1''/V_1) \quad \cdots
\end{array}$$

where, for each $n \geq 1$, $\mu_n \colon P_{n-1}''/V_{n-1} \longrightarrow P_n' \oplus P_n''$ is a monomorphism such that $\operatorname{Im}\mu_n \subseteq JP_n' \oplus JP_n''$, $p_n \colon P_n'' \longrightarrow P_n''/V_n$ is the canonical projection, $\underbrace{0 \oplus \cdots \oplus 0}_{n+1} \oplus V_n$ is a pseudocomplement of $M_n = P_0' \oplus \cdots \oplus P_{n-1}' \oplus (P_{n-1}''/V_{n-1})$ in $P_0' \oplus \cdots \oplus P_n' \oplus P_n''$ with canonical inclusion $\iota_n \colon V_n \longrightarrow P_n''$ and $f_n = (\underbrace{1, \ldots, 1}_{n+1}, p_n) \circ (\underbrace{1, \ldots, 1}_{n}, \mu_n) \colon M_n \longrightarrow M_{n+1}$ is an essential monomorphism.

Now for each $n \geq 1$, it can be easily seen that

$$\begin{aligned} V_0 &= \operatorname{Ker}(P_0'' \longrightarrow P_1' \oplus P_1'' \longrightarrow \cdots \longrightarrow P_1' \oplus \cdots \oplus P_n' \oplus P_n'') \\ &= \operatorname{Ker}(P_0'' \longrightarrow P_1') \cap \operatorname{Ker}(P_0'' \longrightarrow P_1'' \longrightarrow P_2') \cap \cdots \cap \operatorname{Ker}(P_0'' \longrightarrow \cdots P_n'') \end{aligned}$$

where $P_i'' \longrightarrow P_{i+1}'$ and $P_i'' \longrightarrow P_{i+1}''$ (the components of $\mu_i p_i \colon P_i'' \longrightarrow P_{i+1}' \oplus P_{i+1}''$) have images contained in JP_{i+1}' and JP_{i+1}'' respectively. As a result, the sequence $P_0'' \longrightarrow P_1'' \longrightarrow P_2'' \longrightarrow \cdots$ has the property that $\operatorname{Im}(P_i'' \longrightarrow P_{i+1}'') \subseteq JP_{i+1}''$ and from this follows that if we take $F = \varinjlim_n (P_n'', \{P_n'' \longrightarrow P_{n+1}''\})$, then $F = JF$. Consequently, the left T-nilpotency of J yields $F = 0$ and so Lemma 1 applies. That is, for n sufficiently large $P_0'' \longrightarrow \cdots \longrightarrow P_n''$ is zero. Hence,

$$V_0 = \operatorname{Ker}(P_0'' \longrightarrow P_1') \cap \operatorname{Ker}(P_0'' \longrightarrow P_1'' \longrightarrow P_2') \cap \cdots \cap \operatorname{Ker}(P_0'' \longrightarrow \cdots P_{n-1}'' \longrightarrow P_n')$$

and so the top row of the diagram

$$\begin{array}{ccccccc} P_0' \oplus P_0'' & \longrightarrow & \cdots & \longrightarrow & P_0' \oplus P_1' \oplus \cdots \oplus P_n' \oplus P_n'' & \xrightarrow{\pi_1} & P_0' \oplus P_1' \oplus \cdots \oplus P_n' \\ \downarrow & & & & \downarrow & & \\ M_1 & \xrightarrow{f_1} & \cdots & \xrightarrow{f_n} & M_{n+1} & \xrightarrow{\pi_1'} & P_0' \oplus P_1' \oplus \cdots \oplus P_n' \end{array}$$

has kernel $0 \oplus V_0$, where π_1 and π_1' are the canonical projections onto the first $n+1$ components. Therefore, the composition in the bottom row has to be a monomorphism, from which it follows, since $f_n \circ \cdots \circ f_1$ is an essential monomorphism, that $\pi_1' \circ f_n \circ \cdots \circ f_1 \colon M_1 \longrightarrow P_0' \oplus \cdots \oplus P_n'$ is also an essential monomorphism (and even more $P_n''/V_n = 0$).

Finally, $1_P \oplus (\pi_1' \circ f_n \circ \cdots \circ f_1 \circ f_0) : M = P \oplus M_0 \longrightarrow P \oplus P_0' \oplus \cdots \oplus P_n'$ is an essential embedding into a projective module and so R is QF. ∎

Now we can go further and answer Question B in a particular situation.

THEOREM 3. *Let R be a semiregular ring satisfying one of the following two conditions:*

1. $\bar{J}(R) = 0$.
2. *For every sequence $x_1, \ldots, x_n, \ldots$ of elements of $J(R)$, $\bigcap_{n \geq 1} x_1 \cdots x_n R = 0$.*

If every countably generated left R-module embeds in a free module, then R is QF.

PROOF. (1) Take the same F as in the proof of the above theorem. All we need to show is that $F = 0$ and the same argument of that proof would apply. Suppose $F \neq 0$ and consider, since F is a countably generated flat left R-module, a non-zero homomorphism $f \colon F \longrightarrow R$. By taking $I = \operatorname{Im} f$ and bearing in mind that $JF = F$, we get $JI = I$ and from that follows easily that $I \subseteq J^{\alpha}$ for every ordinal α. So $I \subseteq \bar{J}(R)$ which contradicts the assumption that $\bar{J}(R) = 0$.

(2) Let $x_1, \ldots, x_n, \ldots$ be a sequence in $J(R)$ and consider the sequence of homomorphisms ${}_R R \xrightarrow{\rho_1} {}_R R \xrightarrow{\rho_2} {}_R R \longrightarrow \cdots \xrightarrow{\rho_n} {}_R R \longrightarrow \cdots$, where ρ_n is the right multiplication

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by x_n for each $n \geq 1$. By passing to the direct limit, $F' = \varinjlim_n ({}_RR, \rho_n)$ is a countably generated flat left R-module. If we are able to prove that $F' = 0$, Lemma 1 tells us that $x_1 \cdots x_n = 0$ for some $n \geq 1$ and so J will be left T-nilpotent, which implies that R is QF by Theorem 2. We then prove that $F' = 0$. Let $F' \xrightarrow{f} {}_RR$ be any homomorphism. Since $F' \cong R^{(\mathbb{N})}/K$, where K is the submodule of $R^{(\mathbb{N})}$ generated by $(1, -x_1, 0, \ldots)$, $(0, 1, -x_2, 0, \ldots), \ldots (0, \ldots, 0, 1, -x_n, 0, \ldots), \ldots, f$ is given by a homomorphism $\varphi : R^{(\mathbb{N})} \longrightarrow {}_RR$ such that $K \subseteq \operatorname{Ker} \varphi$. Suppose φ is right multiplication by the column matrix $(b_0, b_1, \ldots, b_n, \ldots)^\top$. From $K \subseteq \operatorname{Ker} \varphi$ we get $b_i = x_{i+1} b_{i+1}$ for all $i = 0, 1, \ldots$ and so $b_i \epsilon \bigcap_{n \geq i+1} x_{i+1} x_{i+2} \cdots x_n R$. Condition 2 yields $b_i = 0$ for all $i = 0, 1, \ldots$ and so $f \equiv 0$. Hence $\operatorname{Hom}_R(F', {}_RR) = 0$ and the embedding hypothesis entails that $F' = 0$. ∎

EXAMPLE. For a semiregular ring, both Conditions 1 and 2 in the above theorem are strictly more general than that of left T-nilpotency, as can be seen by considering a (commutative) discrete valuation domain.

In the following two results we just assume the semiregularity of R and try to identify what might provoke a negative answer for Question A.

PROPOSITION 4. *Let R be a semiregular left FGF ring and M a finitely generated left R-module. If no non-zero direct summand of M embeds in the radical of a finitely generated free left module then M is projective and injective.*

PROOF. Let $x \epsilon E(M)$ (the injective hull of M). Then by the FGF assumption, $M + Rx$ embeds in a free module, which by the finite generation of $M + Rx$ can be assumed to be R^m for an integer $m > 0$. Since R is semiregular, M admits a decomposition $M = P \oplus N$ where P is a direct summand of R^m and $N \subseteq JR^m$. By hypothesis $N = 0$ so $M = P$ is projective. Furthermore, M is an essential direct summand of $M + Rx$. Thus $M = M + Rx$ and so M is injective. ∎

From now on $l(X)$ (resp. $r(X)$) will denote the left (resp. right) annihilator of the subset X of R.

COROLLARY 5. *Let R be a semiregular left FGF ring. The following conditions are equivalent:*

1. *${}_RR$ is not injective;*
2. *There exists an idempotent $e \neq 1$ in R and elements $x_1, \ldots, x_n$ in $J(R)$ such that $Re = l(x_1, \ldots, x_n)$;*
3. *There exists a finitely presented left R-module whose projective dimension is exactly 1.*

PROOF. (1) ⇒ (3). By Proposition 4, there is a non-zero direct summand Re of ${}_RR$ and an embedding $\mu : \mathrm{Re} \longrightarrow {}_RR^n$, for some n, such that $\operatorname{Im} \mu \subseteq JR^n$. Now $M = \operatorname{Coker} \mu$ is the desired finitely presented module.

(3) ⇒ (2). The assumption and the semiregularity of R guarantee the existence of an embedding $0 \longrightarrow P_1 \xrightarrow{\mu} P_0$, where P_1 and P_0 are non-zero finitely generated projective

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and $\operatorname{Im}\mu \subseteq JP_0$. Moreover, since every non-zero finitely generated projective module is isomorphic to a direct sum of left ideals of the form Rf, with $f \epsilon R$-$\{0\}$ idempotent [8, Theorem 2.11], it is not restrictive to assume $P_1 = \mathrm{Rf}$ and $P_0 = {}_RR^n$, for some $n \geq 1$. In that case, if $\mu(f) = (x_1, \ldots x_n)$ (hence $x_1, \ldots, x_n \epsilon J$) one easily gets that $R(1-f) = l(f) = l(x_1, \ldots, x_n)$ and thus $e = 1 - f$ is the desired choice.

(2) $\Rightarrow$ (1). Let e and $x_1, \ldots, x_n$ as in (2). Then there exists a well-defined monomorphism $R(1-e) \longrightarrow \oplus_{i=1}^n Rx_i \hookrightarrow JR^n$ given by $r(1-e) \rightsquigarrow (rx_1, \ldots, rx_n)$). If ${}_RR$ is injective then $R(1-e)$ is a direct summand of R^n which is contained in JR^n. This is a contradiction and so ${}_RR$ is not injective. ∎

REMARK. Although we do not know if the above equivalent conditions ever hold, the corollary helps to understand the precise obstacle for Question A to have an affirmative answer. Furthermore, it is definite to state that answer in many partial cases, as the following shows.

COROLLARY 6. *Let R be a semiregular left FGF ring. Each of the following conditions forces R to be QF:*

1. $J \subseteq Z({}_RR)$;
2. $\operatorname{Soc}(R_R)$ *is essential as a left ideal of R;*
3. $\operatorname{Hom}_R(X, R_R) \neq 0$ *for every cyclic finitely presented right R-module X;*
4. *R is left FP-injective;*

PROOF. (1) For elements $x_1, \ldots, x_n$ in J, $l(x_1, \ldots, x_n)$ is an essential left ideal of R. Consequently, it cannot be a non-zero direct summand of R. It follows from Corollary 5 that ${}_RR$ is injective and by [2] or [12], that R is QF.

(2) Since $\operatorname{Soc}(R_R) \subseteq l(J)$ we know that $l(J)$ is an essential left ideal of R which implies that $J \subseteq Z({}_RR)$. The result follows now from (1).

(3) If R is not QF then by Corollary 5 there exist elements $x_1, \ldots, x_n$ in J and $e \neq 1$ an idempotent in R such that $\mathrm{Re} = l(x_1, \ldots, x_n)$. Then $X = (1-e)R / \sum_{i=1}^n x_iR$ is cyclic finitely presented and $\operatorname{Hom}_R(X, R_R) = 0$ (Observe that $x_i \epsilon r(\mathrm{Re}) = (1-e)R$).

(4) When R is left FP-injective every sequence $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$, with P_0 and P_1 finitely generated projective, splits. Hence Condition 3 in Corollary 5 fails, which implies that R is left self-injective and so QF. ∎

Now we go back to impose some preconditions, but strictly weaker than the T-nilpotency of J.

THEOREM 7. *Let R be a semiregular left FGF ring and suppose that, for every sequence* $x_1, \ldots, x_n, \ldots$ *in* J-$\{0\}$, *there exists* $n \geq 1$ *such that* $l(x_1 \cdots x_n) \neq l(x_1 \cdots x_{n+1})$. *Then every indecomposable injective left R-module is isomorphic to a direct summand of* ${}_RR$.

PROOF. Let E be an indecomposable injective left R-module and take U_0 a finitely generated submodule of E. If $E = E(U_0)$ is not projective then U_0 is not projective and injective so by Proposition 4, U_0 embeds in the radical of a finitely generated free module. In fact, it is possible to embed U_0 in J. Indeed, assume $\lambda \colon U_0 \hookrightarrow R^n$ is an embedding such

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that $\operatorname{Im}(\lambda) \subseteq JR^n$ and for $i = 1, \ldots, n$ let $\pi_i : R^n \longrightarrow R$ be the canonical projections. Then the fact that $0 = \operatorname{Ker}(\lambda) = \bigcap_{i=1}^{n} \operatorname{Ker}(\pi_i \circ \lambda)$ implies that $\operatorname{Ker}(\pi_j \circ \lambda) = 0$ for some $j \epsilon \{1, \ldots, n\}$, because U_0 is uniform. Thus $\mu_0 = \pi_j \circ \lambda : U_0 \longrightarrow R$ is a monomorphism which clearly satisfies $\operatorname{Im}(\mu_0) \subseteq J$ as desired. Now we adapt the Proof of Theorem 2 and, taking a pseudocomplement V_0 of $\operatorname{Im}(\mu_0)$ in R, we can define an essential monomorphism $U_0 \xrightarrow{\mu_0} R \xrightarrow{p_0} R/V_0$ where p_0 is the canonical projection. It follows that $E(U_0) = E(R/V_0)$ and so $U_1 = R/V_0$ is a finitely generated uniform module such that $E(U_1)$ is not projective. We can repeat this argument to construct a diagram as follows:

$$
\begin{array}{ccccc}
 & & V_0 & & \\
 & & \big\downarrow{\scriptstyle \iota_0} & & \\
U_0 & \xrightarrow{\mu_0} & R & & V_1 \\
 & & \big\downarrow{\scriptstyle p_0} & & \big\downarrow{\scriptstyle \iota_1} \\
 & & U_1 & \xrightarrow{\mu_1} & R \\
 & & & & \big\downarrow{\scriptstyle p_1} \\
 & & & & U_2 \quad \cdots
\end{array}
$$

where for each $i \epsilon \mathbb{N}$, $\mu_i : U_i \longrightarrow R$ is a monomorphism such that $\operatorname{Im}(\mu_i) \subseteq J$, V_i is a pseudocomplement of $\operatorname{Im}(\mu_i)$ in R with canonical inclusion $\iota_i : V_i \longrightarrow R$, $U_i = R/V_{i-1}$ and $p_i : R \longrightarrow U_{i+1}$ is the canonical projection. Now for each $i \epsilon \mathbb{N}$, $\mu_i \circ p_{i-1} : R \longrightarrow R$ is right multiplication by an element $x_i \epsilon J$. Therefore

$$l(x_1 \cdots x_n) = \operatorname{Ker}(\mu_n \circ p_{n-1} \circ \mu_{n-1} \circ \cdots \circ \mu_1 \circ p_0)$$

for every $n \geq 1$. We claim that

$$\operatorname{Ker}(\mu_{n+1} \circ p_n \circ \mu_n \circ \cdots \circ \mu_1 \circ p_0) = \operatorname{Ker}(\mu_n \circ p_{n-1} \circ \mu_{n-1} \circ \cdots \circ \mu_1 \circ p_0).$$

One inclusion is clear. To see the other we take $x \in \operatorname{Ker}(\mu_{n+1} \circ p_n \circ \cdots \circ \mu_1 \circ p_0)$. Since μ_{n+1} is a monomorphism

$$(\mu_n \circ p_{n-1} \circ \cdots \circ \mu_1 \circ p_0)(x) \epsilon \operatorname{Ker}(p_n) \cap \operatorname{Im}(\mu_n) = V_n \cap U_n = 0.$$

Hence $x \epsilon \operatorname{Ker}(\mu_n \circ p_{n-1} \circ \cdots \circ \mu_1 \circ p_0)$ as desired. It follows that for each $n \geq 1$ $l(x_1 \cdots x_n) = l(x_1 \cdots x_{n+1})$ which is a contradiction. As a consequence, E is a projective module. Moreover, since every projective is isomorphic to a direct sum of left ideals of the form Re, with $e \epsilon R$ idempotent, it follows that E is isomorphic to a direct summand of ${}_R R$. ∎

REMARK. The annihilator hypothesis of Theorem 7 is trivially satisfied when J is left T-nilpotent. But it is not the only case. If R is left (cyclic) $\aleph_0$-injective then, for every pair (x_1, x_2) of elements of $J - \{0\}$, the inequality $l(x_1) \neq l(x_1 \cdot x_2)$ holds. Indeed, let x_1 and x_2 be non-zero elements in J and assume $l(x_1) = l(x_1 \cdot x_2)$. Then $\varphi : Rx_1x_2 \longrightarrow Rx_1$ defined by $\varphi(rx_1x_2) = rx_1$ $(r \epsilon R)$ is a well-defined isomorphism. Since R is left (cyclic) $\aleph_0$-injective there exists a homomorphism $h : R \longrightarrow R$ such that $h \circ i = j \circ \varphi$ where

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$i\colon Rx_1x_2 \hookrightarrow R$ and $j\colon Rx_1 \hookrightarrow R$ are the canonical inclusions. Now h is right multiplication by an element $y\epsilon R$, so for each $r\epsilon R$ we have that $rx_1 = \varphi(rx_1x_2) = h(rx_1x_2) = rx_1x_2y$. Taking $r = 1$ it follows that $x_1(1 - x_2y) = 0$ and since $x_2y\epsilon J$ then $1 - x_2y$ is invertible. Hence $x_1 = 0$ which yields a contradiction.

EXAMPLE. Every local left self-injective ring which is not left perfect satisfies the annihilator hypothesis of Theorem 7 and its Jacobson radical cannot be left T-nilpotent (For an example of a local left self-injective ring which is not left perfect see [9, Example 1]).

Given a ring R, we shall denote by $\Omega(R)$, $\mathcal{I}(R)$ and $\mathcal{P}(R)$, respectively, the sets of isomorphism classes of simple, indecomposable injective and indecomposable projective left R-modules. On the other hand, $\mathcal{C}(R)$ will stand for the set of isomorphism classes of simple left R-modules which are isomorphic to minimal left ideals of R. We shall make an abuse of notation and use the same letter to denote a module and its isomorphism class. Then the "injective envelope map" $E(-)\colon \Omega(R) \longrightarrow \mathcal{I}(R)$ is an injective map and, when R is semiregular, so is the "top map" $\overline{(-)}\colon \mathcal{P}(R) \longrightarrow \Omega(R)$ that takes P onto $\bar{P} = P/JP$, since every indecomposable projective is local [8, Corollary 2.13].

COROLLARY 8. *Let R be a ring as in Theorem 7. Then each of the following conditions forces R to be QF:*

1. *$\Omega(R)$ is a finite set;*
2. *R/J is left CS;*
3. *$\bigoplus_{P\epsilon\mathcal{P}(R)} P$ is a self-generator (see e.g., [13, p. 120]).*

PROOF. (1) By Theorem 7 we have a composition of injective mappings

$$\Omega(R) \xrightarrow{E(-)} \mathcal{I}(R) \subseteq \mathcal{P}(R) \xrightarrow{\overline{(-)}} \Omega(R).$$

If $\Omega(R)$ is finite then this composition must be bijective. Consequently, every simple left R-module has a projective cover and so R is semiperfect. Moreover, R is left self-injective since $\mathcal{I}(R) = \mathcal{P}(R)$ and $R = \bigoplus_{i=1}^{n} \mathrm{Re}_i$ where each e_i $(i = 1, \ldots, n)$ is a local idempotent of R. It follows from [2] or [12] that R is QF.

(2) For every $P\epsilon\mathcal{P}(R)$ we know that $P/JP\,\epsilon\,\mathcal{C}(R/J)$. Then by Theorem 7,

$$\Omega(R/J) = \Omega(R) \xrightarrow{E(-)} \mathcal{I}(R) \subseteq \mathcal{P}(R) \xrightarrow{\overline{(-)}} \mathcal{C}(R/J) \subseteq \Omega(R/J)$$

is a composition of injective mappings which implies that the cardinality of $\Omega(R)$ coincides with that of $\mathcal{C}(R/J)$. Now since R/J is regular and left CS it follows from [5](see note below) that $\Omega(R)$ must be finite. Consequently, by (1), R is QF.

(3) Let S be a simple left R-module. By Theorem 7, $E(S)\epsilon\mathcal{P}(R)$ and so S is isomorphic to a submodule of $\bigoplus_{P\epsilon\mathcal{P}(R)} P$. Since $\bigoplus_{P\epsilon\mathcal{P}(R)} P$ is a self-generator, S is a factor of some $P\epsilon\mathcal{P}(R)$. Consequently $S \cong P/JP$ thus showing that every simple left R-module has a projective cover. Hence R is semiperfect and, again by (1), R is QF. ∎

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NOTE. In Lemma 2.3 of [5] the authors give a modified proof of a result of Osofsky [9], essentially stating that if Q is regular and left self-injective then $|C(Q)|$ infinite implies $|C(Q)| < |\Omega(Q)|$ ($|X|$ denotes the cardinality of the set X). We have checked that Gómez Pardo and Guil Asensio's proof works "mutatis mutandi" when "self-injective" is replaced by "CS". In other words, the following is true:

If Q is a regular left CS ring such that $C(Q)$ is an infinite set, then $|C(Q)| < |\Omega(Q)|$.

This is the result that we have used in the Proof of Corollary 8(2).

In the following result, $\mathrm{Tr}_R(I)$ denotes the trace ideal of I in R, i.e., $\mathrm{Tr}_R(I) = \sum\{\mathrm{Im}\, f : f \epsilon \operatorname{Hom}_R(I, {}_RR)\}$.

COROLLARY 9. *Let R be a ring as in Theorem 7 with the extra property that $\mathrm{Tr}_R(I) = IR$ for every minimal left ideal of R. If $\mathrm{Soc}({}_RR)$ is essential as a left ideal of R then R is QF.*

PROOF. All we need to prove is that $\mathrm{Soc}({}_RR)J = 0$ for then $l(J)$ is an essential ideal of R and so $J \subseteq Z({}_RR)$ which, by Corollary 6, implies the statement.

Take a minimal left ideal I of R and assume first that $I \subseteq \mathrm{Re}$ for some idempotent $e \epsilon R$ with the property that Re is injective. If $Ix \neq 0$ for some $x \epsilon J$ then $\rho_x : I \longrightarrow Ix$ defined by $\rho_x(y) = yx$ for each $y \epsilon I$ is an isomorphism with inverse map $\lambda : Ix \longrightarrow I$ (given by $yx \rightsquigarrow y$). Now, due to the injective condition of Re, there exists a homomorphism $\hat{\lambda} : {}_RR \longrightarrow \mathrm{Re}$ making the following diagram commute:

$$\begin{array}{ccc} Ix & \hookrightarrow & {}_RR \\ \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \hat{\lambda}} \\ I & \hookrightarrow & \mathrm{Re}. \end{array}$$

Choose $b \epsilon \mathrm{Re}$ such that $\hat{\lambda}(r) = rb$ for all $r \epsilon R$. Then for all $y \epsilon I$, $yxb = y$ and so $y(1 - xb) = 0$. Since $x \epsilon J$, $1 - xb$ is an invertible element and, as a consequence, $I = 0$ which is a contradiction. Hence, $IJ = 0$ in this case.

Let us come back now to the general case in which I is an arbitrary minimal left ideal of R. We know, by Theorem 7, that $E(I) \cong \mathrm{Re}$ for certain local idempotent $e \epsilon R$. Then there exists a monomorphism $f : I \longrightarrow R$ such that $f(I) \subseteq \mathrm{Re}$. Applying our assumption, bearing in mind that we have a composition $f(I) \xrightarrow{\sim} I \hookrightarrow {}_RR$, we get that $I \subseteq f(I)R$. Thus, $IJ \subseteq f(I)J = 0$ and so $\mathrm{Soc}({}_RR)J = 0$. ∎

EXAMPLE. As two particular examples in which the trace hypothesis of the foregoing corollary holds we can give:

1. When $\mathrm{Ext}^1_R(R/I, {}_RR) = 0$, for every minimal left ideal I of R (*e.g.* if R is left (cyclic) $\aleph_0$-injective). Hence, in particular, if R is semiregular left (cyclic) $\aleph_0$-injective and $\mathrm{Soc}({}_RR)$ is left essential, then R left FGF implies R QF.
2. If ${}_RR$ contains exactly one isomorphic copy of each simple left R-module.

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Departamento de Matemáticas
Universidad de Los Andes
Mérida, Venezuela
e-mail: juanrada@ciens.ula.ve

Departamento de Matemáticas
Universidad de Murcia. Aptdo. 4021
30100 Espinardo, Murcia
Spain
e-mail: msaorinc@fcu.um.es

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