



RESEARCH ARTICLE

On local Galois deformation rings[†]

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Abstract

We show that framed deformation rings of mod p representations of the absolute Galois group of a p -adic local field are complete intersections of expected dimension. We determine their irreducible components and show that they and their special fibres are normal and complete intersection. As an application, we prove density results of loci with prescribed p -adic Hodge theoretic properties.

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1. Introduction

Let p denote any prime number, let F be a finite extension of \mathbb{Q}_p and let G_F denote its absolute Galois group. Let L be another finite extension of \mathbb{Q}_p with ring of integers \mathcal{O} , uniformizer ϖ and residue field $k = \mathcal{O}/\varpi$. Fix a continuous representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_d(k)$ and denote by $D_{\bar{\rho}}^{\square} : \mathfrak{A}_{\mathcal{O}} \rightarrow \mathrm{Sets}$ the functor from the category $\mathfrak{A}_{\mathcal{O}}$ of local Artinian \mathcal{O} -algebras with residue field k to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D_{\bar{\rho}}^{\square}(A)$ is the set of continuous representations $\rho_A : G_F \rightarrow \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \bar{\rho}$. The functor $D_{\bar{\rho}}^{\square}$ of framed deformations of $\bar{\rho}$ is pro-represented by a complete local Noetherian \mathcal{O} -algebra $R_{\bar{\rho}}^{\square}$ (with residue field k).

Our first main result completely settles a folklore conjecture on ring-theoretic properties of $R_{\bar{\rho}}^{\square}$ that can be traced back to the foundational work of Mazur [37, Section 1.10]:

Theorem 1.1 (Corollary 3.38). *The ring $R_{\bar{\rho}}^{\square}$ is a local complete intersection, flat over \mathcal{O} and of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$. In particular, every continuous representation $\bar{\rho} : G_F \rightarrow \mathrm{GL}_d(k)$ has a lift to characteristic zero.*

Obstruction theory provides a presentation $R_{\bar{\rho}}^{\square} = \mathcal{O}\langle\langle x_1, \dots, x_r \rangle\rangle / (f_1, \dots, f_s)$ with r equal to the dimension of the tangent space and s equal to $\dim H^2(G_F, \mathrm{ad} \bar{\rho})$. The Euler–Poincaré characteristic formula from local class field theory gives

$$r - s = d^2 + d^2[F : \mathbb{Q}_p].$$

Our theorem proves that $\dim R_{\bar{\rho}}^{\square}/\varpi$ is given by this cohomological quantity, the *expected dimension* in the spirit of the *Dimension Conjecture* of Gouvêa from [27, Lecture 4]. Having the expected dimension implies that ϖ, f_1, \dots, f_s is a regular sequence and that $R_{\bar{\rho}}^{\square}$ is a local complete intersection. It also implies (see [26, Lemma 7.5]) that the derived deformation ring of $\bar{\rho}$ as introduced by Galatius and Venkatesh in [26] (see also [12]) is homotopy discrete, which means the derived deformation theory of $\bar{\rho}$ does not contain more information than the usual deformation theory of $\bar{\rho}$. Theorem 1.1 is used in the forthcoming work of Matthew Emerton, Toby Gee and Xinwen Zhu on derived stacks of global Galois representations.

Our second main result completely describes the connected components of the space $\mathrm{Spec} R_{\bar{\rho}}^{\square}[1/p]$ as envisioned in [8]. Let $\mu := \mu_{p^\infty}(F) \subset F^\times$ be the p -power torsion subgroup and suppose that L is sufficiently large. Let $R_{\det \bar{\rho}}$ denote the universal deformation ring of the one-dimensional representation $\det \bar{\rho}$.

Theorem 1.2 (Corollaries 4.5, 4.15, 4.19, 4.21, Proposition 5.12). *The natural map $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$, induced by sending a deformation of $\bar{\rho}$ to its determinant, is flat and induces a bijection of connected components*

$$\pi_0(\mathrm{Spec} R_{\bar{\rho}}^{\square}[1/p]) \rightarrow \pi_0(\mathrm{Spec} R_{\det \bar{\rho}}[1/p]). \tag{1}$$

Labeling these components in a natural way by characters $\chi : \mu \rightarrow \mathcal{O}^\times$, the connected components of $\mathrm{Spec} R_{\bar{\rho}}^{\square}[1/p]$ are in natural bijection with the irreducible components $\mathrm{Spec} R_{\bar{\rho}}^{\square, \chi}$ of $\mathrm{Spec} R_{\bar{\rho}}^{\square}$, and the rings $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are normal domains and complete intersections.

As a consequence, we obtain the following useful Corollary.

Corollary 1.3 (Corollary 4.22). *$R_{\bar{\rho}}^{\square}$ is reduced and $R_{\bar{\rho}}^{\square}[1/p]$ is normal.*

We would like to highlight the following result for the amusement of the reader.

Theorem 1.4 (Corollary 4.25). *If $\bar{\rho}$ is absolutely irreducible, then $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are factorial, except in the case $d = 2, F = \mathbb{Q}_3$ and $\bar{\rho} \cong \bar{\rho}(1)$.*

Let $\psi : G_F \rightarrow \mathcal{O}^\times$ be a continuous character lifting $\det \bar{\rho}$. Let $R_{\bar{\rho}}^{\square, \psi}$ be the quotient of $R_{\bar{\rho}}^{\square}$ parameterizing deformations with determinant equal to ψ .

Theorem 1.5 (Corollary 5.4, Theorem 5.6). *The rings $R_{\bar{\rho}}^{\square, \psi}$, $R_{\bar{\rho}}^{\square, \psi} / \varpi$ are normal domains and complete intersections of dimension $\dim R_{\bar{\rho}}^{\square} - \dim R_{\det \bar{\rho}} + 1$ and $\dim R_{\bar{\rho}}^{\square} - \dim R_{\det \bar{\rho}}$, respectively. Moreover, $R_{\bar{\rho}}^{\square, \psi}$ is \mathcal{O} -flat.*

Our work builds in an essential way on the work of GB–Juschka [9] on the special fibres of the deformation rings of pseudo-characters (i.e., pseudo-representations) of G_F . The paper [9] draws its inspiration from the work of Chenevier [16], who studied rigid analytic generic fibres of these rings. Our results in turn imply that the rigid analytic spaces appearing in [16] are normal (Corollaries 4.27, 5.10).

The knowledge of irreducible components of $R_{\bar{\rho}}^{\square}$ allows us to refine the existing results on the Zariski density of the locus with prescribed p -adic Hodge theoretic properties.

Theorem 1.6 (Theorem 6.1). *Suppose that p does not divide $2d$. Let Σ be a subset of the maximal spectrum of $R_{\bar{\rho}}^{\square}[1/p]$ parameterizing any of the following sets of lifts of $\bar{\rho}$ to characteristic zero:*

- (1) *crystalline lifts with regular Hodge–Tate weights;*
- (2) *potentially crystabelline lifts with fixed regular Hodge–Tate weights;*
- (3) *potentially crystalline supercuspidal lifts with fixed regular Hodge–Tate weights.*

Then Σ is Zariski dense in $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$.

The assumption $p \nmid 2d$ enters via our use of the patched module M_{∞} constructed in [14]. The paper [14] is applicable whenever $\bar{\rho}$ has a potentially diagonalisable lift. It has been proved recently by Emerton–Gee [24], using the Emerton–Gee stack, that this holds for all $\bar{\rho}$. The rest of our paper is independent of [24]. We show that the action of $R_{\bar{\rho}}^{\square}$ on M_{∞} is faithful (Theorem 6.8), which allows us to deduce Theorem 1.6 from [25].

Partial results towards Theorem 1.1 and also towards the more recent question solved by Theorem 1.2 appear in many places (e.g., [3], [8], [6], [20], [30], [40]) in special cases. However, these papers either compute with equations defining the rings or impose assumptions on $\bar{\rho}$ so that the deformation theory of $\bar{\rho}$ is essentially unobstructed, which leads to only one irreducible component. Although there is some overlap in ideas with [30], the argument in our paper is rather different as we do not compute with equations. We refer the reader to Section 6 for a more detailed discussion of the previous results on Zariski density of specific loci in $\text{Spec } R_{\bar{\rho}}^{\square}$ and to Remark 6.10 for a detailed explanation of the relation between Theorem 1.6 and our more recent results in [7].

Remark 1.7. In the theorems above, we work with framed deformation rings. Our results also carry over to the versal deformation rings (which coincide with the universal deformation rings if $\bar{\rho}$ has only scalar endomorphisms) by exploiting the fact that framed deformation rings are formally smooth over versal deformation rings (see, for example, [30, Lemma 2.1]) and using [11, Theorem 2.3.6, Corollary 2.2.23 (a)].

1.1. Complete intersection

We now give an overview of the proof of Theorem 1.1. To do so, we introduce two further key players. The first are determinant laws, which we refer to as *pseudo-characters* throughout the paper, and their deformations. Let $\bar{D} : k[G_F] \rightarrow k$ be the pseudo-character attached to $\bar{\rho}$ as defined in [18]. Let $D^{\text{ps}} : \mathfrak{A}_{\mathcal{O}} \rightarrow \text{Sets}$ be the functor mapping $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$ to the set $D^{\text{ps}}(A)$ of continuous A -valued d -dimensional pseudo-characters $D : A[G_F] \rightarrow A$ with $\bar{D} = D \pmod{\mathfrak{m}_A}$. The functor D^{ps} is pro-representable by a complete local Noetherian \mathcal{O} -algebra $(R^{\text{ps}}, \mathfrak{m}_{R^{\text{ps}}})$; see [18, Section 3.1]. The ring R^{ps} has been well understood in the recent work of GB–Juschka [9], who have determined the dimension of its special fibre and showed that the absolutely irreducible locus is dense in the special fibre. In particular, they show the following:

Theorem 1.8 (GB–Juschka [9, Theorem 5.5.1(a)]). *The ring R^{ps}/ϖ is equi-dimensional of dimension $1 + d^2[F : \mathbb{Q}_p]$.*

Mapping a lifting of $\bar{\rho}$ to its associated pseudo-character induces a natural transformation $D_{\bar{\rho}}^{\square} \rightarrow D^{\text{ps}}$ and thus a map of local \mathcal{O} -algebras $R^{\text{ps}} \rightarrow R_{\bar{\rho}}^{\square}$. Our basic idea is to study $R_{\bar{\rho}}^{\square}$ by studying the fibres of this map. Our initial observation was that the difference between the expected dimension of $R_{\bar{\rho}}^{\square}/\varpi$ and the dimension computed in Theorem 1.8 is $d^2 - 1$, which is the dimension of PGL_d . However, a fibre at a point corresponding to an absolutely irreducible pseudo-character can be shown to be isomorphic to PGL_d . This led us naturally to study fibres at other points. In fact, it is technically more convenient to introduce an intermediate ring $R^{\text{ps}} \rightarrow A^{\text{gen}} \rightarrow R_{\bar{\rho}}^{\square}$, depending on \bar{D} and not on $\bar{\rho}$ itself, such that A^{gen} is of finite type over R^{ps} and $R_{\bar{\rho}}^{\square}$ is a completion of A^{gen} at a maximal ideal. This is our second key player.

To describe A^{gen} , let $D^u : R^{\text{ps}}[[G_F]] \rightarrow R^{\text{ps}}$ be the universal pseudo-character lifting \bar{D} and let $\text{CH}(D^u)$ be the closed two-sided ideal of $R^{\text{ps}}[[G_F]]$ defined in [18, Section 1.17], so that

$$E := R^{\text{ps}}[[G_F]]/\text{CH}(D^u)$$

is the largest quotient of $R^{\text{ps}}[[G_F]]$ for which the Cayley–Hamilton theorem for D^u holds. Following [18, Section 1.17], we will call such algebras Cayley–Hamilton R^{ps} -algebras of degree d . By [50, Proposition 3.6], the ring E is a finitely generated R^{ps} -module. Now a construction of Procesi [44] gives a commutative R^{ps} -algebra A^{gen} together with a homomorphism

$$j : E \rightarrow M_d(A^{\text{gen}})$$

of Cayley–Hamilton R^{ps} -algebras satisfying the following *universal property*: if $f : E \rightarrow M_d(B)$ is a map of Cayley–Hamilton R^{ps} -algebras for a commutative R^{ps} -algebra B , then there is a unique map $\tilde{f} : A^{\text{gen}} \rightarrow B$ of R^{ps} -algebras such that $f = M_d(\tilde{f}) \circ j$. We give further details in Lemma 3.1 in the main text. The superscript *gen* in A^{gen} stands for *generic matrices*.

Since E is finitely generated as an R^{ps} -module, the construction of Procesi shows that A^{gen} is of finite type over R^{ps} . Moreover, one has an algebraic action of GL_d on $X^{\text{gen}} := \text{Spec } A^{\text{gen}}$ which, for every R^{ps} -algebra B and point $f : E \rightarrow M_d(B)$ in $X^{\text{gen}}(B)$, is simply given by conjugation of matrices. Wang-Erickson has studied the quotient stack $[X^{\text{gen}}/\text{GL}_d]$ in his thesis [49], [50] and X^{gen} is isomorphic to $\text{Rep}_{\bar{D}}^{\square} = \text{Rep}_{E, D^u}^{\square}$ as defined in [50, Theorem 3.8]. It is an important observation that to $\pi : X^{\text{gen}} \rightarrow X^{\text{ps}} := \text{Spec } R^{\text{ps}}$ we can apply geometric invariant theory (GIT). As shown in [50, Theorem 2.20], the induced morphism $X^{\text{gen}} // G \rightarrow X^{\text{ps}}$ is an adequate homeomorphism in the sense of [1, Definition 3.3.1].

Our first important result on dimensions is for $\bar{X}^{\text{gen}} := \text{Spec } A^{\text{gen}}/\varpi$.

Theorem 1.9 (Theorem 3.31, Lemma 3.23). *We have*

$$\dim X^{\text{gen}}[1/p] \leq \dim \bar{X}^{\text{gen}} \leq d^2 + d^2[F : \mathbb{Q}_p].$$

To prove the second inequality of Theorem 1.9, we decompose the base of the finite type morphism $\bar{\pi} : \bar{X}^{\text{gen}} \rightarrow \bar{X}^{\text{ps}} = \text{Spec } R^{\text{ps}}/\varpi$ as a finite union $\bar{X}^{\text{ps}} = \bigcup_i U_i$ of locally closed subschemes U_i . The points of the U_i correspond to semi-simple degree d representations of G_F with certain (degree) conditions on the irreducible constituents. The work [9] gives dimension estimates on the U_i . We combine them with bounds on the dimensions of the fibres at the closed points of U_i , obtained using GIT, and with results on $\bar{\pi}^{-1}(U_i) \rightarrow U_i$ from commutative algebra. In Subsection 3.2, we analyze in detail the dimensions of the fibres of π at points y of X^{ps} valued either in finite fields containing k or local fields whose residue fields contain k . The analysis at such points suffices for all results in this paper. The commutative algebra results, used to analyze $\bar{\pi}^{-1}(U_i) \rightarrow U_i$ and to give the first inequality, are proved in Subsection 3.4. The key technical improvement, working with X^{gen} instead of $\text{Spec } R_{\bar{\rho}}^{\square}$ directly, is that the fibres are of finite type over $\kappa(y)$.

We apply the bounds from Theorem 1.9 to the study of lifting rings of continuous residual representations $\rho_x : G_F \rightarrow \mathrm{GL}_d(\kappa(x))$, where x is a point of X^{gen} whose residue field $\kappa(x)$ is a finite or a local field. We distinguish three cases:

- (1) If $\kappa(x)$ is a finite extension k' of k , then we set Λ to be the ring of integers \mathcal{O}' of the unramified extension L' of L with residue field k' .
- (2) If $\kappa(x)$ is a finite extension of L , then we set Λ to be $\kappa(x)$.
- (3) If $\kappa(x)$ is a local field that contains k and if k' denotes its residue field, then we take as Λ a Cohen ring of $\kappa(x)$ (with the natural topology) tensored over the Witt vector ring $W(k')$ with \mathcal{O}' .

Let \mathfrak{A}_Λ be the category of local Artinian Λ -algebras (A, \mathfrak{m}_A) with residue field $\kappa(x)$. We equip the rings A with a natural topology, and we consider the functor $D_{\rho_x}^\square : \mathfrak{A}_\Lambda \rightarrow \mathrm{Sets}$ such that $D_{\rho_x}^\square(A)$ is the set of continuous group homomorphisms $\rho : G_F \rightarrow \mathrm{GL}_d(A)$, such that $\rho \pmod{\mathfrak{m}_A} = \rho_x$. In cases (1) and (2), such functors occur in the work of Mazur and Kisin, respectively. The formulation in case (3) appears to be new. In all cases, the functor $D_{\rho_x}^\square$ is pro-represented by a complete local Noetherian Λ -algebra $R_{\rho_x}^\square$ with residue field $\kappa(x)$. The arguments of Mazur and Kisin carry over to the case when $\kappa(x)$ is a local field of characteristic p and yield a presentation

$$R_{\rho_x}^\square \cong \Lambda[[x_1, \dots, x_r]]/(f_1, \dots, f_s) \tag{2}$$

with $r = \dim_{\kappa(x)} Z^1(G_F, \mathrm{ad} \rho_x)$ and $s = \dim_{\kappa(x)} H^2(G_F, \mathrm{ad} \rho_x)$; here, $\mathrm{ad} \rho_x$ is the adjoint representation of G_F on $\mathrm{End}_{\kappa(x)}(\rho_x)$ by conjugation. By a suitable version of Tate local duality results, one finds $r - s = d^2 + d^2[F : \mathbb{Q}_p]$. From this, Theorem 1.9 and some commutative algebra results that relate the completion of A^{gen} at x to the ring $R_{\rho_x}^\square$, we deduce the following result.

Corollary 1.10 (Corollaries 3.38 and 3.44). *For x as above, the following hold:*

- (1) $R_{\rho_x}^\square$ is a flat Λ -algebra of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;
- (2) if $\mathrm{char}(\kappa(x)) = p$, then $R_{\rho_x}^\square/\varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.

At first glance, one might expect that for closed points x of X^{gen} , the residue field $\kappa(x)$ is always finite. However, as we show in Example 3.22, $\kappa(x)$ can also be a local field of characteristic 0 or p . In Subsection 3.5, we show that this exhausts all possibilities.

Corollary 1.10 gives us a handle on the completions of the local rings $\mathcal{O}_{X^{\mathrm{gen}}, x}$ (resp. $\mathcal{O}_{\overline{X^{\mathrm{gen}}}, x}$) at closed points $x \in X^{\mathrm{gen}}$ (resp. $x \in \overline{X^{\mathrm{gen}}}$), which allows us to deduce the following result.

Corollary 1.11 (Corollaries 3.40 and 3.45). *The following hold:*

- (1) A^{gen} is \mathcal{O} -torsion free, equi-dimensional of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection;
- (2) A^{gen}/ϖ is equi-dimensional of dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection.

We end Section 3 with a result on the density of (certain) absolutely irreducible points in R_{ρ}^\square and in R_{ρ}^\square/ϖ . This is motivated by and relies on similar results for R^{ps} . A point x in $X^{\mathrm{ps}} = \mathrm{Spec} R^{\mathrm{ps}}$ is called *absolutely irreducible* if the associated semisimple representation $\rho_x : G_F \rightarrow \mathrm{GL}_d(\overline{\kappa(x)})$ (which is unique up to isomorphism) is irreducible. It follows from the main theorem of [16] that the locus of absolutely irreducible points is dense open in the generic fibre $X^{\mathrm{ps}}[1/p] = \mathrm{Spec} R^{\mathrm{ps}}[1/p]$, and this is extremely useful because such points are regular on $X^{\mathrm{ps}}[1/p]$.

A key role in the study of the regular locus in the special fibre $\overline{X}^{\mathrm{ps}} = \mathrm{Spec} R^{\mathrm{ps}}/\varpi$ in [9] is played by a class of absolutely irreducible points, which are called *non-special*. We extend this notion slightly in Appendix A. We say that an absolutely irreducible point x in $\overline{X}^{\mathrm{ps}}$ with finite or local residue field is *Kummer-reducible* if there exists a degree p Galois extension F' of $F(\zeta_p)$ such that $\rho_x|_{G_{F'}}$ is reducible, and *Kummer-irreducible* if not. If $\zeta_p \in F$, then $x \in \overline{X}^{\mathrm{ps}}$ is Kummer-irreducible if and only if it is *non-special* in the sense of [9, Section 5]. We show that if x is Kummer-irreducible, then $H^2(G_F, \mathrm{ad}^0 \rho_x) = 0$,

where $\text{ad}^0 \rho_x$ is the subrepresentation of $\text{ad} \rho_x$ of trace zero matrices. Much more importantly for us, we also show that the locus of Kummer-irreducible $x \in \overline{X}^{\text{ps}}$ is dense open. At these points, \overline{X}^{ps} is not necessarily smooth, but it is relatively smooth over $\text{Spec } R_{\det \bar{\rho}}$. Here we prove the following:

Proposition 1.12 (Proposition 3.55 and Corollaries 3.59 and 3.61). *We have the following.*

- (1) *The set of absolutely irreducible points $x \in \text{Spec } R_{\bar{\rho}}^{\square}[1/p]$ with $\kappa(x)$ finite over L is dense in $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$.*
- (2) *The set of Kummer-irreducible points $x \in \text{Spec } R_{\bar{\rho}}^{\square}/\varpi$ with $\kappa(x)$ a local field is dense in $\text{Spec } R_{\bar{\rho}}^{\square}/\varpi$.*

In particular, every continuous representation $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$ has an absolutely irreducible lift to characteristic zero.

1.2. Irreducible components

From here on, we assume that L contains F , so that, in particular, L contains all roots of unity contained in F . We now give a more detailed overview of Theorem 1.2 on components of $R_{\bar{\rho}}^{\square}$. The homomorphism $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$ from that theorem is induced by the natural transformation $D_{\bar{\rho}}^{\square} \rightarrow D_{\det \bar{\rho}}$ that to a representation assigns its determinant, and it induces the map (1) on components.

Via the Artin map $F^{\times} \rightarrow G_F^{\text{ab}}$ from local class field theory, the inclusion $\mu \subset F^{\times}$ and the identification of $R_{\det \bar{\rho}}$ with the completed group ring of the pro- p completion of G_F^{ab} , the ring $R_{\det \bar{\rho}}$ becomes an $\mathcal{O}[\mu]$ -algebra. It is well-known that $R_{\det \bar{\rho}}$ is a power series ring over $\mathcal{O}[\mu]$ in $[F : \mathbb{Q}_p] + 1$ formal variables. The components of the étale L -algebra $\mathcal{O}[\mu][1/p] = L[\mu]$ are in bijection with the characters $\chi : \mu \rightarrow \mathcal{O}^{\times}$. Setting $R_{\bar{\rho}}^{\square, \chi} = R_{\bar{\rho}}^{\square} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$, we obtain a decomposition $\text{Spec } R_{\bar{\rho}}^{\square}[1/p] = \bigsqcup_{\chi} \text{Spec } R_{\bar{\rho}}^{\square, \chi}[1/p]$, where χ ranges over the characters $\mu \rightarrow \mathcal{O}^{\times}$.

The main step in the proof of the bijectivity of the map (1) in Theorem 1.2 is to show that the rings $R_{\bar{\rho}}^{\square, \chi}$ are normal by verifying Serre’s criterion for normality. We first present $R_{\bar{\rho}}^{\square}$ over $R_{\det \bar{\rho}}$ (Proposition 4.3) by imitating Kisin’s method of presenting global rings over local rings. Since $R_{\det \bar{\rho}}^{\chi} := R_{\det \bar{\rho}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$ is formally smooth, by applying $\otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$, we obtain a presentation of $R_{\bar{\rho}}^{\square, \chi}$ over $R_{\det \bar{\rho}}^{\chi}$ analogous to the presentation (2). Since $R_{\bar{\rho}}^{\square, \chi}$ has the same dimension as $R_{\bar{\rho}}^{\square}$, the presentation yields that $R_{\bar{\rho}}^{\square, \chi}$ is complete intersection of expected dimension and hence satisfies Serre’s condition (S2). We then show that $X^{\text{gen}, \chi} := \text{Spec}(A^{\text{gen}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O})$ and its special fibre $\overline{X}^{\text{gen}, \chi}$ are regular in codimension 1 by showing that the Kummer-irreducible locus in $\overline{X}^{\text{gen}, \chi}$ (resp. absolutely irreducible locus in $X^{\text{gen}, \chi}[1/p]$) is regular, and its complement has codimension at least 2 if either $F \neq \mathbb{Q}_p$, or $d > 2$ or \overline{D} is absolutely irreducible. The case $F = \mathbb{Q}_p$, $d = 2$ and \overline{D} reducible requires an extra analysis of the reducible locus. Since $R_{\bar{\rho}}^{\square, \chi}$ is a completion of a local ring at a closed point of $X^{\text{gen}, \chi}$, we deduce that $R_{\bar{\rho}}^{\square, \chi}$ is regular in codimension 1. We thus deduce that $R_{\bar{\rho}}^{\square, \chi}$ is normal. Since $R_{\bar{\rho}}^{\square, \chi}$ is a local ring, it is an integral domain. A similar argument works for the special fibre.

Theorem 1.5 on $R_{\bar{\rho}}^{\square, \psi}$ is proved by reduction to the results on $R_{\bar{\rho}}^{\square, \chi}$ where $\chi : \mathcal{O}[\mu] \rightarrow \mathcal{O}^{\times}$ is the restriction of ψ to μ via the Artin map. To give an idea of the proof, let $\mathcal{X} : \mathfrak{A}_{\mathcal{O}} \rightarrow \text{Sets}$ be the functor, which sends (A, \mathfrak{m}_A) to the group $\mathcal{X}(A)$ of continuous characters $\theta : G_F \rightarrow 1 + \mathfrak{m}_A$ such that θ restricted to μ is trivial, and let $\mathcal{O}(\mathcal{X})$ be the complete local Noetherian \mathcal{O} -algebra pro-representing \mathcal{X} . Local class field theory gives an isomorphism $\mathcal{O}(\mathcal{X}) \cong \mathcal{O}[[y_1, \dots, y_{[F:\mathbb{Q}_p]+1}]]$. Let $\varphi_d : \mathcal{O}(\mathcal{X}) \rightarrow \mathcal{O}(\mathcal{X})$ be the morphism induced by the d -power map $\mathcal{X} \rightarrow \mathcal{X}$, $\theta \mapsto \theta^d$. Our key technical result is Proposition 5.1, which yields a natural isomorphism

$$R_{\bar{\rho}}^{\square, \chi} \otimes_{\mathcal{O}(\mathcal{X}), \varphi_d} \mathcal{O}(\mathcal{X}) \cong R_{\bar{\rho}}^{\square, \psi} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X})$$

that comes from an analogous isomorphism of functors. It allows us to compare the sets of points x with $\kappa(x)$ finite or local at which $H^2(G_F, \text{ad}^0 \rho_x)$ is non-zero on both sides. We also show that the

map $\text{Spec } R_{\bar{\rho}}^{\square, \chi} \otimes_{\mathcal{O}(\mathcal{X}), \varphi_d} \mathcal{O}(\mathcal{X}) \rightarrow \text{Spec } R_{\bar{\rho}}^{\square, \chi}$ induces a homeomorphism on special fibres, and a finite covering on generic fibres. Then we use topological arguments to obtain the dimension of $R_{\bar{\rho}}^{\square, \psi}$ and bound the codimension of its singular locus from the analogous results on $R_{\bar{\rho}}^{\square, \chi}$.

We also prove analogs of Theorem 1.2 (resp. Theorem 1.6) for spaces $X^{\text{gen}, \chi}$ and $\bar{X}^{\text{gen}, \chi}$ for characters $\chi : \mu \rightarrow \mathcal{O}^\times$ (resp. $X^{\text{gen}, \psi}, \bar{X}^{\text{gen}, \psi}$ for characters $\psi : G_F \rightarrow \mathcal{O}^\times$ lifting $\det \bar{\rho}$); see Corollaries 4.6, 4.18, 4.26 (resp. Corollaries 5.8, 5.9). We expect that our results will be useful in the study of the geometry of the Emerton–Gee stack and its derived versions.

It is natural to ask whether our results generalize to deformations valued in reductive groups other than GL_d . This question will be addressed in the forthcoming joint work of VP and Julian Quast.

1.3. Overview by section

In Section 2, we briefly review GIT. A key result that gets used later on is Lemma 2.2. In Section 3, we introduce X^{gen} and its special fibre \bar{X}^{gen} . In Subsection 3.2, we bound the dimensions of the fibres of the map $X^{\text{gen}} \rightarrow X^{\text{ps}}$. In Subsection 3.4, we combine this with results of [9] to bound the dimension of X^{gen} and \bar{X}^{gen} . In Subsection 3.5, we relate the completions of local rings at closed points x of $X^{\text{gen}}, \bar{X}^{\text{gen}}$ to the deformation theory of Galois representations $\rho_x : G_F \rightarrow \text{GL}_d(\kappa(x))$ and prove Theorem 1.1. In Section 3.6, we bound the maximally reducible semi-simple locus in X^{gen} and \bar{X}^{gen} . This computation later on gets used only in the case $d = 2, F = \mathbb{Q}_2$ and \bar{D} is reducible. In Subsection 3.7, we prove the Zariski density of the Kummer-irreducible locus in \bar{X}^{gen} and absolutely irreducible locus in $X^{\text{gen}}[1/p]$ and also establish lower bounds for the dimension of their complements. These bounds are used to establish normality later on. In Section 4, we present $R_{\bar{\rho}}^{\square}$ over $R_{\det \bar{\rho}}$ and prove Theorem 1.2. In Section 5, we prove Theorem 1.5. In Section 6, we prove Theorem 1.6. In Appendix A, we introduce the notion of Kummer-irreducible points in $\text{Spec } R^{\text{ps}}/\varpi$, which slightly generalizes the notion of non-special points defined in [9]. This technical improvement is needed in Section 5 when $\zeta_p \notin F$.

1.4. Notation

Let F be a finite extension of \mathbb{Q}_p and let G_F be its absolute Galois group. Let L be another finite extension of \mathbb{Q}_p , such that $\text{Hom}_{\mathbb{Q}_p\text{-alg}}(F, L)$ has cardinality $[F : \mathbb{Q}_p]$. Let \mathcal{O} be the ring of integers in L , ϖ a uniformiser, and k the residue field. We will denote by ζ_p a primitive p -th root of unity in a fixed algebraic closure of F . For a commutative ring R , we let $P_1 R = \{\mathfrak{p} \in \text{Spec } R : \dim R/\mathfrak{p} = 1\}$.

We fix a representation $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$ and assume that all its irreducible subquotients are absolutely irreducible. We note that we may always achieve that after enlarging k , since the image of $\bar{\rho}$ is a finite group. Let $\text{ad } \bar{\rho}$ be the adjoint representation of G_F and $\text{ad}^0 \bar{\rho}$ the subspace of trace zero endomorphisms, so that G_F acts on $\text{End}_k(\bar{\rho})$ by conjugation. We will denote the dimension as a k -vector space of cohomology groups $H^i(G_F, \text{ad } \bar{\rho})$ by h^i .

2. Geometric invariant theory

We first recall the setup of [47]. Let R be a Noetherian ring and let $S = \text{Spec } R$. Let G be a reductive group scheme over S , so that G is an affine group scheme over S , $G \rightarrow S$ is smooth and the geometric fibres are connected reductive groups. In the application, $G = S \times_{\text{Spec } \mathbb{Z}} \text{GL}_d$ and $G = S \times_{\text{Spec } \mathbb{Z}} \mathbb{G}_m^r$ so that these conditions hold.

Let V be a free R -module of finite rank r endowed with a G -module structure, let $\check{V} = \text{Hom}_R(V, R)$ and let $\text{Sym}(\check{V})$ be the symmetric algebra. The G -module structure on V induces an action of G on $\text{Spec}(\text{Sym}(\check{V})) = \mathbb{A}_S^r$. Let X be a closed G -invariant subscheme of $\text{Spec}(\text{Sym}(\check{V}))$. The G -action on X induces an action on B , the ring of functions on X . The GIT quotient $X // G$ is represented by the ring of invariants B^G .

Lemma 2.1. *Every irreducible component Z of X , equipped with its reduced subscheme structure, is G -invariant.*

Proof. This fact is mentioned (and a proof is sketched) in [47, Section 4], but we give a full proof for the convenience of the reader. We have to show that $\varphi(G \times_S Z) \subset Z$, where $\varphi : G \times_S X \rightarrow X$ is the action map. In terms of rings, this amounts to showing that the kernel of $\varphi^\sharp : B \rightarrow \mathcal{O}(G) \otimes_R B/\mathfrak{p}$ is equal to \mathfrak{p} , where $\mathcal{O}(G)$ is the ring of functions of G and \mathfrak{p} is a prime of B such that $Z = V(\mathfrak{p})$. Since the identity element in G maps Z to itself, $\ker \varphi^\sharp$ is contained in \mathfrak{p} . Since Z is an irreducible component of X , it is enough to show that $\mathcal{O}(G) \otimes_R B/\mathfrak{p}$ is an integral domain, as then $\ker \varphi^\sharp$ is a prime of B and therefore has to equal to \mathfrak{p} .

Since $G \rightarrow S$ is geometrically connected and smooth, $G \times_S \eta$ is integral for every geometric point η of S . Thus, $\mathcal{O}(G) \otimes_R \overline{\kappa(\mathfrak{p})}$ is an integral domain, where $\overline{\kappa(\mathfrak{p})}$ is an algebraic closure of the fraction field of B/\mathfrak{p} . Since $G \rightarrow S$ is smooth, it is also flat. Thus, $\mathcal{O}(G) \otimes_R B/\mathfrak{p}$ is a subring of $\mathcal{O}(G) \otimes_R \overline{\kappa(\mathfrak{p})}$ and hence is an integral domain. □

Let $y = \text{Spec } \kappa$ be a geometric point of $X // G$. We may identify the fibre X_y with a closed G -invariant subscheme of X .

Lemma 2.2. *Let $x \in X_y(\kappa)$ be such that the orbit $G \cdot x$ is closed in X_y ; then*

$$\dim X_y \leq \dim_\kappa T_x(X_y).$$

Proof. Let Z be an irreducible component of X_y such that $\dim Z = \dim X_y$. By Lemma 2.1, Z is closed in X_y and G -invariant. Then by [47, Theorem 3], both Z and X_y have a unique closed G -orbit; hence, those orbits must be equal. Therefore, $x \in Z$, so since Z is irreducible,

$$\dim X_y = \dim Z \leq \dim_\kappa T_x(Z) \leq \dim_\kappa T_x(X_y). \quad \square$$

3. $R_{\overline{\rho}}^\square$ is complete intersection

Let $\overline{\rho} : G_F \rightarrow \text{GL}_d(k)$ be a continuous representation. Let $D_{\overline{\rho}}^\square : \mathfrak{A}_{\mathcal{O}} \rightarrow \text{Sets}$ be the functor from the category of local Artinian \mathcal{O} -algebras with residue field k to the category of sets, such that for $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$, $D_{\overline{\rho}}^\square(A)$ is the set of continuous representations $\rho_A : G_F \rightarrow \text{GL}_d(A)$ such that $\rho_A \pmod{\mathfrak{m}_A} = \overline{\rho}$. The functor $D_{\overline{\rho}}^\square$ is pro-represented by a complete local Noetherian \mathcal{O} -algebra $R_{\overline{\rho}}^\square$. The main goal of this section is to establish inequalities

$$\dim R_{\overline{\rho}}^\square \leq 1 + d^2 + d^2[F : \mathbb{Q}_p], \quad \dim R_{\overline{\rho}}^\square/\varpi \leq d^2 + d^2[F : \mathbb{Q}_p]. \quad (3)$$

It is well known to the experts that these bounds imply that the rings $R_{\overline{\rho}}^\square$ and $R_{\overline{\rho}}^\square/\varpi$ are complete intersection. Namely, the proof of [38, Proposition 21.1] shows that the tangent space to $D_{\overline{\rho}}^\square$ is $Z^1(G_F, \text{ad } \overline{\rho})$, and it follows from the proof of Proposition 2 in [37, Section 1.6] that there is a presentation

$$R_{\overline{\rho}}^\square \cong \mathcal{O}[[x_1, \dots, x_r]]/(f_1, \dots, f_s), \quad (4)$$

where $r = \dim_k Z^1(G_F, \text{ad } \overline{\rho}) = d^2 - h^0 + h^1$ and $s = h^2$ and $h^i = \dim_k H^i(G_F, \text{ad } \overline{\rho})$. The Euler–Poincaré characteristic formula implies that $r - s = d^2 + d^2[F : \mathbb{Q}_p]$. Thus, ϖ, f_1, \dots, f_s can be extended to a system of parameters in a regular ring $\mathcal{O}[[x_1, \dots, x_r]]$ and hence form a regular sequence.

Let $\overline{D} : k[G_F] \rightarrow k$ be the determinant law attached to $\overline{\rho}$ in the sense of [18], so that \overline{D} is equal to the composition of the polynomial laws induced by $k[G_F] \xrightarrow{\overline{\rho}} M_d(k)$ and $M_d(k) \xrightarrow{\det} k$. In this paper, we will refer to determinant laws as pseudo-characters. Let $D^{\text{ps}} : \mathfrak{A}_{\mathcal{O}} \rightarrow \text{Sets}$ be the functor sending an object $(A, \mathfrak{m}_A) \in \mathfrak{A}_{\mathcal{O}}$ to the set $D^{\text{ps}}(A)$ of continuous A -valued d -dimensional pseudo-characters

of $A[G_F]$ which reduce to \overline{D} modulo \mathfrak{m}_A . The functor D^{ps} is pro-representable by a complete local Noetherian \mathcal{O} -algebra $(R^{\text{ps}}, \mathfrak{m}_{R^{\text{ps}}})$ by [18, Proposition 3.3].

Mapping a deformation of $\overline{\rho}$ to its determinant induces a natural transformation $D^{\square}_{\overline{\rho}} \rightarrow D^{\text{ps}}$ and thus a map of local \mathcal{O} -algebras $R^{\text{ps}} \rightarrow R^{\square}_{\overline{\rho}}$. The ring R^{ps} has been well understood in the recent work of GB–Juschka [9], who have determined its dimension and showed that the absolutely irreducible locus is dense in the special fibre. Our basic idea is to study $R^{\square}_{\overline{\rho}}$ by studying the fibres of this map. In fact, it is technically more convenient to introduce an intermediate ring $R^{\text{ps}} \rightarrow A^{\text{gen}} \rightarrow R^{\square}_{\overline{\rho}}$ (see the next subsection), depending on \overline{D} and not on $\overline{\rho}$ itself, such that A^{gen} is of finite type over R^{ps} and $R^{\square}_{\overline{\rho}}$ is a completion of A^{gen} at a maximal ideal. Since $\dim R^{\square}_{\overline{\rho}} \leq \dim A^{\text{gen}}$, it is enough to bound the dimension of A^{gen} . In fact, we first bound the dimension of its special fibre (Theorem 3.31).

3.1. Generic matrices

Let $D^u : R^{\text{ps}}[[G_F]] \rightarrow R^{\text{ps}}$ be the universal pseudo-character lifting \overline{D} . Let $\text{CH}(D^u)$ be the Cayley–Hamilton ideal, which is a closed two-sided ideal of $R^{\text{ps}}[[G_F]]$ defined in [18, Section 1.17] in such a way that

$$E := R^{\text{ps}}[[G_F]]/\text{CH}(D^u)$$

is the largest quotient of $R^{\text{ps}}[[G_F]]$ for which the Cayley–Hamilton theorem for D^u holds. Following [18, Section 1.17], we will call such algebras *Cayley–Hamilton R^{ps} -algebras of degree d* . Then E is a finitely generated R^{ps} -module, [50, Proposition 3.6]. If $f : E \rightarrow M_d(B)$ is a homomorphism of R^{ps} -algebras for a commutative R^{ps} -algebra B , then we say f is a *homomorphism of Cayley–Hamilton algebras* if $\det \circ f : E \rightarrow B$ is equal to the specialization of D^u along $R^{\text{ps}} \rightarrow B$.

The superscript *gen* in A^{gen} stands for *generic matrices*, and the following construction appears in the work of Procesi [44]; Lemmas 3.1, 3.2, 3.4 are contained in [50, Theorem 3.8], but one needs to translate from the language of groupoids and stacks used in *op. cit.* to access them.

Lemma 3.1. *There is a finitely generated commutative R^{ps} -algebra A^{gen} together with a homomorphism of Cayley–Hamilton R^{ps} -algebras $j : E \rightarrow M_d(A^{\text{gen}})$, satisfying the following universal property: if $f : E \rightarrow M_d(B)$ is a map of Cayley–Hamilton R^{ps} -algebras for a commutative R^{ps} -algebra B , then there is a unique map $\tilde{f} : A^{\text{gen}} \rightarrow B$ of R^{ps} -algebras such that $f = M_d(\tilde{f}) \circ j$.*

Proof. By writing down a generic $d \times d$ -matrix for each R^{ps} -generator of E and quotienting out by the relations the generators satisfy in E , one obtains a commutative R^{ps} -algebra C and a homomorphism of R^{ps} -algebras $j : E \rightarrow M_d(C)$. More formally, C is a quotient of $R^{\text{ps}} \otimes_{\mathbb{Z}} \text{Sym}(W)$, where W is a direct sum of n copies of $\text{End}(\text{Std})^*$, where Std is the standard representation of GL_d over \mathbb{Z} , n is the size of a generating set of E as an R^{ps} -module and $\text{Sym}(W)$ is the symmetric algebra over \mathbb{Z} . If we were to only require the maps to be R^{ps} -algebras homomorphisms (i.e., if we did not impose the Cayley–Hamilton condition), then the map $j : E \rightarrow M_d(C)$ would satisfy the required universal property. To ensure that the Cayley–Hamilton condition is satisfied, we have to consider the quotient of C constructed as follows. Let $\Lambda_i : E \rightarrow R^{\text{ps}}$, $0 \leq i \leq d$ be the coefficients of the characteristic polynomial of D^u ; these are homogeneous polynomial laws satisfying $D^u(t - a) = \sum_{i=0}^n (-1)^i \Lambda_i(a) t^{d-i}$ in $R^{\text{ps}}[t]$ as explained in [18, Section 1.10]. For each $a \in E$, let $c_i(j(a))$ be the i -th coefficient of the characteristic polynomial of the matrix $j(a) \in M_d(C)$. Let I be the ideal of C generated by $\Lambda_i(a) - c_i(j(a))$ for all $a \in E$ and $0 \leq i \leq d$ and let $A^{\text{gen}} := C/I$. Since [18, Corollary 1.14] and [49, 1.1.9.15] imply that the coefficients of the characteristic polynomial determine pseudo-characters uniquely, the composition $E \rightarrow M_d(C) \rightarrow M_d(A^{\text{gen}})$ is a map of Cayley–Hamilton algebras, and the universal property of $j : E \rightarrow M_d(C)$ implies the universal property for $j : E \rightarrow M_d(A^{\text{gen}})$. Since E is finitely generated as R^{ps} -module, C and hence A^{gen} are of finite type over R^{ps} . □

Let us make a connection to GIT as described in Section 2. If E is generated by n generators as an R^{ps} -module, then as explained in the proof of Lemma 3.1, A^{gen} is a quotient of $R^{\text{ps}} \otimes_{\mathbb{Z}} \text{Sym}(W)$. The group $G := \text{GL}_d$ acts on W by conjugation, and this induces an action of GL_d on $X^{\text{gen}} := \text{Spec } A^{\text{gen}}$. For every R^{ps} -algebra B , a point in $X^{\text{gen}}(B)$ corresponds to an n -tuple of $d \times d$ -matrices with entries in B satisfying certain relations, and $\text{GL}_d(B)$ acts on $X^{\text{gen}}(B)$ by conjugating the matrices. The scheme X^{gen} is isomorphic to $\text{Rep}_{\overline{D}}^{\square} = \text{Rep}_{E, D^u}^{\square}$ as defined in [50, Theorem 3.8].

The GIT quotient $X^{\text{gen}} // G$ is represented by the ring of invariants $(A^{\text{gen}})^G$. The map $R^{\text{ps}} \rightarrow (A^{\text{gen}})^G$ is G -invariant and induces a homomorphism $R^{\text{ps}} \rightarrow (A^{\text{gen}})^G$. It follows from [50, Theorem 2.20] that the induced map

$$X^{\text{gen}} // G \rightarrow X^{\text{ps}} := \text{Spec } R^{\text{ps}} \tag{5}$$

is an adequate homeomorphism in the sense of [1, Definition 3.3.1] (i.e., an integral, universal homeomorphism which is a local isomorphism around points with characteristic zero residue field). We denote by $\overline{X}^{\text{gen}}$ and \overline{X}^{ps} the special fibres of X^{gen} and X^{ps} , respectively. The same argument shows that

$$\overline{X}^{\text{gen}} // G \rightarrow \overline{X}^{\text{ps}}$$

is an adequate homeomorphism.

We equip R^{ps} with the $\mathfrak{m}_{R^{\text{ps}}}$ -adic topology. Since the ring is Noetherian and the residue field is finite, R^{ps} is a compact ring with respect to this topology.

Lemma 3.2. *Let B be a topological R^{ps} -algebra. If $f : E \rightarrow M_d(B)$ is any (not a priori continuous) homomorphism of R^{ps} -algebras, then the composition $G_F \rightarrow E^{\times} \xrightarrow{f} \text{GL}_d(B)$ defines a continuous representation of G_F .*

Proof. Since R^{ps} is a compact ring [2, Corollary 1.10] implies that for every finitely generated R^{ps} -module M there is a unique Hausdorff topology on M making M into a topological R^{ps} -module.

We equip $R^{\text{ps}}[[G_F]]$ with its projective limit topology, E with the quotient topology, and its group of units E^{\times} with the subspace topology via the embedding $E^{\times} \hookrightarrow E \times E, x \mapsto (x, x^{-1})$. Since the map $G_F \rightarrow R^{\text{ps}}[[G_F]]$ is continuous, the map $G_F \rightarrow E^{\times}$ is also continuous. Moreover, since $\text{CH}(D^u)$ is a closed ideal, the topology on E is Hausdorff.

Since E is a finitely generated R^{ps} -module, its topology coincides with $\mathfrak{m}_{R^{\text{ps}}}$ -adic topology (this is also proved in [50, Proposition 3.6]). Let $M := f(E) \subset M_d(B)$, let τ_1 be the subspace topology on M and let τ_2 be the unique Hausdorff topology on M such that the action of R^{ps} is continuous. We claim that the identity map $(M, \tau_2) \rightarrow (M, \tau_1)$ is continuous. We will now prove the claim. Since B is a topological R^{ps} -algebra, the action of R^{ps} on $M_d(B)$, and hence on M , is continuous with respect to τ_1 . Since M is a finitely generated R^{ps} -module, we may pick a continuous surjection $\varphi : (R^{\text{ps}})^n \rightarrow M$ for some $n \geq 1$. Since R^{ps} is Noetherian, the kernel of φ is finitely generated and hence a closed submodule of $(R^{\text{ps}})^n$. Thus, the quotient topology on M induced via φ is Hausdorff and therefore must coincide with τ_2 , which proves the claim.

The same argument shows that τ_2 coincides with the quotient topology via $E \twoheadrightarrow M$, and the claim implies that the map $f : E \rightarrow M_d(B)$ is continuous and hence induces a continuous group homomorphism $E^{\times} \rightarrow M_d(B)^{\times} = \text{GL}_d(B)$. □

Lemma 3.3. *The composition $R^{\text{ps}}[G_F] \rightarrow R^{\text{ps}}[[G_F]] \twoheadrightarrow E$ is surjective.*

Proof. Since $R^{\text{ps}}[G_F]$ is dense in $R^{\text{ps}}[[G_F]]$, its image will be dense in E for the topologies introduced in the proof of Lemma 3.2. The image is also closed, as it is an R^{ps} -submodule of E . Hence, the map is surjective. □

The representation $\overline{\rho} : G_F \rightarrow \text{GL}_d(k)$ induces a map of R^{ps} -algebras $E \rightarrow M_d(k)$ and thus a homomorphism of R^{ps} -algebras $A^{\text{gen}} \rightarrow k$. It follows from the universal property of A^{gen} that $R_{\overline{\rho}}^{\square}$ is

isomorphic to the completion of A^{gen} with respect to the kernel of this map; see Proposition 3.34 for a more precise statement. Conversely, we have the following Lemma.

Lemma 3.4. *Let $x \in X^{\text{gen}}$ be a closed point above the unique closed point of X^{ps} and let $\rho_x : G_F \rightarrow \text{GL}_d(\kappa(x))$ be the representation obtained by composing*

$$G_F \rightarrow R^{\text{ps}}[[G_F]] \rightarrow E \xrightarrow{j} M_d(A^{\text{gen}}) \rightarrow M_d(\kappa(x)).$$

Then the pseudo-character associated to ρ_x is equal to $\overline{D} \otimes_k \kappa(x)$. In particular, ρ_x and $\overline{\rho} \otimes_k \kappa(x)$ have the same semi-simplification.

Proof. Since $D^u \otimes_{R^{\text{ps}}} k = \overline{D}$, the first part follows immediately from the definition of A^{gen} . The second part follows from [18, Theorem 2.12]. Note that since we have assumed that all irreducible subquotients of $\overline{\rho}$ are absolutely irreducible, it is enough to prove that ρ_x and $\overline{\rho}$ have the same semi-simplification after extending scalars to the algebraic closure of k . □

Remark 3.5. We note that one needs to impose the Cayley–Hamilton condition in the definition of A^{gen} for Lemma 3.4 to hold. For example, if $\overline{D} = \chi_1 + \chi_2$, where $\chi_1, \chi_2 : G_F \rightarrow k^\times$ are distinct characters, then $E \otimes_{R^{\text{ps}}} k \cong k \times k$ by Equation (8) in the proof of [5, Lemma 1.4.3], let $\pi_1 : E \rightarrow k$ be the map obtained by projecting to the first component. Then the map $E \rightarrow M_2(k)$, $a \mapsto \text{diag}(\pi_1(a), \pi_1(a))$ is a map of R^{ps} -algebras, and hence induces a map of R^{ps} -algebras $x : C \rightarrow k$, where C is the algebra introduced in the proof of Lemma 3.1. The representation ρ_x obtained by specializing $j : E \rightarrow M_2(C)$ at x is isomorphic to $\chi_1 + \chi_1$; hence, ρ_x is not equal to $\chi_1 + \chi_2$.

3.2. Bounding the dimension of the fibres

Let \mathfrak{p} be a prime ideal of R^{ps} such that $\dim R^{\text{ps}}/\mathfrak{p} \leq 1$. Its residue field $\kappa(\mathfrak{p})$ is either k or a local field by Lemma 3.17 below. Let κ be an algebraic closure $\kappa(\mathfrak{p})$ equipped with its natural topology and let $y : R^{\text{ps}} \rightarrow \kappa$ denote the corresponding homomorphism. The goal of this subsection (Proposition 3.15) is to bound the dimension of the fibre

$$X_y^{\text{gen}} := X^{\text{gen}} \times_{X^{\text{ps}}, y} \text{Spec } \kappa.$$

Let D_y be the specialization of the universal pseudo-character along $y : R^{\text{ps}} \rightarrow \kappa$ and let

$$E_y := E \otimes_{R^{\text{ps}}, y} \kappa \cong (R^{\text{ps}}[[G_F]] \otimes_{R^{\text{ps}}, y} \kappa) / \text{CH}(D_y), \tag{6}$$

where the last isomorphism follows from [18, Section 1.22] or [49, Lemma 1.1.8.6].

Since E is a finitely generated R^{ps} -module, E_y is a finite dimensional κ -algebra. It follows from the proof of Lemma 3.2 that the natural map $G_F \rightarrow E_y^\times$ is continuous for the topology on E_y induced by the topology on κ . Thus, if W is an E_y -module on a finite dimensional κ -vector space, then the induced G_F -action on W is continuous.

Since κ is algebraically closed, we may write¹

$$D_y = \prod_{i=1}^r D_i,$$

where each D_i is an irreducible pseudo-character² of dimension d_i . We define an equivalence relation on the set $\{D_i : 1 \leq i \leq r\}$ by $D_i \sim D_j$ if $D_i = D_j(m)$ for some $m \in \mathbb{Z}$. Let k be the number of the equivalence classes and let n_i be the number of elements in the i -th equivalence class.

¹We follow the convention of [18], so that a pseudo-character of a direct sum of representations is a product of their pseudo-characters; the papers [9] and [50] refer to a direct sum instead.

²We refer the reader to [9, Section 4.1] for the fundamentals of pseudo-characters.

Moreover, for $1 \leq i \leq r$, we fix representations $\rho_i : G_F \rightarrow \mathrm{GL}_{d_i}(\kappa)$ such that D_i is the pseudo-character associated to ρ_i . These representations are uniquely determined up to an isomorphism by [18, Theorem 2.12], but by ρ_i , we really mean a group homomorphism into $\mathrm{GL}_{d_i}(\kappa)$ and not the equivalence class.

If V is a continuous representation of G_F on a finite dimensional κ -vector space such that its semi-simplification is isomorphic to $\bigoplus_{i=1}^r \rho_i$, then the pseudo-character associated to V is equal to D_y and the action of G_F on V extends to an action of $R^{\mathrm{ps}}[[G_F]]$ and then to an action of $R^{\mathrm{ps}}[[G_F]] \otimes_{R^{\mathrm{ps},y}} \kappa$, which factors through the Cayley–Hamilton quotient. It follows from (6) that V and any G_F -invariant subquotient of V is an E_y -module. In particular, we may apply this to $V = \bigoplus_{i=1}^r \rho_i$ to deduce that each ρ_i is an E_y -module.

Lemma 3.6. *If $i \neq j$, then³*

$$\mathrm{Hom}_{E_y}(\rho_i, \rho_j) = \mathrm{Hom}_{G_F}(\rho_i, \rho_j) \quad \text{and} \quad \mathrm{Ext}_{E_y}^1(\rho_i, \rho_j) = \mathrm{Ext}_{G_F}^1(\rho_i, \rho_j),$$

where $\mathrm{Ext}_{G_F}^1(\rho_i, \rho_j)$ is computed in the category of continuous representations of G_F on finite dimensional κ -vector spaces.

Proof. It follows from Lemma 3.3 that the natural map $\kappa[G_F] \rightarrow E_y$ is surjective. This implies the assertion about Hom spaces and gives an inclusion $\mathrm{Ext}_{E_y}^1(\rho_i, \rho_j) \subset \mathrm{Ext}_{G_F}^1(\rho_i, \rho_j)$. To prove the reverse inclusion, consider an extension $0 \rightarrow \rho_j \rightarrow W \rightarrow \rho_i \rightarrow 0$ of G_F -representations and let $V = W \oplus \bigoplus_{l \neq i,j} \rho_l$. As explained above, the G_F -action on V will factor through the action of E_y . Hence, W is a representation of E_y , which implies that $\mathrm{Ext}_{E_y}^1(\rho_i, \rho_j) = \mathrm{Ext}_{G_F}^1(\rho_i, \rho_j)$. \square

Since (5) is an adequate homeomorphism, there is a unique point $y' \in X^{\mathrm{gen}} // G$ above y and $X_{y'}^{\mathrm{gen}} \rightarrow X_y^{\mathrm{gen}}$ is a homeomorphism. The group G acts on $X_{y'}^{\mathrm{gen}}$. Moreover, $X_{y'}^{\mathrm{gen}}$ is of finite type over κ and $X_{y'}^{\mathrm{gen}}(\kappa)$ is in bijection with the set of continuous representations $\rho : G_F \rightarrow \mathrm{GL}_d(\kappa)$ such that the semi-simplification of ρ is isomorphic to $\rho_1 \oplus \dots \oplus \rho_r$.

Lemma 3.7. *The fibre X_y^{gen} is connected, and the unique closed G -orbit in X_y^{gen} corresponds to the semi-simple representations. If the ρ_i are pairwise non-isomorphic, then its dimension is equal to $d^2 - r$.*

Proof. It follows from [47, Theorem 3] that $X_{y'}^{\mathrm{gen}}$ (and hence X_y^{gen} , by the remark in the paragraph above) contains a unique closed G -orbit. Thus, it is enough to show that the closure of every G -orbit will contain a semi-simple representation. If $x \in X_y^{\mathrm{gen}}(\kappa)$, then after conjugation we may assume that x corresponds to a representation $\rho : G_F \rightarrow \mathrm{GL}_d(\kappa)$ such that the image of ρ is block-upper-triangular, and the blocks on the diagonal are given by $\mathrm{diag}(\rho_{\sigma(1)}(g), \dots, \rho_{\sigma(r)}(g))$ for some permutation $\sigma \in S_r$. By extending scalars to $\kappa[T]$, conjugating ρ by $\mathrm{diag}(T^{r-1} \mathrm{id}_{d_{\sigma(1)}}, T^{r-2} \mathrm{id}_{d_{\sigma(2)}}, \dots, \mathrm{id}_{d_{\sigma(r)}})$ and specializing at $T = 0$, we see that the closure of the G -orbit will contain a semi-simple representation. The action of G on X_y^{gen} leaves the connected components invariant by Lemma 2.1. Hence, every connected component of X_y^{gen} will contain the closed point corresponding to the representation $g \mapsto \mathrm{diag}(\rho_1(g), \dots, \rho_r(g))$. Thus X_y^{gen} is connected.

The stabilizer of a semi-simple representation with distinct irreducible factors in GL_d is isomorphic to \mathbb{G}_m^r : a copy of \mathbb{G}_m is embedded as scalar matrices inside of each block. Hence, the dimension of the closed G -orbit is given by $\dim \mathrm{GL}_d - \dim \mathbb{G}_m^r = d^2 - r$. \square

In order to analyze X_y^{gen} , we introduce the following notation. We fix a permutation $\sigma \in S_r$ and write P for the block-upper-triangular parabolic subgroup of GL_d with the i -th diagonal block of size $d_{\sigma(i)} \times d_{\sigma(i)}$. We write N for its unipotent radical and L for its Levi subgroup consisting of block

³We do not assume that V is multiplicity free so that ρ_i and ρ_j might be isomorphic as G_F -representations even if $i \neq j$. Note, however, that the statement of the lemma might not hold if $i = j$. For example, if V is irreducible, then $E_y = M_d(\kappa)$ is a semi-simple algebra, but $\mathrm{Ext}_{G_F}^1(\rho_1, \rho_1)$ is non-zero. The proof uses that the pseudo-character associated to $\rho_i \oplus \rho_j$ is a factor of D_y .

diagonal matrices. We let $Z_L \cong \mathbb{G}_m^r$ denote the centre of L . Finally, we denote their Lie algebras by \mathfrak{p} , \mathfrak{n} , \mathfrak{l} and \mathfrak{z}_L , respectively, and write \mathfrak{g} for the Lie algebra of GL_d . We have

$$\dim \mathfrak{g} = d^2, \quad \dim \mathfrak{l} = \sum_{i=1}^r d_i^2, \quad \dim \mathfrak{z}_L = r, \tag{7}$$

$$\dim \mathfrak{n} = \frac{1}{2}(\dim \mathfrak{g} - \dim \mathfrak{l}) = \sum_{1 \leq i < j \leq r} d_i d_j. \tag{8}$$

Remark 3.8. We note that although \mathfrak{p} , \mathfrak{n} , \mathfrak{l} and \mathfrak{z}_L depend on σ , their dimensions do not.

Let $\rho_\sigma : G_F \rightarrow \mathrm{GL}_d(\kappa)$ be the representation $g \mapsto \mathrm{diag}(\rho_{\sigma(1)}(g), \dots, \rho_{\sigma(r)}(g))$. It follows from a calculation with block-upper-triangular matrices that \mathfrak{p} can be given an associative κ -algebra structure such that the inclusion $\mathfrak{p} \subset \mathfrak{g} = M_d(\kappa)$ is an inclusion of Cayley–Hamilton algebras.

Lemma 3.9. *There exists a closed subscheme $X_{y,\sigma}^{\mathrm{gen}} \subset X_y^{\mathrm{gen}}$ representing the functor sending a κ -algebra B to the set of homomorphisms of Cayley–Hamilton κ -algebra $\varphi : E_y \rightarrow \mathfrak{p} \otimes_\kappa B$ such that the projection onto the i th diagonal block is $\rho_{\sigma(i)} \otimes_\kappa B$ for $1 \leq i \leq r$.*

Proof. The universal map $j : E \rightarrow M_d(A^{\mathrm{gen}})$ induces a map

$$j_y : E_y \rightarrow M_d(A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa).$$

Let $I_{\rho,\sigma}$ be the ideal of $A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa$ generated by the matrix entries of $j_y(a)$ for all $a \in E_y$, which lie below the diagonal blocks of P , and by all the elements on the block diagonal of the matrices $(j_y(a) - \rho_\sigma(a))$ for all $a \in E_y$. Let

$$X_{y,\sigma}^{\mathrm{gen}} := \mathrm{Spec}((A^{\mathrm{gen}} \otimes_{R^{\mathrm{ps}},y} \kappa) / I_{\rho,\sigma}).$$

Then $X_{y,\sigma}^{\mathrm{gen}}$ is a closed subscheme of X_y^{gen} , and its defining ideal $I_{\rho,\sigma}$ was constructed precisely so that a B -point of X_y^{gen} factors through $X_{y,\sigma}^{\mathrm{gen}}$ if and only if it lands in $\mathfrak{p} \otimes_\kappa B$ and matches the ρ_i on the diagonals for $1 \leq i \leq r$. □

The adjoint action (i.e., via conjugation) of $Z_L N$ on \mathfrak{p} induces an action of $Z_L N$ on $X_{y,\sigma}^{\mathrm{gen}}$.

Lemma 3.10. *The unique closed Z_L -orbit in $X_{y,\sigma}^{\mathrm{gen}}$ is the singleton $\{\rho_\sigma\}$.*

Proof. This is the same proof as in Lemma 3.7 and uses the same diagonal matrix trick to kill off the unipotent part. □

Proposition 3.11. *Let $x \in X_{y,\sigma}^{\mathrm{gen}}$ be the point corresponding to the representation ρ_σ . Then*

$$\begin{aligned} \dim T_x(X_{y,\sigma}^{\mathrm{gen}}) &= \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{1 \leq i < j \leq r} \dim \mathrm{Hom}_{G_F}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)) \\ &\leq \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}. \end{aligned} \tag{9}$$

Proof. Using Lemma 3.9 and the decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}$, we may identify $T_x(X_{y,\sigma}^{\mathrm{gen}})$ with the space of κ -algebra homomorphisms $\varphi : E_y \rightarrow M_d(\kappa[\varepsilon])$, which can be written as $\varphi = \rho_\sigma + \varepsilon\beta$, where β is a κ -linear map $\beta : E_y \rightarrow \mathfrak{n}$. If $\beta : E_y \rightarrow \mathfrak{n}$ is any κ -linear map, then $\varphi := \rho_\sigma + \varepsilon\beta$ is a homomorphism of κ -algebras if and only if

$$\beta(aa') = \rho_\sigma(a)\beta(a') + \beta(a)\rho_\sigma(a'), \quad \forall a, a' \in E_y. \tag{10}$$

For $1 \leq i \leq r$, we let $\mathbf{1}_i \in M_d(\kappa)$ be the block diagonal matrix with the identity matrix on the i -th block and zeros everywhere else. Since $\rho_\sigma(g)$ commutes with $\mathbf{1}_i$ for all i , we have an isomorphism

$$T_x(X_{y,\sigma}^{\text{gen}}) \cong \bigoplus_{1 \leq i < j \leq r} V_{ij},$$

where V_{ij} is the space of functions $\beta : E_y \rightarrow \mathbf{1}_i \mathfrak{n} \mathbf{1}_j$ satisfying (10). We may identify $\mathbf{1}_i \mathfrak{n} \mathbf{1}_j$ with $\text{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Then V_{ij} is precisely the space of 1-cocycles for the Hochschild cohomology of E_y with values in $\text{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})$. Thus,

$$\begin{aligned} \dim_\kappa V_{ij} &= \dim_\kappa HH^1(E_y, \text{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) + \dim_\kappa \text{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &\quad - \dim_\kappa HH^0(E_y, \text{Hom}_\kappa(\rho_{\sigma(j)}, \rho_{\sigma(i)})) \\ &= \dim_\kappa \text{Ext}_{E_y}^1(\rho_{\sigma(j)}, \rho_{\sigma(i)}) + d_i d_j - \dim_\kappa \text{Hom}_{E_y}(\rho_{\sigma(j)}, \rho_{\sigma(i)}) \\ &= d_i d_j + [F : \mathbb{Q}_p] d_i d_j + \dim_\kappa \text{Ext}_{G_F}^2(\rho_{\sigma(j)}, \rho_{\sigma(i)}), \end{aligned} \tag{11}$$

where the first equality follows from [15, Proposition IX.4.4.1], the second from [15, Corollary IX.4.4.4] and the third from Lemma 3.6 together with the local Euler–Poincaré characteristic formula in this context⁴ (see [9, Theorem 3.4.1 (c)]). Thus,

$$\dim_\kappa T_x(X_{y,\sigma}^{\text{gen}}) = \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{1 \leq i < j \leq r} \dim_\kappa \text{Ext}_{G_F}^2(\rho_{\sigma(j)}, \rho_{\sigma(i)}).$$

It follows from the local duality (see [9, Theorem 3.4.1 (b)]) that

$$\dim_\kappa \text{Ext}_{G_F}^2(\rho_{\sigma(j)}, \rho_{\sigma(i)}) = \dim_\kappa \text{Hom}_{G_F}(\rho_{\sigma(i)}, \rho_{\sigma(j)}(1)).$$

Thus, if this term is non-zero, then it is equal to 1 and $\rho_{\sigma(i)}$ and $\rho_{\sigma(j)}$ belong to the same equivalence class. □

Remark 3.12. If $\text{char}(\kappa) = p$ and $\zeta_p \in F$, then $D_i \sim D_j$ if and only if $D_i = D_j$ and the bound is sharp in this case.

Corollary 3.13. $\dim X_{y,\sigma}^{\text{gen}} \leq \dim_\kappa T_x(X_{y,\sigma}^{\text{gen}}) \leq \dim \mathfrak{n} + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}.$

Proof. This follows from Lemma 3.10 and Lemma 2.2 applied with $G = Z_L$ and $X = X_{y,\sigma}^{\text{gen}}$, noting that $X_{y,\sigma}^{\text{gen}} // Z_L$ is a singleton. □

Lemma 3.14. *If $f : X \rightarrow Y$ is a finite type and dominant morphism of Noetherian Jacobson universally catenary schemes, then $\dim Y \leq \dim X$.*

Proof. Passing to reduced subschemes does not affect Krull dimension, so we may assume that X and Y are both reduced.

First, assume X and Y are irreducible. Pick dense open affines $U \subset Y, V \subset X$ such that $f(V) \subset U$. Since f is dominant, [48, Tag 0CC1] implies that $A := \mathcal{O}_Y(U) \hookrightarrow B := \mathcal{O}_X(V)$ is injective. Since A is an integral domain, Noether normalization [48, Tag 07NA] implies that the map factors as

$$A \hookrightarrow A[x_1, \dots, x_m] \hookrightarrow B' \hookrightarrow B,$$

⁴The results in [9, Theorem 3.4.1] on local Tate duality and a local Euler–Poincaré characteristic formula, when the coefficient field κ is a local field, are based on the work of Nekovář [41].

with B' finite over $A[x_1, \dots, x_m]$ and $B'_g \cong B_g$ for some non-zero $g \in A$. Then [48, Tag 0DRT] and [35, 13.C, Theorem 20] imply that

$$\dim X = \dim B = \dim B_g = \dim B'_g = \dim B' = \dim A + m = \dim Y + m,$$

so $\dim Y \leq \dim X$.

For the general case, we argue as in the proof of [48, Tag 01RM]. Write $X = \bigcup_j Z_j$ as the union of its irreducible components. Because f is dominant, we have $Y = \bigcup_j \overline{f(Z_j)}$. Clearly, the $\overline{f(Z_j)}$ have to be irreducible, and so the irreducible components of Y have to be among them. The Z_j and $\overline{f(Z_j)}$ are again Noetherian, Jacobson and universally catenary, and hence by the case already treated, we have

$$\dim Y = \max_j \dim \overline{f(Z_j)} \leq \max_j \dim Z_j = \dim X. \quad \square$$

Proposition 3.15. $\dim X_y^{\text{gen}} \leq \dim \mathfrak{g} - r + (\dim \mathfrak{n})[F : \mathbb{Q}_p] + \sum_{i=1}^k \binom{n_i}{2}$.

Proof. We want to apply Lemma 3.14 to

$$\bigsqcup_{\sigma \in S_r} G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\text{gen}} \rightarrow X_y^{\text{gen}}, \quad (12)$$

where the actions of $Z_{L_\sigma} N_\sigma$ on $X_{y,\sigma}^{\text{gen}}$ and of G on X_y^{gen} are given by conjugation.

If $x \in X_y^{\text{gen}}(\kappa)$ and $\varphi : E_y \rightarrow M_d(\kappa)$ is the corresponding κ -algebra homomorphism, then there will exist $\sigma \in S_r$ such that κ^d will admit a filtration by subspaces $0 = V_0 \subset V_1 \subset \dots \subset V_r = V$, which is invariant under the action of E_y via φ , satisfying $V_i/V_{i-1} \cong \rho_{\sigma(i)}$ for $1 \leq i \leq r$. Thus, there is $g \in G(\kappa)$ such that $g\varphi g^{-1}$ will lie in $X_{y,\sigma}^{\text{gen}}(\kappa)$, and hence, (12) induces a surjection on κ -points. But (12) is also a map of finite type κ -schemes and therefore is a dominant map of Noetherian Jacobson universally catenary schemes, so we can apply Lemma 3.14.

The fibre bundles $G \times^{Z_{L_\sigma} N_\sigma} X_{y,\sigma}^{\text{gen}}$ have dimension equal to

$$\dim G + \dim X_{y,\sigma}^{\text{gen}} - \dim(Z_{L_\sigma} N_\sigma) = \dim \mathfrak{g} - r + \dim X_{y,\sigma}^{\text{gen}} - \dim \mathfrak{n}.$$

The bound in Corollary 3.13 gives the required assertion. □

Corollary 3.16. *If $r = 1$, then X_y^{gen} is smooth of dimension $\dim \mathfrak{g} - 1$.*

Proof. If $r = 1$ then $E_y \cong M_d(\kappa)$ and thus has a unique irreducible representation ρ (up to isomorphism). Thus, all the points in $X_y^{\text{gen}}(\kappa)$ lie in the same G -orbit. Fix such a point x . Since the G -stabiliser of x is equal to Z_G , we obtain $\dim X_y^{\text{gen}} = \dim G - \dim Z_G = \dim \mathfrak{g} - 1$.

Since E_y is semi-simple, we have $\text{Ext}_{E_y}^1(\rho, \rho) = 0$, and thus an argument as in the proof of Proposition 3.11 gives us

$$\dim_\kappa T_x(X_y^{\text{gen}}) = \dim_\kappa \text{End}_\kappa(\rho) - \dim_\kappa \text{End}_{E_y}(\rho) = \dim X_y^{\text{gen}}.$$

Thus, x is a smooth point of X_y^{gen} , and since G acts transitively on $X_y^{\text{gen}}(\kappa)$, all the points in $X_y^{\text{gen}}(\kappa)$ are smooth. Since X_y^{gen} is of finite type over κ , we deduce that X_y^{gen} is smooth. □

3.3. Commutative algebra preparations

Lemma 3.18 is the key result of this section, and it will be applied repeatedly with $R = R^{\text{ps}}$ and $S = A^{\text{gen}}$ or their reductions modulo ϖ .

We will start with some general commutative algebra lemmas. For a ring R , we set $P_1 R = \{\mathfrak{p} \in \text{Spec } R : \dim R/\mathfrak{p} = 1\}$.

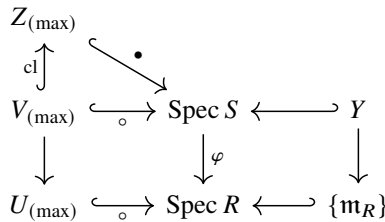
Lemma 3.17. *Let (R, \mathfrak{m}_R) be a complete local Noetherian \mathcal{O} -algebra with finite residue field k' . If $\mathfrak{p} \in P_1 R$, then $\kappa(\mathfrak{p})$ is either a finite extension of L or a local field of characteristic p . Moreover, R/\mathfrak{p} is contained in the ring of integers $\mathcal{O}_{\kappa(\mathfrak{p})}$ of $\kappa(\mathfrak{p})$, and the quotient topology on R/\mathfrak{p} induced by the \mathfrak{m}_R -adic topology on R coincides with the subspace topology induced by the topology on $\mathcal{O}_{\kappa(\mathfrak{p})}$.*

Proof. It follows from Cohen’s structure theorem that if $\text{char}(R/\mathfrak{p}) = 0$, then $\mathcal{O} \subset R/\mathfrak{p}$ and R/\mathfrak{p} is a finitely generated \mathcal{O} -module. Thus, $\kappa(\mathfrak{p})$ is a finite extension of L , and R/\mathfrak{p} is contained in the integral closure of \mathcal{O} in $\kappa(\mathfrak{p})$, which is equal to $\mathcal{O}_{\kappa(\mathfrak{p})}$. If $\text{char}(R/\mathfrak{p}) = p$, then R/\mathfrak{p} is finite over a subring isomorphic to $k'[[t]]$, and the same argument carries over. Moreover, $\mathcal{O}_{\kappa(\mathfrak{p})}$ is a finitely generated R/\mathfrak{p} -module, and this implies that the topologies coincide. \square

Lemma 3.18. *Let (R, \mathfrak{m}_R) be a complete local Noetherian ring and $\varphi : R \rightarrow S$ a ring map of finite type. Let U be a non-empty open subscheme of $U_{\max} := (\text{Spec } R) \setminus \{\mathfrak{m}_R\}$, let V (resp. V_{\max}) be the preimage of U (resp. U_{\max}) in $\text{Spec } S$, let Z (resp. Z_{\max}) be the closure of V (resp. V_{\max}) in $\text{Spec } S$ and let Y be the preimage of $\{\mathfrak{m}_R\}$ in $\text{Spec } S$. Then*

- (1) V is Jacobson;
- (2) the set of closed points of V is $V \cap \{\text{closed points of } V_{\max}\}$;
- (3) if x is a closed point of V , then its image y in $\text{Spec } R$ is a closed point of U and the field extension $\kappa(x)/\kappa(y)$ is finite;
- (4) the set of closed points of U is $U \cap P_1 R$;
- (5) if every irreducible component of $\text{Spec } S$ meets Y nontrivially, then $\dim Z = \dim V + 1$;
- (6) $\dim V \leq \dim U + \max_{y \in U \cap P_1 R} \dim \varphi^{-1}(\{y\})$.

Proof. We summarize the situation in the following diagram.



We will first prove parts (1), (2) and (3). If $R = S$ and if $U = U_{\max}$, then (1) follows from [48, Tag 02IM] and both (2) and (3) hold trivially. If $R = S$ and if U is arbitrary, then $U = V$ and (1), (2) follow from the previous case together with [48, Tag 005W] and (3) holds trivially. The case of general φ now follows from [48, Tag 00GB] together with [48, Tag 01P4] because the map $V \rightarrow U$ induced from φ is of finite type.

Part (4) follows from (2) applied with $S = R$, using that \mathfrak{m}_R is the unique maximal ideal of R , so that the set of closed points of U_{\max} is equal to $P_1 R$.

For (5), note first that since V is open in $\text{Spec } S$, the set of generic points of V is a subset of the set of generic points of $\text{Spec } S$. Thus, Z is union of irreducible components of $\text{Spec } S$. Let $Z' = \text{Spec } S'$ be an irreducible component of Z with the induced reduced subscheme structure so that S' is a domain, let $V' = Z' \cap V$ and let R' be the image of R in S' . The rings R' and S' are excellent and hence universally catenary by [48, Tag 07QW]. If $\mathfrak{q} \in \text{Spec } S'$ and $\mathfrak{p} = \mathfrak{q} \cap R'$, then

$$\begin{aligned}
 \dim S'_q &= \dim R'_p + \text{trdeg}_{R'} S' - \text{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) \\
 &= \dim R' + \text{trdeg}_{R'} S' - \dim R'/\mathfrak{p} - \text{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}),
 \end{aligned}
 \tag{13}$$

where trdeg stands for transcendence degree, the first equality is [48, Tag 02IJ] and the second is [36, Theorem 31.4]. It follows from (13) that

$$\dim S'_q \leq \dim R' + \text{trdeg}_{R'} S', \tag{14}$$

and the equality in (14) holds if and only if \mathfrak{q} maps to the maximal ideal of R' and \mathfrak{q} is a maximal ideal of S' . Since $Z' \cap Y$ is non-empty by assumption, such \mathfrak{q} exists and so

$$\dim S' = \dim R' + \text{trdeg}_{R'} S'.$$

Let \mathfrak{q} be a closed point of V' and let $\mathfrak{p} = \mathfrak{q} \cap R'$. Since V' is open in Z' , we have $\mathcal{O}_{V',\mathfrak{q}} = S'_q$. It follows from (3) that $\text{trdeg}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{q}) = 0$ and $\dim R/\mathfrak{p} = 1$. Thus, (13) gives us

$$\dim \mathcal{O}_{V',\mathfrak{q}} = \dim R' + \text{trdeg}_{R'} S' - 1.$$

Since this holds for all closed points of V' , we deduce that

$$\dim V' = \dim R' + \text{trdeg}_{R'} S' - 1.$$

This implies part (5).

Let x be a closed point of V and let y be its image in U . Then y is also a closed point of U . We have

$$\dim \mathcal{O}_{V,x} \leq \dim \mathcal{O}_{U,y} + \dim(\mathcal{O}_{V,x} \otimes_{\mathcal{O}_{U,y}} \kappa(y)) \leq \dim U + \dim \varphi^{-1}(\{y\}),$$

where the first inequality is given by [36, Theorem 15.1 (i)]. Since

$$\dim V = \max_x \dim \mathcal{O}_{V,x},$$

where the maximum is taken over all closed points x of V , we get (6). □

Remark 3.19. We caution the reader that the equality $\dim Z = \dim V + 1$ might fail if one drops the assumption that Y meets every irreducible component nontrivially. For example, if $R = \mathbb{Z}_p$ and $S = \mathbb{Z}_p[x]/(px - 1) = \mathbb{Q}_p$, then Y is empty and $V_{\max} = Z_{\max} = \text{Spec } S$.

Remark 3.20. Here is another cautionary example. If R and S are as in Lemma 3.18, \mathfrak{q} is a prime of S and S is a domain, then it need not be true that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = \dim S$. For example, if $R = \mathbb{Z}_p$, $S = \mathbb{Z}_p[x]$ and $\mathfrak{q} = (px - 1)$, then $S/\mathfrak{q} = \mathbb{Q}_p$ and $S_{\mathfrak{q}}$ is a DVR, so that $\dim S_{\mathfrak{q}} + \dim S/\mathfrak{q} = 1$ and $\dim S = 2$. We also note that \mathfrak{q} is a closed point of $\text{Spec } S$, but it does not map to a closed point of $\text{Spec } R$. Further, if $\mathfrak{q}' = (p, x)$, then $S/\mathfrak{q}' = \mathbb{F}_p$ and p, x is a regular sequence of parameters in $S_{\mathfrak{q}'}$, and thus $\dim S_{\mathfrak{q}'} = 2$. Thus, \mathfrak{q} and \mathfrak{q}' are closed points of an irreducible scheme, but their local rings have different dimensions.

Lemma 3.21. *Let Y be the preimage of $\{\mathfrak{m}_{R^{\text{ps}}}\}$ in X^{gen} , let W be a closed non-empty GL_d -invariant subscheme of X^{gen} and let Z be an irreducible component of W . Then $Y \cap Z$ is non-empty. Moreover, if x is a closed point of Z , then the following hold:*

- (1) if $x \in Y$, then $\dim \mathcal{O}_{Z,x} = \dim Z$;
- (2) if $x \notin Y$, then $\dim \mathcal{O}_{Z,x} = \dim Z - 1$.

Proof. By Lemma 2.1, each irreducible component Z of W is GL_d -invariant. The image of Z in X^{ps} is closed by Corollary 2 (ii) to [47, Proposition 9], and is nonempty and so must contain $\mathfrak{m}_{R^{\text{ps}}}$ because X^{ps} has a unique closed point. Therefore, $Y \cap Z$ is nonempty.

The claims about $\dim \mathcal{O}_{Z,x}$ follows from the proof of part (5) in Lemma 3.18. □

Example 3.22. Let us illustrate Lemma 3.21 with a concrete example. Let \overline{D} be the pseudo-character of the 2-dimensional trivial representation of the group $\Gamma := \mathbb{Z}_p$. It follows from [18, Theorem 1.15] that $R^{\text{ps}} \cong \mathcal{O}[[t, d]]$ and

$$E \cong \frac{R^{\text{ps}}[[T]]}{((1+T)^2 - (2+t)(1+T) + 1+d)},$$

where the map $\Gamma \rightarrow R^{\text{ps}}[[\Gamma]] \twoheadrightarrow E$ sends a fixed topological generator γ of Γ to $1+T$. Then E is a free R^{ps} -module with basis $1+T, 1$ and so

$$A^{\text{gen}} = \frac{R^{\text{ps}}[x_{11}, x_{12}, x_{21}, x_{22}]}{(x_{11} + x_{22} - (2+t), x_{11}x_{22} - x_{12}x_{21} - (1+d))},$$

and $j : E \rightarrow M_2(A^{\text{gen}})$ sends $1+T$ to the matrix $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Let $x : A^{\text{gen}} \rightarrow L$ be the homomorphism corresponding to the representation $\rho : E \rightarrow M_2(L)$, such that $\rho(\gamma) = \begin{pmatrix} 1 & p^{-1} \\ 0 & 1 \end{pmatrix}$. Then x is a closed point of X^{gen} with residue field L ; thus, it does not map to the closed point in X^{ps} . Indeed, $A^{\text{gen}}/(x_{11} - 1, x_{21}, x_{22} - 1) \cong \mathcal{O}[x_{12}]$, so we are in the situation considered in Remark 3.20.

Lemma 3.23. *Let W be a closed nonempty GL_d -invariant subscheme of X^{gen} and write $W[1/p]$ and \overline{W} for the generic and special fibre. Then $\dim W[1/p] \leq \dim \overline{W}$. In particular, $\dim X^{\text{gen}}[1/p] \leq \dim \overline{X^{\text{gen}}}$.*

Proof. We may assume that $W[1/p]$ is nonempty, and using Lemma 2.1, we may further assume that W is irreducible. Lemma 3.21 implies that there is a closed point $x \in W$, which maps to the closed point in X^{ps} . Lemma 3.18 (5) implies that $\dim W[1/p] = \dim W - 1$.

Since W is irreducible and $W[1/p] \neq \emptyset$, the local ring $\mathcal{O}_{W,x}$ is a domain, and multiplication by ϖ is injective. Since $\text{char}(\kappa(x)) = p$, ϖ cannot be a unit in $\mathcal{O}_{W,x}$. Thus, $\dim \mathcal{O}_{\overline{W},x} = \dim \mathcal{O}_{W,x} - 1$. It follows from Lemma 3.21 that $\dim \overline{W} = \dim W - 1$. □

3.4. Bounding the dimension of the space

The main result of this subsection is Theorem 3.31, which bounds the dimension of $\overline{X^{\text{gen}}}$. As explained earlier, this is an intermediate step in bounding the dimension of R^{\square}_{ρ} .

Recall that $\overline{D} : G_F \rightarrow k$ is the specialization of the universal pseudo-character $D^u : G_F \rightarrow R^{\text{ps}}$ at the maximal ideal of R^{ps} . We may write $\overline{D} = \prod_{i=1}^m \overline{D}_i$, where \overline{D}_i are absolutely irreducible pseudo-characters. Let \mathcal{P} be an (unordered) partition of the set $\{1, \dots, m\}$ into r disjoint subsets Σ_j , and let $\underline{\Sigma} = (\Sigma_1, \dots, \Sigma_r)$ be an ordering of the subsets in \mathcal{P} . For each $1 \leq j \leq r$, let $\overline{D}'_j = \prod_{i \in \Sigma_j} \overline{D}_i$, and let d_j be the dimension of \overline{D}'_j . We define an equivalence relation on the set of pseudo-characters $\{\overline{D}'_j : 1 \leq j \leq r\}$ by $\overline{D}'_j \sim \overline{D}'_{j'}$ if $\overline{D}'_j = \overline{D}'_{j'}(t)$ for some $t \in \mathbb{Z}$. Let k' be the number of the equivalence classes, n'_i be the number of elements in the i -th equivalence class, and c_i be the dimension of the pseudo-characters in the i -th equivalence class. We have

$$\sum_{i=1}^{k'} n'_i = r, \quad \sum_{i=1}^{k'} c_i n'_i = d.$$

We define

$$l_{\mathcal{P}} := \sum_{j=1}^r d_j^2 = \sum_{i=1}^{k'} n'_i c_i^2, \quad p_{\mathcal{P}} := l_{\mathcal{P}} + n_{\mathcal{P}} = \sum_{j=1}^r d_j^2 + \sum_{1 \leq j < j' \leq r} d_j d_{j'}, \tag{15}$$

where

$$n_{\mathcal{P}} = \frac{1}{2}(d^2 - l_{\mathcal{P}}) = \sum_{1 \leq j < j' \leq r} d_j d_{j'} = \sum_{1 \leq i < i' \leq k'} c_i c_{i'} n'_i n'_{i'} + \sum_{i=1}^{k'} c_i^2 \binom{n'_i}{2}. \tag{16}$$

The notation is motivated by (7) and (8); see also Remark 3.8.

For each $1 \leq j \leq r$, let R_j^{ps} be the universal deformation ring of \overline{D}'_j and let $X_j^{\text{ps}} := R_j^{\text{ps}}$. The functor $\mathcal{F}_{\underline{\Sigma}}$, which sends a local Artinian \mathcal{O} -algebra (A, \mathfrak{m}_A) with residue field k to the set of ordered r -tuples (D_1, \dots, D_r) of pseudo-characters with each D_i , a deformation of \overline{D}'_i to A is represented by the completed tensor product

$$R_{\underline{\Sigma}}^{\text{ps}} := R_1^{\text{ps}} \widehat{\otimes}_{\mathcal{O}} \dots \widehat{\otimes}_{\mathcal{O}} R_r^{\text{ps}}.$$

We let $X_{\underline{\Sigma}}^{\text{ps}} := \text{Spec } R_{\underline{\Sigma}}^{\text{ps}}$ and denote by $\overline{X}_{\underline{\Sigma}}^{\text{ps}} := \text{Spec } R_{\underline{\Sigma}}^{\text{ps}} / \varpi$ its special fibre. By mapping an r -tuple of pseudo-characters to their product, we obtain a map

$$t_{\underline{\Sigma}} : \overline{X}_{\underline{\Sigma}}^{\text{ps}} \rightarrow \overline{X}^{\text{ps}}.$$

Lemma 3.24. *The map $R^{\text{ps}} \rightarrow R_{\underline{\Sigma}}^{\text{ps}}$ is finite.*

Proof. By topological Nakayama’s lemma, it is enough to show that the fibre ring $C := k \otimes_{R^{\text{ps}}} R_{\underline{\Sigma}}^{\text{ps}}$ is a finite dimensional k -vector space. Let \mathcal{F} be the closed subfunctor of $\mathcal{F}_{\underline{\Sigma}}$ defined by C . If (A, \mathfrak{m}_A) is a local Artinian k -algebra, then $\mathcal{F}(A)$ is in bijection with the set of r -tuples (D_1, \dots, D_r) , each D_i lifting \overline{D}'_i to A such that $\prod_{i=1}^r D_i = (\prod_{i=1}^r \overline{D}'_i) \otimes_k A$.

Since C is a complete local Noetherian ring, it is enough to show that its Krull dimension is 0. If this is not the case, then there is $\mathfrak{p} \in \text{Spec } C$ such that $\dim C/\mathfrak{p} = 1$. Let $(D_{1,y}, \dots, D_{r,y})$ be the specialization of the universal object of $\mathcal{F}_{\underline{\Sigma}}$ along $y : R_{\underline{\Sigma}}^{\text{ps}} \rightarrow \kappa(\mathfrak{p})$. It follows from [18, Corollary 1.14] that the coefficients of the polynomials $D_{i,y}(t - a)$, for all $a \in E$ and $1 \leq i \leq r$ will generate a dense subring of $R_{\underline{\Sigma}}^{\text{ps}}/\mathfrak{p}$. Since $R_{\underline{\Sigma}}^{\text{ps}}/\mathfrak{p}$ is a complete local k -algebra of dimension 1, there will exist $a \in E$ and index i such that the coefficients of $D_{i,y}(t - a)$ will generate a transcendental extension of k inside $\kappa(\mathfrak{p})$. Since $\mathfrak{p} \in \text{Spec } C$, we have

$$\prod_{i=1}^r D_{i,y}(t - a) = \prod_{i=1}^r \overline{D}'_i(t - a).$$

Thus, all the roots of $D_{i,y}(t - a)$ in the algebraic closure of $\kappa(\mathfrak{p})$ are algebraic over k . Since $D_{i,y}(t - a)$ is a monic polynomial, we conclude that all the coefficients are also algebraic over k , giving a contradiction. \square

Let $\overline{X}_{\mathcal{P}}^{\text{ps}}$ be the scheme theoretic image of $t_{\underline{\Sigma}}$. We note that $\overline{X}_{\mathcal{P}}^{\text{ps}}$ depends only on \mathcal{P} and not on the chosen ordering $\underline{\Sigma}$. It follows from Lemma 3.24 that

$$\dim \overline{X}_{\mathcal{P}}^{\text{ps}} = \dim \overline{X}_{\underline{\Sigma}}^{\text{ps}} = \sum_{i=1}^r \dim \overline{X}_i^{\text{ps}} = r + l_{\mathcal{P}}[F : \mathbb{Q}_p], \tag{17}$$

where the last equality is obtained by applying [9, Theorem 5.4.1(a)] to each $\overline{X}_i^{\text{ps}}$.

We define a partial order on the set of partitions of $\{1, \dots, m\}$ by $\mathcal{P} \leq \mathcal{P}'$ if \mathcal{P}' is a refinement of \mathcal{P} . The partition \mathcal{P}_{\min} consisting of one part is the minimal element, and the partition \mathcal{P}_{\max} consisting of m parts is the maximal element with respect to this partial ordering. If $\mathcal{P} \leq \mathcal{P}'$, then $\overline{X}_{\mathcal{P}'}^{\text{ps}}$ is a closed subscheme of $\overline{X}_{\mathcal{P}}^{\text{ps}}$ and $\overline{X}_{\mathcal{P}_{\min}}^{\text{ps}} = \overline{X}^{\text{ps}}$. Let

$$U_{\mathcal{P}} := \overline{X}_{\mathcal{P}}^{\text{ps}} \setminus (\{\mathfrak{m}_{R^{\text{ps}}}\} \cup \bigcup_{\mathcal{P}' < \mathcal{P}} \overline{X}_{\mathcal{P}'}^{\text{ps}}),$$

let $V_{\mathcal{P}}$ be the preimage of $U_{\mathcal{P}}$ in $\overline{X}^{\text{gen}}$ and let $Z_{\mathcal{P}}$ be the closure of $V_{\mathcal{P}}$ in $\overline{X}^{\text{gen}}$. Let $\overline{X}_{\mathcal{P}}^{\text{gen}}$ be the preimage of $\overline{X}_{\mathcal{P}}^{\text{ps}}$ in $\overline{X}^{\text{gen}}$. Then $\overline{X}_{\mathcal{P}}^{\text{gen}}$ is closed in $\overline{X}^{\text{gen}}$ and contains $V_{\mathcal{P}}$; hence, we are in the situation of Lemma 3.18 with $\text{Spec } R = \overline{X}_{\mathcal{P}}^{\text{ps}}$ and $\text{Spec } S = Z_{\mathcal{P}}$. Note that Lemma 3.21 implies that every irreducible component

of $\overline{X}_{\mathcal{P}}^{\text{gen}}$ contains a closed point mapping to $\mathfrak{m}_{R^{\text{ps}}}$. Thus, the condition in part (5) of Lemma 3.18 is satisfied, and hence, $\dim Z_{\mathcal{P}} = \dim V_{\mathcal{P}} + 1$; the same conclusion applies to closures of various loci considered below. Moreover, we have

$$\overline{X}_{\mathcal{P}}^{\text{gen}} = Y \cup \bigcup_{\mathcal{P}' \leq \mathcal{P}} Z_{\mathcal{P}'}, \tag{18}$$

where Y is the preimage of $\{\mathfrak{m}_{R^{\text{ps}}}\}$ in $\overline{X}^{\text{gen}}$.

We will also need a variant of the situation above. Let us assume that $r > 1$ and let i and j be distinct indices with $1 \leq i, j \leq r$. Let $\mathcal{F}_{\Sigma}^{ij}$ be a subfunctor of \mathcal{F}_{Σ} parameterizing the deformations (D_1, \dots, D_r) of the ordered r -tuple $(\overline{D}'_1, \dots, \overline{D}'_r)$ such that $D_i = D_j(1)$. Then $\mathcal{F}_{\Sigma}^{ij}$ is a closed subfunctor of \mathcal{F}_{Σ} , and we let $R_{\Sigma}^{\text{ps},ij}$ be the quotient of R_{Σ}^{ps} representing it. If $\overline{D}'_i \neq \overline{D}'_j(1)$, then $R_{\Sigma}^{\text{ps},ij}$ is the zero ring; otherwise, it follows from Equation (17) and another application of [9, Theorem 5.4.1(a)] that

$$\begin{aligned} \dim R_{\Sigma}^{\text{ps},ij}/\varpi &= \dim R_{\Sigma}^{\text{ps}}/\varpi - \dim R_i^{\text{ps}}/\varpi \leq r + l_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}] - (1 + d_i^2[F : \mathbb{Q}_{\mathcal{P}}]) \\ &\leq r + l_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}] - (1 + [F : \mathbb{Q}_{\mathcal{P}}]). \end{aligned}$$

Let $\overline{X}_{\mathcal{P}}^{\text{ps},ij}$ be the scheme theoretic image of $\text{Spec } R_{\Sigma}^{\text{ps},ij}$ in \overline{X}^{ps} under ι_{Σ} . Then

$$\dim \overline{X}_{\mathcal{P}}^{\text{ps},ij} \leq r + l_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}] - (1 + [F : \mathbb{Q}_{\mathcal{P}}]). \tag{19}$$

Let $U_{\mathcal{P}}^{ij} := U_{\mathcal{P}} \cap \overline{X}_{\mathcal{P}}^{\text{ps},ij}$, let $V_{\mathcal{P}}^{ij}$ be the preimage of $U_{\mathcal{P}}^{ij}$ in $\overline{X}^{\text{gen}}$ and let $Z_{\mathcal{P}}^{ij}$ be the closure of $V_{\mathcal{P}}^{ij}$ in $\overline{X}^{\text{gen}}$.

Lemma 3.25. *If y is a geometric closed point of $U_{\mathcal{P}}$, then*

$$\dim X_y^{\text{gen}} \leq d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}] + \sum_{i=1}^{k'} \binom{n'_i}{2}.$$

If we additionally assume that $y \notin U_{\mathcal{P}}^{ij}$ for any $i \neq j$, then

$$\dim X_y^{\text{gen}} \leq d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}].$$

Proof. We may write $D_y = D_1 + \dots + D_r$ with D_i lifting \overline{D}'_i . We note that all the D_i are absolutely irreducible since otherwise, $y \in X_{\mathcal{P}'}^{\text{ps}}$ for some $\mathcal{P}' > \mathcal{P}$. Let k and n_i be the numbers defined in Section 3.2. Proposition 3.15 implies that

$$\dim X_y^{\text{gen}} \leq d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_{\mathcal{P}}] + \sum_{i=1}^k \binom{n_i}{2}.$$

If $D_i = D_j(m)$ for some $m \in \mathbb{Z}$, then also $\overline{D}'_i = \overline{D}'_j(m)$. This implies that

$$\sum_{i=1}^k \binom{n_i}{2} \leq \sum_{i=1}^{k'} \binom{n'_i}{2},$$

which implies the first assertion. We note that if a_1, \dots, a_s are positive integers, then

$$\sum_{i=1}^s \binom{a_i}{2} \leq \binom{\sum_{i=1}^s a_i}{2}.$$

If $y \notin U_{\mathcal{P}}^{ij}$ for any $i \neq j$, then $D_i \neq D_j(1)$ for any $i \neq j$, and the Hom terms in (9) vanish. The assertion follows from Proposition 3.15 using this improved bound. \square

Proposition 3.26. $\dim Z_{\mathcal{P}}^{ij} \leq d^2 + p_{\mathcal{P}}[F : \mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2} - (1 + [F : \mathbb{Q}_p])$.

Proof. It follows from Lemma 3.18 (5) that the closure of $U_{\mathcal{P}}^{ij}$ has dimension $\dim U_{\mathcal{P}}^{ij} + 1$. Thus,

$$\dim U_{\mathcal{P}}^{ij} + 1 \leq \dim X_{\mathcal{P}}^{\text{ps},ij} \leq r + l_{\mathcal{P}}[F : \mathbb{Q}_p] - (1 + [F : \mathbb{Q}_p]),$$

where the last inequality is (19). Parts (5) and (6) of Lemma 3.18 together with Lemma 3.25 imply that

$$\dim Z_{\mathcal{P}}^{ij} \leq (r + l_{\mathcal{P}}[F : \mathbb{Q}_p] - (1 + [F : \mathbb{Q}_p])) + (d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}),$$

which imply the assertion. \square

Proposition 3.27. Let $\delta_{\mathcal{P}} = \max\{0, \sum_{i=1}^{k'} \binom{n'_i}{2} - (1 + [F : \mathbb{Q}_p])\}$. Then

$$\dim Z_{\mathcal{P}} \leq d^2 + p_{\mathcal{P}}[F : \mathbb{Q}_p] + \delta_{\mathcal{P}}.$$

Proof. Let $U'_{\mathcal{P}} := U_{\mathcal{P}} \setminus \bigcup_{i \neq j} U_{\mathcal{P}}^{ij}$, let $V'_{\mathcal{P}}$ be the preimage of $U'_{\mathcal{P}}$ in $\overline{X}^{\text{gen}}$ and let $Z'_{\mathcal{P}}$ denote the closure of $V'_{\mathcal{P}}$ in $\overline{X}^{\text{gen}}$. If y is a closed point of $U'_{\mathcal{P}}$, then $\dim \overline{X}_y^{\text{gen}} \leq d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_p]$ by Lemma 3.25. Thus, Lemma 3.18 implies that

$$\dim Z'_{\mathcal{P}} \leq \dim \overline{X}_{\mathcal{P}}^{\text{ps}} + (d^2 - r + n_{\mathcal{P}}[F : \mathbb{Q}_p]) = d^2 + p_{\mathcal{P}}[F : \mathbb{Q}_p]. \tag{20}$$

Since $Z_{\mathcal{P}} = Z'_{\mathcal{P}} \cup \bigcup_{i \neq j} Z_{\mathcal{P}}^{ij}$, we have $\dim Z_{\mathcal{P}} = \max_{i \neq j} \{\dim Z'_{\mathcal{P}}, \dim Z_{\mathcal{P}}^{ij}\}$, and the assertion follows from Proposition 3.26. \square

Proposition 3.28. $\dim Z_{\mathcal{P}_{\min}} \leq d^2 + d^2[F : \mathbb{Q}_p]$.

Proof. In this case, $r = 1$, so $Z_{\mathcal{P}} = Z'_{\mathcal{P}}$, and the assertion follows from (20). \square

Lemma 3.29. Assume that $\mathcal{P} \neq \mathcal{P}_{\min}$. If $d = 2$, then

$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq [F : \mathbb{Q}_p],$$

and

$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq 1 + [F : \mathbb{Q}_p],$$

otherwise.

Proof. Proposition 3.27 implies that

$$d^2 + d^2[F : \mathbb{Q}_p] - \dim Z_{\mathcal{P}} \geq n_{\mathcal{P}}[F : \mathbb{Q}_p] - \delta_{\mathcal{P}}.$$

If $d > 2$, then $n_{\mathcal{P}} \geq 2$, and if $d = 2$, then $n_{\mathcal{P}} = 1$, which implies the assertion if $\delta_{\mathcal{P}} = 0$. Let us assume that $\delta_{\mathcal{P}} \neq 0$. Then using (16), we may write

$$n_{\mathcal{P}}[F : \mathbb{Q}_p] - \delta_{\mathcal{P}} = \sum_{1 \leq i < j \leq k'} c_i c_j n'_i n'_j [F : \mathbb{Q}_p] + \sum_{i=1}^{k'} (c_i^2 [F : \mathbb{Q}_p] - 1) \binom{n'_i}{2} + 1 + [F : \mathbb{Q}_p],$$

which implies the assertion. \square

Lemma 3.30. *Let Y be the preimage of $\{\mathfrak{m}_{R^{\text{ps}}}\}$ in $\overline{X}^{\text{gen}}$. Then*

$$\dim Y \leq d^2 + n_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] + n_{\mathcal{P}_{\max}} - 1.$$

In particular, $d^2 + d^2[F : \mathbb{Q}_p] - \dim Y \geq 1 + l_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] \geq 1 + 2[F : \mathbb{Q}_p]$.

Proof. Proposition 3.15 implies that

$$\dim Y \leq d^2 - m + n_{\mathcal{P}_{\max}}[F : \mathbb{Q}_p] + \sum_{i=1}^{k'} \binom{n'_i}{2}.$$

As already explained in the proof of Lemma 3.29, we have $\sum_{i=1}^{k'} \binom{n'_i}{2} \leq n_{\mathcal{P}_{\max}}$. This implies the assertion. □

Theorem 3.31. $\dim \overline{X}^{\text{gen}} \leq d^2 + d^2[F : \mathbb{Q}_p]$.

Proof. Since $\overline{X}^{\text{ps}} = \{\mathfrak{m}_{R^{\text{ps}}}\} \cup \bigcup_{\mathcal{P}} U_{\mathcal{P}}$, we have $\overline{X}^{\text{gen}} = Y \cup \bigcup_{\mathcal{P}} Z_{\mathcal{P}}$. Since these are closed in $\overline{X}^{\text{gen}}$, we have

$$\dim \overline{X}^{\text{gen}} = \max_{\mathcal{P}} \{\dim Y, \dim Z_{\mathcal{P}}\} \leq d^2 + d^2[F : \mathbb{Q}_p],$$

by Proposition 3.28 and Lemmas 3.29 and 3.30. □

Theorem 3.31 is the main input to Corollary 3.38, which proves Theorem 1.1. The missing ingredient is a description of the relationship between X^{gen} and $R_{\mathcal{P}}^{\square}$, which is the subject of the next subsection.

3.5. Completions at maximal ideals and deformation problems

Let $Y \subset X^{\text{gen}}$ be the preimage of the closed point of X^{ps} , let x be either a closed point of Y or a closed point of $X^{\text{gen}} \setminus Y$ and let y be its image in $\text{Spec } R^{\text{ps}}$. It follows from Lemmas 3.17 and 3.18 that $\kappa(x)$ is a finite extension of $\kappa(y)$ and there are the following possibilities:

- (1) if $x \in Y$, then $\kappa(x)$ is a finite extension of k ;
- (2) if $x \in X^{\text{gen}}[1/p]$, then $\kappa(x)$ is a finite extension of L ;
- (3) if $x \in \overline{X}^{\text{gen}} \setminus Y$, then $\kappa(x)$ is a local field of characteristic p .

The universal property of A^{gen} gives us a continuous Galois representation

$$\rho_x : G_F \rightarrow \text{GL}_d(\kappa(x)).$$

In this section, we want to relate the completion of the local ring $\mathcal{O}_{X^{\text{gen}},x}$ to a deformation problem for ρ_x .

We will introduce some notation to formulate the deformation problem for ρ_x . More generally, let $\rho : G_F \rightarrow \text{GL}_d(\kappa)$ be a continuous representation, where κ is either a finite extension of k , a finite extension of L or a local field of characteristic p containing k equipped with natural topology. We first define a ring of coefficients Λ over which the deformation problem is defined.

- (1) If κ is a finite field, then pick an unramified extension L' of L with residue field κ and let $\Lambda := \mathcal{O}_{L'}$ denote the ring of integers in L' .
- (2) If κ is a finite extension of L , then let $\Lambda := \kappa$, let Λ^0 be the ring of integers in Λ and let $t = \varpi$.
- (3) If κ is a local field of characteristic p , then let \mathcal{O}_{κ} be the ring of integers in κ and let k' be its residue field. Since $\text{char}(\kappa) = p$, by choosing a uniformizer, we obtain an isomorphism $\mathcal{O}_{\kappa} \cong k'[[t]]$. Let L' be an unramified extension of L with residue field k' , let $\Lambda^0 := \mathcal{O}_{L'}[[t]]$ and let Λ be the p -adic completion of $\Lambda^0[1/t]$. Then Λ is a complete DVR with uniformiser ϖ and residue field κ . We equip Λ^0 with its (ϖ, t) -adic topology. This induces a topology on $\Lambda^0[1/t]$ and $\Lambda^0[1/t]/p^n \Lambda^0[1/t]$ for all $n \geq 1$. We equip $\Lambda = \varprojlim_n \Lambda^0[1/t]/p^n \Lambda^0[1/t]$ with the projective limit topology.

Remark 3.32. In case (3), if $\widehat{\Lambda}'$ is an \mathcal{O} -algebra, which is a complete DVR with uniformiser ϖ and residue field κ , then it follows from [10, Ch. IX, §2.3, Prop. 4] that $\widehat{\Lambda}'$ is non-canonically isomorphic to Λ . We will refer to $\widehat{\Lambda}'$ (and Λ) as an \mathcal{O} -Cohen ring of κ .

Let \mathfrak{A}_Λ be the category of local Artinian Λ -algebras with residue field κ . Let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$.

- (1) In case (1), A is a finite \mathcal{O}/ϖ^n -module for some $n \geq 1$, and we just put the discrete topology on A .
- (2) In case (2), A is a finite dimensional L -vector space, and we put the p -adic topology on A .
- (3) In case (3), A is a $\Lambda^0[1/t]/\varpi^n\Lambda^0[1/t]$ -module of finite length for some $n \geq 1$, and we put the induced topology on A .

Let $D_\rho^\square(A)$ be the set of continuous group homomorphisms $\rho_A : G_F \rightarrow \mathrm{GL}_d(A)$, such that $\rho_A \pmod{\mathfrak{m}_A} = \rho$.

Proposition 3.33. *The functor $D_\rho^\square : \mathfrak{A}_\Lambda \rightarrow \text{Sets}$ is pro-represented by a complete local Noetherian Λ -algebra R_ρ^\square . Moreover, there is a presentation*

$$R_\rho^\square \cong \Lambda\llbracket x_1, \dots, x_r \rrbracket / (f_1, \dots, f_s) \tag{21}$$

with $r = \dim_\kappa Z^1(G_F, \mathrm{ad} \rho)$ and $s = \dim_\kappa H^2(G_F, \mathrm{ad} \rho)$.

Proof. If κ is a finite field, then this is a well-known consequence of the obstruction theory due to Mazur, [37, Section 1.6]. (We revisit the argument in the proof of Proposition 4.3.) If κ is a local field, then essentially the same argument works, except that one has to work harder to justify why the 2-cocycle constructed out of an obstruction to lifting is continuous. Lecture 6 in [21] contains a very nice exposition of the result if κ is a finite extension of L . The same argument works if κ is a local field of characteristic p . □

If we let $h^i := \dim_\kappa H^i(G_F, \mathrm{ad} \rho)$, then

$$r - s = \dim_\kappa(\mathrm{ad} \rho) - h^0 + h^1 - h^2 = d^2 + d^2[F : \mathbb{Q}_p], \tag{22}$$

where the last equality follows from Euler–Poincaré characteristic formula, which by [9, Theorem 3.4.1(c)] holds in all of the three settings under consideration.

Proposition 3.34. *Let \mathfrak{q} be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}} \rightarrow \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}\bar{a},$$

where $\bar{\lambda}$ and \bar{a} denote the images of λ and a in $\kappa(x)$. Then the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R_{\rho_x}^\square$.

Proof. We will prove the proposition, when $\kappa(x)$ is a local field of characteristic p . The other cases are similar and are left to the reader.

Let \widehat{B} be the completion of $(\Lambda \otimes_{\mathcal{O}} A^{\mathrm{gen}})_{\mathfrak{q}}$. It follows from Lemma 3.36 below that $\widehat{B}/\varpi\widehat{B}$ (and hence \widehat{B}) is Noetherian. Thus, $\widehat{B}/\mathfrak{q}^n\widehat{B} \in \mathfrak{A}_\Lambda$ for all $n \geq 1$. The composition

$$\Lambda \otimes_{\mathcal{O}} E \xrightarrow{\mathrm{id} \otimes j} \Lambda \otimes_{\mathcal{O}} M_d(A^{\mathrm{gen}}) \rightarrow M_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$$

induces a continuous representation $G_F \rightarrow \mathrm{GL}_d(\widehat{B}/\mathfrak{q}^n\widehat{B})$ by Lemma 3.2, which is a deformation of ρ_x to $\widehat{B}/\mathfrak{q}^n\widehat{B}$, and hence a map of local Λ -algebras $R_{\rho_x}^\square \rightarrow \widehat{B}/\mathfrak{q}^n\widehat{B}$. By passing to the projective limit over n , we obtain a continuous representation $\widehat{\rho} : G_F \rightarrow \mathrm{GL}_d(\widehat{B})$ and a map of local Λ -algebras $R_{\rho_x}^\square \rightarrow \widehat{B}$.

Let $(A, \mathfrak{m}_A) \in \mathfrak{A}_\Lambda$ and let $\rho : G_F \rightarrow \mathrm{GL}_d(A)$ be a continuous representation such that $\rho \pmod{\mathfrak{m}_A} = \rho_x$. We claim that there is a unique homomorphism of local Λ -algebras $\varphi : \widehat{B} \rightarrow A$, such that ρ is equal to the composition $\mathrm{GL}_d(\varphi) \circ \widehat{\rho}$. The claim implies that the map $R_{\rho_x}^\square \rightarrow \widehat{B}$ constructed above is an isomorphism.

The proof of the claim is based on [31, Proposition 9.5]. Following its proof, we may construct an ascending chain of local open Λ^0 -subalgebras A_n^0 of A for $n \geq 1$, such that for all n , the following hold: $A_n^0[1/t] = A$, the image of A_n^0 under the projection $b : A \rightarrow \kappa(x)$ is equal to $\mathcal{O}_{\kappa(x)}$ and $\bigcup_{n \geq 1} A_n^0 = b^{-1}(\mathcal{O}_{\kappa(x)})$. Let $M \in \mathrm{GL}_d(\kappa(x))$ be a matrix such that the image of G_F under $M\rho_x M^{-1}$ is contained in $\mathrm{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' \in X^{\mathrm{gen}}$ correspond to the representation $M\rho_x M^{-1}$. Then $\kappa(x') = \kappa(x)$ and the image of $x' : A^{\mathrm{gen}} \rightarrow \kappa(x)$ is contained in $\mathcal{O}_{\kappa(x)}$. Let $z \in X^{\mathrm{gen}}$ be the composition $z : A^{\mathrm{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \rightarrow k'$, where k' is the residue field of $\mathcal{O}_{\kappa(x)}$, let $\tilde{M} \in \mathrm{GL}_d(A)$ be a matrix lifting M and let $\rho' := \tilde{M}\rho\tilde{M}^{-1}$. Since G_F is compact, $\rho'(G_F)$ will be contained in some $\mathrm{GL}_d(A_n^0)$ for $n \gg 0$. We may consider $\rho' : G_F \rightarrow \mathrm{GL}_d(A_n^0)$ as a deformation of ρ_z to A_n^0 . Since the pseudo-character of ρ_z is equal to $\bar{D} \otimes_k k'$ by Lemma 3.4, the pseudo-character of $\rho' : G_F \rightarrow \mathrm{GL}_d(A_n^0)$ is a deformation of $\bar{D} \otimes_k k'$ to A_n^0 and hence induces a map of local \mathcal{O} -algebras $R^{\mathrm{ps}} \rightarrow A_n^0$. Thus, ρ' factors through the map $\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[[G_F]] \rightarrow M_d(A_n^0)$, which will factor through the Cayley–Hamilton quotient $(\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[[G_F]])/\mathrm{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u) \rightarrow M_d(A_n^0)$. It follows from [18, Section 1.22] or [49, Lemma 1.1.8.6] that

$$\Lambda^0 \otimes_{\mathcal{O}} E \cong (\Lambda^0 \otimes_{\mathcal{O}} R^{\mathrm{ps}}[[G_F]])/\mathrm{CH}(\Lambda^0 \otimes_{\mathcal{O}} D^u).$$

After inverting t and conjugating by \tilde{M}^{-1} , we obtain a map of $\Lambda^0[1/t]$ -algebras $\Lambda^0[1/t] \otimes_{\mathcal{O}} E \rightarrow M_d(A)$, such that if we compose this map with the map induced by $G_F \rightarrow R^{\mathrm{ps}}[[G_F]] \rightarrow E$, then we get back ρ . Since A is an Artinian Λ -algebra, $\varpi^n \Lambda^0[1/t]$ will be mapped to zero for $n \gg 0$, and thus the map extends to a map of Λ -algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \rightarrow M_d(A)$. The universal property of $j : E \rightarrow M_d(A^{\mathrm{gen}})$ implies that there is a unique map of Λ -algebras $\varphi : \hat{B} \rightarrow A$, such that $M_d(\varphi) \circ (\mathrm{id} \otimes j) = \alpha$.

It remains to show the uniqueness of the map φ , which is equivalent to showing that there is at most one map of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}$ -algebras $\alpha : \Lambda \otimes_{\mathcal{O}} E \rightarrow M_d(A)$ such that the composition with $G_F \rightarrow \Lambda \otimes_{\mathcal{O}} E$ gives ρ . It follows from the Cayley–Hamilton theorem in $M_d(A)$ and [18, Corollary 1.14] that the map $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}} \rightarrow \Lambda \otimes_{\mathcal{O}} E \xrightarrow{\alpha} A$ is uniquely determined by ρ . Thus, α is uniquely determined on the image of $\Lambda \otimes_{\mathcal{O}} R^{\mathrm{ps}}[G_F]$ in $\Lambda \otimes_{\mathcal{O}} E$. The map $R^{\mathrm{ps}}[G_F] \rightarrow E$ is surjective, since the image is dense and closed as E is a finitely generated R^{ps} -module; hence, α is uniquely determined by ρ . \square

The following Lemma is a mild generalization of [9, Lemma 3.3.5].

Lemma 3.35. *Let R be a complete local Noetherian k -algebra with residue field k , let A be a finitely generated R -algebra, let $\mathfrak{p} \in \mathrm{Spec} A$ such that its image in $\mathrm{Spec} R$ lies in $P_1 R$, and let \mathfrak{q} be the kernel of the map*

$$B := \kappa(\mathfrak{p}) \otimes_k A \rightarrow \kappa(\mathfrak{p}), \quad x \otimes a \mapsto x(a + \mathfrak{p}).$$

Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[[T]]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.

Proof. Let \mathfrak{p}' be the image of \mathfrak{p} in $\mathrm{Spec} R$. Since by assumption $\mathfrak{p}' \in P_1 R$, the residue field $\kappa(\mathfrak{p}')$ is a local field of characteristic p . Since A is finitely generated over R , $\kappa(\mathfrak{p})$ is a finite extension of $\kappa(\mathfrak{p}')$ and thus is also a local field of characteristic p . The proof of [9, Lemma 3.3.4] goes through verbatim by replacing R with A everywhere. \square

Lemma 3.36. *Let R be a complete local Noetherian \mathcal{O} -algebra with residue field k , let A be a finitely generated R -algebra, let $\mathfrak{p} \in \mathrm{Spec} A$ such that $\kappa(\mathfrak{p})$ is a local field of characteristic p and let \mathfrak{q} be the kernel of the map*

$$B := \Lambda \otimes_{\mathcal{O}} A \rightarrow \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}).$$

Then $\hat{B}_{\mathfrak{q}} \cong \hat{A}_{\mathfrak{p}}[[T]]$. In particular, $A_{\mathfrak{p}}$ is regular (resp. complete intersection) if and only if $\hat{B}_{\mathfrak{q}}$ is.

Proof. We first observe that $\hat{B}_{\mathfrak{q}}$ is flat over $\hat{A}_{\mathfrak{p}}$. This can be seen as follows. Since Λ is \mathcal{O} -flat, B is A -flat. Since $B_{\mathfrak{q}}$ is B -flat and $\hat{B}_{\mathfrak{q}}$ is $B_{\mathfrak{q}}$ -flat, we conclude that $\hat{B}_{\mathfrak{q}}$ is A -flat. Thus, $\hat{B}_{\mathfrak{q}} \otimes_A \hat{A}_{\mathfrak{p}} = \hat{A}_{\mathfrak{p}}$ -flat. This ring

is isomorphic to $\hat{B}_q \otimes_{\hat{A}_p} \hat{A}_p$. Since the map $A_p/\mathfrak{p}^n A_p \rightarrow \hat{A}_p/\mathfrak{p}^n \hat{A}_p$ is an isomorphism for all $n \geq 1$, \hat{B}_q is a completion of $\hat{B}_q \otimes_{\hat{A}_p} \hat{A}_p$ at \mathfrak{q} , which implies the claim.

It follows from Lemma 3.35 that the map $A \rightarrow B, a \mapsto 1 \otimes a$ induces a map of local rings $\hat{A}_p \rightarrow \hat{B}_q$, such that $\hat{B}_q/\varpi \cong (\hat{A}_p/\varpi)[[T]]$. By choosing $b \in \hat{B}_q$, which maps to T under this isomorphism, we obtain a map $\varphi : \hat{A}_p[[T]] \rightarrow \hat{B}_q$, which induces an isomorphism modulo ϖ . Thus, φ is a homomorphism of pseudo-compact \hat{A}_p -modules, which induces an isomorphism after applying $\otimes_{\hat{A}_p} \kappa$. Thus, $(\text{coker } \varphi) \otimes_{\hat{A}_p} \kappa = 0$, and since \hat{B}_q is \hat{A}_p -flat, $(\ker \varphi) \otimes_{\hat{A}_p} \kappa = 0$. Topological Nakayama's lemma⁵ for pseudo-compact modules implies that $\text{coker } \varphi$ and $\ker \varphi$ are both zero. \square

Lemma 3.37. *Let R be a complete local Noetherian \mathcal{O} -algebra with residue field k , let A be a finitely generated R -algebra and let $\mathfrak{p} \in \text{Spec } A$ such that $\kappa(\mathfrak{p})$ is either a finite extension of L or a finite extension of k . Let \mathfrak{q} be the kernel of the map*

$$B := \Lambda \otimes_{\mathcal{O}} A \rightarrow \kappa(\mathfrak{p}), \quad \lambda \otimes a \mapsto \bar{\lambda}(a + \mathfrak{p}).$$

Then $\hat{B}_q \cong \hat{A}_p$.

Proof. The completion of $\Lambda \otimes_{\mathcal{O}} \Lambda$ with respect to the kernel of $\Lambda \otimes_{\mathcal{O}} \Lambda \rightarrow \Lambda, x \otimes y \mapsto xy$ is just Λ (and that is why we do not get an extra variable T like in Lemma 3.35; see [9, Lemma 3.3.5].) The rest of the proof is the same as the proof of Lemma 3.35. \square

Corollary 3.38. *Let x be either a closed point of Y or a closed point of $X^{\text{gen}} \setminus Y$. Then the following hold:*

- (1) $R_{\rho_x}^{\square}$ is a flat Λ -algebra of relative dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;
- (2) if $\text{char}(\kappa(x)) = p$, then $R_{\rho_x}^{\square}/\varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.

Proof. Let us assume that $\kappa(x)$ is a finite extension of k . It follows from Proposition 3.34 and Lemma 3.37 that $R_{\rho_x}^{\square}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\text{gen}},x}$, the completion of the local ring of $\overline{X}^{\text{gen}}$ at x with respect to the maximal ideal. We have $\dim \widehat{\mathcal{O}}_{\overline{X}^{\text{gen}},x} = \dim \mathcal{O}_{\overline{X}^{\text{gen}},x} \leq \dim \overline{X}^{\text{gen}}$, and thus by Theorem 3.31, we obtain the bound

$$\dim R_{\rho_x}^{\square}/\varpi \leq \dim \overline{X}^{\text{gen}} \leq d^2 + d^2[F : \mathbb{Q}_p] = r - s,$$

where the last equality is (22). It follows from (21) that $\dim R_{\rho_x}^{\square}/\varpi \geq r - s$ and $\dim R_{\rho_x}^{\square} \geq 1 + r - s$. Thus, the lower bounds of the dimensions are equalities, and ϖ, f_1, \dots, f_s are a part of system of parameters in $\Lambda[[x_1, \dots, x_r]]$. Thus, they form a regular sequence in $\Lambda[[x_1, \dots, x_r]]$ and so $R_{\rho_x}^{\square}$ and $R_{\rho_x}^{\square}/\varpi$ are complete intersections of the claimed dimensions. Moreover, since Λ is a DVR with uniformiser ϖ , flatness is equivalent to ϖ -torsion equal to zero, and hence, $R_{\rho_x}^{\square}$ is flat over Λ .

Let us assume that $\kappa(x)$ is a local field of characteristic p . Proposition 3.34 and Lemma 3.35 imply that $R_{\rho_x}^{\square}/\varpi \cong \widehat{\mathcal{O}}_{\overline{X}^{\text{gen}},x}[[T]]$, and Lemma 3.21 applied with $W = \overline{X}^{\text{gen}}$ implies that $\dim \widehat{\mathcal{O}}_{\overline{X}^{\text{gen}},x} \leq \dim \overline{X}^{\text{gen}} - 1$. Thus, $\dim R_{\rho_x}^{\square}/\varpi \leq \dim \overline{X}^{\text{gen}}$, and the same argument as above goes through.

If $\kappa(x)$ is a finite extension of L , then Proposition 3.34 and Lemma 3.37 imply that

$$R_{\rho_x}^{\square} \cong \widehat{\mathcal{O}}_{X^{\text{gen}},x} = \widehat{\mathcal{O}}_{X^{\text{gen}}[1/p],x}.$$

Corollary 3.23 implies that $\dim R_{\rho_x}^{\square} \leq \dim X^{\text{gen}}[1/p] \leq \dim \overline{X}^{\text{gen}}$. Then the same argument goes through. \square

Corollary 3.39. *Let x be either a closed point in Y or a closed point in $X^{\text{gen}} \setminus Y$ and let $\widehat{\mathcal{O}}_{X^{\text{gen}},x}$ be the completion with respect to the maximal ideal of the local ring at x . If $\kappa(x)$ is a finite extension of k or L , then $\widehat{\mathcal{O}}_{X^{\text{gen}},x} \cong R_{\rho_x}^{\square}$. If $\kappa(x)$ is a local field of characteristic p , then $R_{\rho_x}^{\square} \cong \widehat{\mathcal{O}}_{X^{\text{gen}},x}[[T]]$.*

⁵Lemma 0.3.3 in Exposé VII_B in SGA3.

Proof. If $\kappa(x)$ is a finite extension of k or L , then the assertion follows from Proposition 3.34 and Lemma 3.37. If $\kappa(x)$ is a local field of characteristic p , then the assertion follows from Proposition 3.34 and Lemma 3.36. □

Corollary 3.40. *The following hold:*

- (1) A^{gen} is \mathcal{O} -torsion free, is equi-dimensional of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection;
- (2) A^{gen}/ϖ is equi-dimensional of dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection.

Proof. Let us prove (1) as the proof of (2) is identical. Corollary 3.39 together with Corollary 3.38 implies that the local rings at closed points of X^{gen} are \mathcal{O} -torsion free and complete intersection. This implies that A^{gen} is \mathcal{O} -torsion free and A^{gen} is locally complete intersection by [48, Tag 09Q5].

Let Z be an irreducible component of X^{gen} . Lemma 3.21 implies that there is a closed point $x \in Z$ such that x maps to the closed point of X^{ps} . Moreover, $\dim Z = \dim \mathcal{O}_{Z,x}$. Since $\mathcal{O}_{X^{\text{gen}},x}$ is complete intersection, it is equi-dimensional, and thus, $\dim \mathcal{O}_{Z,x} = \dim \mathcal{O}_{X^{\text{gen}},x} = d^2 + d^2[F : \mathbb{Q}_p] + 1$, where the last equality follows from Corollaries 3.38 and 3.39. □

Proposition 3.41. *Let $x \in P_1 R_{\bar{\rho}}^{\square}$, where $R_{\bar{\rho}}^{\square}$ is the framed deformation ring of $\bar{\rho} : G_F \rightarrow \text{GL}_d(k')$, where k' is finite extension of k . Let $\rho_x : G_F \rightarrow \text{GL}_d(\kappa(x))$ be the representation obtained by specializing the universal framed deformation of $\bar{\rho}$ at x . Let \mathfrak{q} be the kernel of the map*

$$\Lambda \otimes_{\mathcal{O}} R_{\bar{\rho}}^{\square} \rightarrow \kappa(x), \quad \lambda \otimes a \mapsto \bar{\lambda}a,$$

where Λ is the ring defined at the beginning of the subsection. Then the completion of $(\Lambda \otimes_{\mathcal{O}} R_{\bar{\rho}}^{\square})_{\mathfrak{q}}$ with respect to the maximal ideal is naturally isomorphic to $R_{\rho_x}^{\square}$.

Proof. The proof is similar to the proof of Proposition 3.34, but easier, since the setting is much closer to the setting of [31, Proposition 9.5] or [9, Theorem 3.3.1], where an analogous result is proved for versal deformation rings. We leave the details to the reader. □

Let x be a closed point of $X^{\text{gen}} \setminus Y$, so that $\kappa(x)$ is a local field. Since G_F is compact, there is a matrix $M \in \text{GL}_d(\kappa(x))$, such that the image of $M\rho_x M^{-1}$ is contained in $\text{GL}_d(\mathcal{O}_{\kappa(x)})$. Let $x' : A^{\text{gen}} \rightarrow \mathcal{O}_{\kappa(x)}$ be the R^{ps} -algebra homomorphism corresponding to the representation $E \rightarrow M_d(\mathcal{O}_{\kappa(x)})$, $a \mapsto M\rho_x(a)M^{-1}$. We will denote the corresponding Galois representation by $\rho_{x'}^0 : G_F \rightarrow \text{GL}_d(\mathcal{O}_{\kappa(x)})$

and let $\rho_{x'}$ be the composition $\rho_{x'} : G_F \xrightarrow{\rho_{x'}^0} \text{GL}_d(\mathcal{O}_{\kappa(x)}) \rightarrow \text{GL}_d(\kappa(x))$. We note that $\kappa(x') = \kappa(x)$ and let Λ be the coefficient ring defined at the beginning of the subsection. Let k' be the residue field of $\mathcal{O}_{\kappa(x)}$ and let $\rho_z : G_F \rightarrow \text{GL}_d(k')$ be the representation corresponding to $z : A^{\text{gen}} \xrightarrow{x'} \mathcal{O}_{\kappa(x)} \rightarrow k'$. Then $\rho_{x'}^0$ is a deformation of ρ_z to $\mathcal{O}_{\kappa(x)}$; thus, the map $x' : A^{\text{gen}} \rightarrow \mathcal{O}_{\kappa(x)}$ factors through $x' : R_{\rho_z}^{\square} \rightarrow \mathcal{O}_{\kappa(x)}$.

Corollary 3.42. *There is an isomorphism of local Λ -algebras between $R_{\rho_x}^{\square}$, $R_{\rho_{x'}}^{\square}$ and the completion of $(\Lambda \otimes_{\mathcal{O}} R_{\rho_z}^{\square})_{\mathfrak{q}}$ with respect to the maximal ideal, where \mathfrak{q} is as in Proposition 3.41 with respect to $x' : R_{\rho_z}^{\square} \rightarrow \mathcal{O}_{\kappa(x)}$.*

Proof. Let \tilde{M} be any lift of M to $M_d(\Lambda)$. Since Λ is a local ring, $\det \tilde{M}$ is a unit in Λ and hence $\tilde{M} \in \text{GL}_d(\Lambda)$. Conjugation by \tilde{M} induces an isomorphism between the deformation problems for ρ_x and $\rho_{x'}$ and hence between the deformation rings. Proposition 3.41 implies that these rings are also isomorphic to the completion of $(\Lambda \otimes_{\mathcal{O}} R_{\rho_z}^{\square})_{\mathfrak{q}}$. □

Remark 3.43. Corollary 3.42 enables us to study local properties of X^{gen} by studying the completions of local rings at closed points above $\mathfrak{m}_{R^{\text{ps}}}$. For example, if we could show that $R_{\rho_z}^{\square}$ is regular, we could

conclude that the local ring at x' , $(R_{\rho_z}^\square)_{x'}$ is regular, and hence that the completion $\widehat{(R_{\rho_z}^\square)}_{x'}$ is regular. If $\kappa(x)$ is a local field of characteristic p , then Proposition 3.34, Corollary 3.42 and Lemma 3.36 imply that

$$\widehat{\mathcal{O}}_{X^{\text{gen}},x}[[T]] \cong R_{\rho_x}^\square \cong R_{\rho_{x'}}^\square \cong \widehat{(R_{\rho_z}^\square)}_{x'}[[T]].$$

If $\kappa(x)$ is a finite extension of L , then Proposition 3.34, Corollary 3.42 and Lemma 3.37 imply that

$$\widehat{\mathcal{O}}_{X^{\text{gen}},x} \cong R_{\rho_x}^\square \cong R_{\rho_{x'}}^\square \cong \widehat{(R_{\rho_z}^\square)}_{x'}.$$

Thus, in both cases we can deduce that $\widehat{\mathcal{O}}_{X^{\text{gen}},x}$, and hence $\mathcal{O}_{X^{\text{gen}},x}$, are regular. Thus, if we can show that $R_{\rho_z}^\square$ is regular for all closed points $z \in X^{\text{gen}}$ above $\mathfrak{m}_{R^{\text{ps}}}$, then we can conclude that $\mathcal{O}_{X^{\text{gen}},x}$ is regular for all closed points $x \in X^{\text{gen}}$, and thus X^{gen} is regular.

Of course, one may also reverse the logic of this argument: if X^{gen} is regular, then all its local rings and their completions are regular, and hence, $R_{\rho_z}^\square$ is regular for all closed points $z \in X^{\text{gen}}$ above $\mathfrak{m}_{R^{\text{ps}}}$.

Corollary 3.44. *Let $\rho : G_F \rightarrow \text{GL}_d(\kappa)$ be a continuous representation with κ a local field. Then the conclusion of Corollary 3.38 holds for R_ρ^\square .*

Proof. After conjugation, we may assume that $\rho(G_F) \subset \text{GL}_d(\mathcal{O}_\kappa)$. Let $\bar{\rho}$ be the representation obtained by reducing the matrix entries modulo a uniformizer of \mathcal{O}_κ and let \bar{D} be the associated pseudo-character. Corollary 3.38 applies to $R_{\bar{\rho}}^\square$. Since ρ corresponds to an $x \in P_1 R_{\bar{\rho}}^\square$, Proposition 3.41 together with Lemmas 3.37, 3.36 allows us to bound the dimension of R_ρ^\square from above. Then the proof of Corollary 3.38 carries over. □

Corollary 3.45. *Every representation $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$ can be lifted to characteristic zero.*

Proof. It follows from Corollary 3.38 that $R_{\bar{\rho}}^\square[1/p]$ is non-zero. We may obtain a lift by specializing the universal framed deformation along any \mathcal{O} -algebra homomorphism $x : R^\square \rightarrow \bar{\mathbb{Q}}_p$. □

3.6. Bounding the maximally reducible semi-simple locus

Writing $\bar{D} = \prod_{i=1}^m \bar{D}_i$ with \bar{D}_i absolutely irreducible pseudo-characters, we now take $\mathcal{P} = \mathcal{P}_{\text{max}}$ and consider the finite (by Lemma 3.24) R^{ps} -algebra $R_{\underline{\Sigma}}^{\text{ps}}$, where $\underline{\Sigma}$ amounts to some choice of ordering of $\{1, \dots, m\}$. Note that if $\bar{\rho}_i : G_F \rightarrow \text{GL}_d(k)$ is an (absolutely irreducible) representation with pseudo-character \bar{D}_i , then

$$R_{\underline{\Sigma}}^{\text{ps}} \cong R_{\bar{\rho}_1} \widehat{\otimes}_{\mathcal{O}} \cdots \widehat{\otimes}_{\mathcal{O}} R_{\bar{\rho}_m},$$

where $R_{\bar{\rho}_i}$ denotes the universal deformation ring of $\bar{\rho}_i$. So let $\rho_i^{\text{univ}} : G_F \rightarrow \text{GL}_{d_i}(R_{\bar{\rho}_i})$ denote a representative of the strict equivalence class of the universal representation for each $i = 1, \dots, m$. If we let M denote the universal invertible matrix in $\text{GL}_d(\mathcal{O}_{\text{GL}_d}(\text{GL}_d))$, then the representation

$$M \times \text{diag}(\rho_1^{\text{univ}}, \dots, \rho_m^{\text{univ}}) \times M^{-1} : G_F \rightarrow \text{GL}_d(R_{\underline{\Sigma}}^{\text{ps}} \otimes_{\mathcal{O}} \mathcal{O}_{\text{GL}_d}(\text{GL}_d))$$

gives rise to a map of Cayley–Hamilton algebras $E \rightarrow M_d(R_{\underline{\Sigma}}^{\text{ps}} \otimes_{\mathcal{O}} \mathcal{O}_{\text{GL}_d}(\text{GL}_d))$, which satisfies the universal property of A^{gen} and so defines a map of R^{ps} -schemes

$$\text{GL}_d \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} \rightarrow X^{\text{gen}},$$

which descends to a map of R^{ps} -schemes

$$\eta_{\underline{\Sigma}} : \text{GL}_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} \rightarrow X^{\text{gen}},$$

where $L := L_{\underline{\Sigma}}$ denotes the standard Levi subgroup of GL_d with blocks corresponding to $\underline{\Sigma}$, and Z_L denotes its center.

Definition 3.46. The maximally reducible semi-simple locus $X^{\text{mrs}} \subset X^{\text{gen}}$ is the scheme-theoretic image of $\eta_{\underline{\Sigma}} : GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} \rightarrow X^{\text{gen}}$.

Lemma 3.47. Let $x \in X^{\text{gen}}$ and let y be the image of x in X^{ps} . If y lies in $X_{\mathcal{P}_{\text{max}}}^{\text{ps}}$ and ρ_x is semi-simple, then $x \in X^{\text{mrs}}$. Moreover, such points are dense in X^{mrs} .

Proof. We first note that if $x \in X^{\text{gen}}$ maps to $X_{\mathcal{P}_{\text{max}}}^{\text{ps}}$ and ρ_x is semi-simple, then $\rho_x \cong \rho_1 \oplus \dots \oplus \rho_m$, with each ρ_i an irreducible representation of G_F lifting $\bar{\rho}_i$. By conjugating by an element of $h \in GL_d(\kappa(x))$, we may ensure that $h^{-1}\rho_x(g)h = \text{diag}(\rho_1(g), \dots, \rho_m(g))$ for all $g \in G_F$, and this implies that $x \in X^{\text{mrs}}$.

Since $\eta_{\underline{\Sigma}}$ is a map of affine schemes, it is affine and hence quasi-compact; see [48, Tag 01S5]. It follows from [48, Tag 01R8] that the set theoretic image of $\eta_{\underline{\Sigma}}$ is dense in X^{mrs} . \square

Proposition 3.48. $\dim X^{\text{mrs}} \leq 1 + d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^m d_i^2$.

Proof. The open subscheme $U_{\text{max}} = X^{\text{ps}} \setminus \{\mathfrak{m}_{R^{\text{ps}}}\} \subset \bar{X}^{\text{ps}}$ is Jacobson by Lemma 3.18, as is $V_{\text{max}} := X^{\text{mrs}} \times_{X^{\text{ps}}} U_{\text{max}}$. Let Z_{max} denote the closure of V_{max} in X^{mrs} . The formation of scheme-theoretic images commutes with restriction to opens, so the map

$$(GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}}) \times_{X^{\text{ps}}} U_{\text{max}} \rightarrow V_{\text{max}}$$

is a dominant map of Jacobson Noetherian excellent schemes. Applying Lemma 3.14, we see that

$$\dim V_{\text{max}} \leq \dim((GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}}) \times_{X^{\text{ps}}} U_{\text{max}}).$$

Since X^{mrs} is by definition a nonempty closed GL_d -invariant subscheme of X^{gen} , Lemma 3.21 implies that every irreducible component of X^{mrs} has a point in common with the preimage of $\mathfrak{m}_{R^{\text{ps}}}$ in X^{mrs} . Therefore, Lemma 3.18 (5) implies that

$$\dim Z_{\text{max}} = \dim V_{\text{max}} + 1.$$

Furthermore, GL_d/Z_L is flat over $\text{Spec } \mathcal{O}$ with geometrically irreducible fibres, so the projection $GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} \rightarrow X_{\underline{\Sigma}}^{\text{ps}}$ is a flat (and hence open) map with irreducible fibres. It follows from [48, Tag 037A] that this map induces a bijection between the sets of irreducible components. Since $R_{\underline{\Sigma}}^{\text{ps}}$ is a local ring, we deduce that $GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}}$ satisfies the assumptions of Lemma 3.18 (5), and thus, Lemma 3.18 (5) implies that

$$\dim GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} = \dim((GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}}) \times_{X^{\text{ps}}} U_{\text{max}}) + 1.$$

Since $\dim X_{\underline{\Sigma}}^{\text{ps}} = 1 + \sum_{i=1}^m (1 + d_i^2 [F : \mathbb{Q}_p])$ and the relative dimension of GL_d/Z_L over \mathcal{O} is $d^2 - m$, we get that

$$\dim Z_{\text{max}} \leq \dim GL_d/Z_L \times_{\mathcal{O}} X_{\underline{\Sigma}}^{\text{ps}} = 1 + d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^m d_i^2.$$

Let Y^{mrs} be the scheme theoretic image of $GL_d/Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\text{ps}}}\} \rightarrow Y$. Since Y is of finite type over k , the same argument as above shows that

$$\dim Y^{\text{mrs}} \leq \dim(GL_d/Z_L \times_{\mathcal{O}} \{\mathfrak{m}_{R^{\text{ps}}}\}) = d^2 - m.$$

Now $Z_{\max} \cup Y^{\text{mrs}}$ is a closed subscheme of X^{gen} containing the image of η_{Σ} . It follows from Lemma 3.47 that $Z_{\max} \cup Y^{\text{mrs}}$ will contain X^{mrs} . Hence,

$$\dim X^{\text{mrs}} \leq \max\{\dim Z_{\max}, \dim Y^{\text{mrs}}\} = \dim Z_{\max}. \quad \square$$

Corollary 3.49. $\dim \bar{X}^{\text{mrs}} = \dim X^{\text{mrs}} - 1 \leq d^2 + [F : \mathbb{Q}_p] \sum_{i=1}^m d_i^2$.

Proof. It follows from Corollary 3.38 that R_{Σ}^{ps} is \mathcal{O} -torsion free, which implies that $\text{GL}_d/Z_L \times_{\mathcal{O}} X_{\Sigma}^{\text{ps}}$ is flat over $\text{Spec } \mathcal{O}$, and the same applies for X^{mrs} . (Here, we are simply saying that a subring of \mathcal{O} -torsion free ring is \mathcal{O} -torsion free.) Thus, for all $x \in \bar{X}^{\text{mrs}}$, ϖ is a regular element in $\mathcal{O}_{X^{\text{mrs}},x}$ and so $\dim \mathcal{O}_{\bar{X}^{\text{mrs}},x} = \dim \mathcal{O}_{X^{\text{mrs}},x} - 1$. This implies $\dim \bar{X}^{\text{mrs}} = \dim X^{\text{mrs}} - 1$, and the inequality follows from Proposition 3.48. \square

Remark 3.50. One could study the closure of the reducible semi-simple locus corresponding to more general partitions using a similar argument. We do not pursue this here, since we need the bound only for $d = 2$ and $F = \mathbb{Q}_2$ when we apply it to Case 3 in the proof of Proposition 4.13 below.

3.7. Density of the irreducible locus

Let us first unravel the definitions of $U_{\mathcal{P}_{\min}}$ and $V_{\mathcal{P}_{\min}}$ in Section 3.4. We have that $U_{\mathcal{P}_{\min}}$ is an open subscheme of \bar{X}^{ps} such that the closed points of $U_{\mathcal{P}_{\min}}$ are in bijection with $\mathfrak{p} \in P_1(R^{\text{ps}}/\varpi)$, such that the specialization of the universal pseudo-character along $R^{\text{ps}} \rightarrow \kappa(\mathfrak{p})$ is absolutely irreducible. Now $V_{\mathcal{P}_{\min}}$ is the preimage of $U_{\mathcal{P}_{\min}}$ in \bar{X}^{gen} , so that it is an open subscheme of \bar{X}^{gen} and its closed points are in bijection $\mathfrak{q} \in \bar{X}^{\text{gen}}$, which map to $P_1(R^{\text{ps}}/\varpi)$ in \bar{X}^{ps} , such that the representation

$$E \xrightarrow{j} M_d(A^{\text{gen}}) \rightarrow M_d(\kappa(\mathfrak{q}))$$

is absolutely irreducible.

Proposition 3.51. $V_{\mathcal{P}_{\min}}$ is dense in \bar{X}^{gen} .

Proof. We have

$$\bar{X}^{\text{gen}} \setminus V_{\mathcal{P}_{\min}} = Y \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}},$$

and it follows from Lemmas 3.29, 3.30 that $\bar{X}^{\text{gen}} \setminus V_{\mathcal{P}_{\min}}$ has positive codimension in \bar{X}^{gen} . Since \bar{X} is equi-dimensional by Corollary 3.40, we conclude that $V_{\mathcal{P}_{\min}}$ is dense in \bar{X}^{gen} . In particular, the inequality in Proposition 3.28 is an equality. \square

We will now prove a stronger version of the above result. Following [9, Definition 5.1.2], we call $y \in U_{\mathcal{P}_{\min}}$ *special* if either $\zeta_p \notin F$ and $D_y = D_y(1)$ or $\zeta_p \in F$ and the restriction D_y to $G_{F'}$ is reducible for some degree p Galois extension F' of F . Otherwise, y is called *non-special*. According to [9, Lemma 5.1.3], there is a closed subscheme U^{spcl} of $U_{\mathcal{P}_{\min}}$ such that the closed points of U^{spcl} are precisely the closed special points of $U_{\mathcal{P}_{\min}}$. Let V^{spcl} denote the preimage of U^{spcl} in \bar{X}^{gen} and let Z^{spcl} denote the closure of V^{spcl} .

Similarly, let $U^{\text{Kirr}} \subset U_{\mathcal{P}_{\min}}$ be the Kummer-irreducible locus defined in Appendix A. Let U^{Kred} denote its complement in $U_{\mathcal{P}_{\min}}$, let V^{Kred} be the preimage of U^{Kred} in \bar{X}^{gen} and let Z^{Kred} denote the closure of V^{Kred} . We have $V^{\text{spcl}} \subseteq V^{\text{Kred}}$ with equality if $\zeta_p \in F$, and thus, $Z^{\text{spcl}} \subseteq Z^{\text{Kred}}$.

Lemma 3.52. *We have*

$$\dim \bar{X}^{\text{gen}} - \dim Z^{\text{spcl}} \geq \frac{1}{2} [F : \mathbb{Q}_p] d^2, \quad \dim \bar{X}^{\text{gen}} - \dim Z^{\text{Kred}} \geq [F : \mathbb{Q}_p] d.$$

Proof. It follows from [9, Theorem 5.4.1 (a)] that the dimension of the Zariski closure of U^{spcl} in \overline{X}^{ps} is at most $1 + \frac{1}{2}[F : \mathbb{Q}_p]d^2$. If $y \in U^{\text{spcl}}$, then its fibre X_y^{gen} has dimension $d^2 - 1$ by Corollary 3.16. Thus, Lemma 3.18 implies that

$$\dim Z^{\text{spcl}} \leq d^2 + \frac{1}{2}[F : \mathbb{Q}_p]d^2.$$

Since $\dim \overline{X}^{\text{gen}} = d^2 + d^2[F : \mathbb{Q}_p]$ by Corollary 3.40, the assertion follows. Similarly, Proposition A.9 implies that the dimension of the closure of U^{Kred} in \overline{X}^{ps} is at most $1 + (d^2 - d)[F : \mathbb{Q}_p]$. The same argument gives the required bound for the codimension of Z^{Kred} . \square

Let $U^{n\text{-spcl}} := U_{\mathcal{P}_{\min}} \setminus U^{\text{spcl}}$ and let $V^{n\text{-spcl}}$ the preimage of $U^{n\text{-spcl}}$ in $\overline{X}^{\text{gen}}$. Let V^{Kirr} be the preimage of U^{Kirr} in $\overline{X}^{\text{gen}}$. We have an inclusion $V^{\text{Kirr}} \subset V^{n\text{-spcl}}$, and the subschemes coincide if $\zeta_p \in F$.

Proposition 3.53. *V^{Kirr} is Zariski dense in $\overline{X}^{\text{gen}}$. Moreover, the following hold:*

- (1) if $d = 2$, then $\dim \overline{X}^{\text{gen}} - \dim(\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}}) \geq [F : \mathbb{Q}_p]$;
- (2) if $d > 2$, then $\dim \overline{X}^{\text{gen}} - \dim(\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}}) \geq 1 + [F : \mathbb{Q}_p]$.
- (3) if $d > 1$ is arbitrary but \overline{D} is absolutely irreducible (i.e., $m = 1$), then

$$\dim \overline{X}^{\text{gen}} - \dim(\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}}) \geq d[F : \mathbb{Q}_p].$$

Proof. Since $V_{\mathcal{P}_{\min}}$ is dense in $\overline{X}^{\text{gen}}$ by Proposition 3.51, we have $\overline{X}^{\text{gen}} = Z_{\mathcal{P}_{\min}} = Z^{\text{Kred}} \cup Z^{\text{Kirr}}$, where Z^{Kirr} is the closure of V^{Kirr} . Since $\dim Z^{\text{Kred}} < \dim \overline{X}^{\text{gen}}$ by Lemma 3.52 and $\overline{X}^{\text{gen}}$ is equi-dimensional, we get that $\overline{X}^{\text{gen}} = Z^{\text{Kirr}}$. Moreover,

$$\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}} = Y \cup Z^{\text{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} Z_{\mathcal{P}},$$

and claims (1) and (2) follow from the dimension estimates in Lemmas 3.30, 3.29, 3.52. If \overline{D} is absolutely irreducible, then $\{\mathcal{P} : \mathcal{P}_{\min} < \mathcal{P}\} = \emptyset$, and claim (3) follows from Lemmas 3.30 and 3.52. \square

We now want to transfer the density results from $\overline{X}^{\text{gen}}$ to $R_{\rho}^{\square}/\varpi$.

Lemma 3.54. *Let $A \rightarrow B$ be a flat ring homomorphism, let U be an open subscheme of $\text{Spec } A$ and let V be the preimage of U in $\text{Spec } B$. If U is dense in $\text{Spec } A$, then V is dense in $\text{Spec } B$.*

Proof. Let \mathfrak{q} be a minimal prime of B and let \mathfrak{p} be its image in $\text{Spec } A$. Since the map is flat, it satisfies going down, and so \mathfrak{p} is a minimal prime of A . Since U is dense, it will contain \mathfrak{p} ; hence, V will contain \mathfrak{q} . Thus, V contains all the minimal primes of B and so is dense in $\text{Spec } B$. \square

Proposition 3.55. *Let $(\text{Spec}(R_{\rho}^{\square}/\varpi))^{\text{Kirr}}$ be the preimage of V^{Kirr} in $\text{Spec}(R_{\rho}^{\square}/\varpi)$. Then $(\text{Spec}(R_{\rho}^{\square}/\varpi))^{\text{Kirr}}$ is dense in $\text{Spec}(R_{\rho}^{\square}/\varpi)$.*

Proof. The map $A^{\text{gen}}/\varpi \rightarrow R_{\rho}^{\square}/\varpi$ is flat since it is a localization followed by a completion. The assertion follows from Lemma 3.54 and Proposition 3.53. \square

Remark 3.56. Since $(\text{Spec}(R_{\rho}^{\square}/\varpi))^{\text{Kirr}}$ is also the preimage of U^{Kirr} in $\text{Spec } R_{\rho}^{\square}/\varpi$, we may characterise it as an open subscheme of $\text{Spec } R_{\rho}^{\square}/\varpi$, such that its closed points are in bijection with $x \in P_1(R_{\rho}^{\square}/\varpi)$, which map to $P_1(R^{\text{ps}}/\varpi)$ in $\text{Spec } R^{\text{ps}}$ and for which the representation

$$\rho_x : G_F \rightarrow \text{GL}_d(R_{\rho}^{\square}/\varpi) \rightarrow \text{GL}_d(\kappa(x))$$

remains absolutely irreducible after restriction to $G_{F'}$ for all degree p Galois extensions F' of $F(\zeta_p)$. Lemma A.2 implies that $H^2(G_F, \text{ad}^0 \rho_x) = 0$ for such x .

We will now prove similar results for the generic fibres. For each partition \mathcal{P} as in Section 3.4, let $X_{\mathcal{P}}^{\text{PS}}$ be the scheme theoretic image of X^{PS} inside X_{Σ}^{PS} and let $X_{\mathcal{P}}^{\text{gen}}$ be the preimage of $X_{\mathcal{P}}^{\text{PS}}$ in X^{gen} . We warn the reader that, contrary to our usual notational conventions, it is not clear that $\overline{X}_{\mathcal{P}}^{\text{PS}}$ considered in Section 3.4 is the special fibre of $X_{\mathcal{P}}^{\text{PS}}$. However, the following still holds.

Lemma 3.57. $\dim X_{\mathcal{P}}^{\text{gen}}[1/p] \leq \dim \overline{X}_{\mathcal{P}}^{\text{gen}}$.

Proof. Let $\mathfrak{a}_{\mathcal{P}}$ be the R^{PS} -annihilator of R_{Σ}^{PS} and let $\mathfrak{b}_{\mathcal{P}}$ be the R^{PS} -annihilator of $R_{\Sigma}^{\text{PS}}/\varpi$. We may write

$$X_{\mathcal{P}}^{\text{gen}} = \text{Spec } A^{\text{gen}}/\mathfrak{a}_{\mathcal{P}}A^{\text{gen}}, \quad \overline{X}_{\mathcal{P}}^{\text{gen}} = \text{Spec } A^{\text{gen}}/\mathfrak{b}_{\mathcal{P}}A^{\text{gen}}.$$

Since R_{Σ}^{PS} is a finite R^{PS} -module by Lemma 3.24, we have $\sqrt{\mathfrak{b}_{\mathcal{P}}} = \sqrt{(\mathfrak{a}_{\mathcal{P}}, \varpi)}$. In particular, the special fibre of $X_{\mathcal{P}}^{\text{gen}}$ has dimension equal to $\dim \overline{X}_{\mathcal{P}}^{\text{gen}}$. The assertion follows from Lemma 3.23. \square

Proposition 3.58. *Let*

$$V^{\text{irr}} := X^{\text{gen}}[1/p] \setminus \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} X_{\mathcal{P}}^{\text{gen}}[1/p].$$

Then V^{irr} is an open dense subset of $X^{\text{gen}}[1/p]$. Moreover, the following hold:

- (1) *if $d = 2$, then $\dim X^{\text{gen}}[1/p] - \dim(X^{\text{gen}}[1/p] \setminus V^{\text{irr}}) \geq [F : \mathbb{Q}_p]$;*
- (2) *if $d > 2$, then $\dim X^{\text{gen}}[1/p] - \dim(X^{\text{gen}}[1/p] \setminus V^{\text{irr}}) \geq 1 + [F : \mathbb{Q}_p]$;*
- (3) *if $d > 1$ is arbitrary but \overline{D} is absolutely irreducible (i.e., $m = 1$), then $X^{\text{gen}}[1/p] = V^{\text{irr}}$.*

Proof. It follows from Corollary 3.40 that $\dim X^{\text{gen}}[1/p] = d^2 + d^2[F : \mathbb{Q}_p] = \dim \overline{X}^{\text{gen}}$. Lemmas 3.57 and 3.29 together with (18) imply that for $\mathcal{P} > \mathcal{P}_{\min}$ we have

$$\dim X^{\text{gen}}[1/p] - \dim X_{\mathcal{P}}^{\text{gen}}[1/p] \geq \dim \overline{X}^{\text{gen}} - \dim \overline{X}_{\mathcal{P}}^{\text{gen}}. \tag{23}$$

It follows from (18) that $\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}} = Y \cup Z^{\text{Kred}} \cup \bigcup_{\mathcal{P}_{\min} < \mathcal{P}} \overline{X}_{\mathcal{P}}^{\text{gen}}$. Thus, it follows from (23) and the definition of V^{irr} that

$$\dim X^{\text{gen}}[1/p] - \dim(X^{\text{gen}}[1/p] \setminus V^{\text{irr}}) \geq \dim \overline{X}^{\text{gen}} - \dim(\overline{X}^{\text{gen}} \setminus V^{\text{Kirr}}), \tag{24}$$

and the lower bounds for the codimension of $X^{\text{gen}}[1/p] \setminus V^{\text{irr}}$ follow from Proposition 3.53.

Thus, the dimension of the closure of V^{irr} is equal to $\dim X^{\text{gen}}[1/p]$. Since A^{gen} is \mathcal{O} -torsion free and equi-dimensional by Corollary 3.40 (1), $X^{\text{gen}}[1/p]$ is equi-dimensional, and so V^{irr} is dense in $X^{\text{gen}}[1/p]$.

If \overline{D} is absolutely irreducible, then ρ_x is absolutely irreducible for all closed points $x \in X^{\text{gen}}[1/p]$ and so $X^{\text{gen}}[1/p] = V^{\text{irr}}$. \square

Corollary 3.59. *Let $(\text{Spec } R_{\rho}^{\square}[1/p])^{\text{irr}}$ be the preimage of V^{irr} in $\text{Spec } R_{\rho}^{\square}[1/p]$. Then $(\text{Spec } R_{\rho}^{\square}[1/p])^{\text{irr}}$ is dense in $\text{Spec } R_{\rho}^{\square}[1/p]$.*

Proof. As explained in the proof of Proposition 3.55, the map $A^{\text{gen}} \rightarrow R_{\rho}^{\square}$ is flat. Hence, the localization $A^{\text{gen}}[1/p] \rightarrow R_{\rho}^{\square}[1/p]$ is also flat. The assertion follows from Lemma 3.54 and Proposition 3.58. \square

Remark 3.60. Similarly to Remark 3.56, we may characterize $(\text{Spec } R_{\rho}^{\square}[1/p])^{\text{irr}}$ as an open subscheme of $\text{Spec } R_{\rho}^{\square}[1/p]$ such that its closed points correspond to maximal ideals \mathfrak{p} of $R_{\rho}^{\square}[1/p]$ for which the representation

$$\rho_{\mathfrak{p}} : G_F \rightarrow \text{GL}_d(R_{\rho}^{\square}[1/p]) \rightarrow \text{GL}_d(\kappa(\mathfrak{p}))$$

is absolutely irreducible.

Corollary 3.61. *The characteristic zero lift of $\bar{\rho}$ in Corollary 3.45 may be chosen to be absolutely irreducible.*

Proof. It follows from Corollary 3.45 that $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$ is nonempty, and Corollary 3.59 implies that $(\text{Spec } R_{\bar{\rho}}^{\square}[1/p])^{\text{irr}}$ is nonempty. A closed point in $(\text{Spec } R_{\bar{\rho}}^{\square}[1/p])^{\text{irr}}$ gives the desired lift of $\bar{\rho}$ to characteristic zero. □

Corollary 3.62. *Let $\Sigma \subset \text{m-Spec } R_{\bar{\rho}}^{\square}[1/p]$ be dense in $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$. Then*

$$\Sigma^{\text{irr}} := \Sigma \cap (\text{Spec } R_{\bar{\rho}}^{\square}[1/p])^{\text{irr}}$$

is also dense in $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$.

Proof. It follows from the proof of Proposition 3.58 that $\Sigma \setminus \Sigma^{\text{irr}}$ is contained in a closed subset of $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$ of positive codimension. Since $\text{Spec } R_{\bar{\rho}}^{\square}[1/p]$ is equi-dimensional, Σ^{irr} is dense. □

4. Irreducible components

The aim of this section is to determine the irreducible components of $\text{Spec } R_{\bar{\rho}}^{\square}$ for any $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$ and study their geometry. It is instructive to consider first the one-dimensional case. Let $\bar{\psi} : G_F \rightarrow k^{\times}$ denote any continuous character and write $\psi^{\text{univ}} : G_F \rightarrow \text{GL}_1(R_{\bar{\psi}})$ for its universal deformation. Local class field theory gives a group homomorphism

$$\mu \rightarrow F^{\times} \xrightarrow{\text{Art}_F} G_F^{\text{ab}} \xrightarrow{\psi^{\text{univ}}} \text{GL}_1(R_{\bar{\psi}}),$$

where $\mu := \mu_{p^{\infty}}(F)$ is the subgroup of p -power roots of unity in F . We note that μ is a finite cyclic p -group. The map induces a homomorphism of \mathcal{O} -algebras $\mathcal{O}[\mu] \rightarrow R_{\bar{\psi}}$, where $\mathcal{O}[\mu]$ is the group algebra of μ over \mathcal{O} .

Lemma 4.1. $R_{\bar{\psi}} \cong \mathcal{O}[\mu][[y_1, \dots, y_{[F:\mathbb{Q}_p]+1}]]$.

Proof. It follows from local class field theory that the pro- p completion of G_F^{ab} is isomorphic to $\mu_{p^{\infty}}(F) \times \mathbb{Z}_p^{[F:\mathbb{Q}_p]+1}$, and the assertion follows from [27, Proposition 3.13]. □

It follows immediately from Lemma 4.1 that the set of irreducible components of $\text{Spec } R_{\bar{\psi}}$ is in bijection with the group of characters $\chi : \mu \rightarrow \mathcal{O}^{\times}$, and the irreducible component corresponding to χ is given by $R_{\bar{\psi}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$, which is formally smooth over \mathcal{O} .

Let us return to the general case $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$. Mapping a deformation of $\bar{\rho}$ to its determinant induces a natural map $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$, which makes $R_{\bar{\rho}}^{\square}$ into an $\mathcal{O}[\mu]$ -algebra by applying the above discussion to $\bar{\psi} = \det \bar{\rho}$. The algebra $\mathcal{O}[\mu][1/p]$ is semi-simple and its maximal ideals are in bijection with characters $\chi : \mu \rightarrow \mathcal{O}^{\times}$. We thus have

$$R_{\bar{\rho}}^{\square}[1/p] \cong \prod_{\chi: \mu \rightarrow \mathcal{O}^{\times}} R_{\bar{\rho}}^{\square, \chi}[1/p], \tag{25}$$

where $R_{\bar{\rho}}^{\square, \chi} := R_{\bar{\rho}}^{\square} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$. So our goal is to show that the rings $R_{\bar{\rho}}^{\square, \chi}$ are \mathcal{O} -torsion free integral domains, which we do by showing in Corollary 4.19 that they are normal. Since we already know that $R_{\bar{\rho}}^{\square}$ is \mathcal{O} -torsion free by Corollary 3.38, this implies that the map $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$ induces a bijection between the sets of irreducible components, which answers affirmatively a question raised by GB–Juschka in [8]. Along the way, we will also determine the irreducible components of A^{gen} and R^{ps} .

Warning 4.2. We emphasize that $R_{\bar{\rho}}^{\square, \chi}$ is not a ‘fixed determinant deformation ring’ in the usual sense but is rather constructed by fixing the value of the determinant only on the subgroup $\text{Art}_F(\mu) \subset G_F^{\text{ab}}$: the ring $R_{\bar{\rho}}^{\square, \chi}$ represents the closed subfunctor $D_{\bar{\rho}}^{\square, \chi} \subset D_{\bar{\rho}}^{\square}$ given by

$$D_{\bar{\rho}}^{\square, \chi}(A) = \{\rho_A \in D_{\bar{\rho}}^{\square}(A) : \det \rho_A(\text{Art}_F(x)) = \chi(x), \forall x \in \mu\}.$$

Proposition 4.3. *There is an isomorphism*

$$R_{\det \bar{\rho}} \llbracket x_1, \dots, x_r \rrbracket / (f_1, \dots, f_t) \xrightarrow{\cong} R_{\bar{\rho}}^{\square},$$

where $r := \dim_k Z^1(G_F, \text{ad}^0 \bar{\rho})$ and $t := \dim_k H^2(G_F, \text{ad}^0 \bar{\rho})$ such that the elements f_1, \dots, f_t form a regular sequence in $R_{\det \bar{\rho}} \llbracket x_1, \dots, x_r \rrbracket$. Moreover,

$$r - t = (d^2 - 1)([F : \mathbb{Q}_p] + 1).$$

Proof. This argument is a modification of Kisin’s method of presenting global deformation rings over local ones in [32, Section 4]. Kisin’s argument is an important refinement of Mazur’s obstruction theory in [37, Section 1.6].

The exact sequence $0 \rightarrow \text{ad}^0 \bar{\rho} \rightarrow \text{ad} \bar{\rho} \xrightarrow{\text{tr}} k \rightarrow 0$ of Galois representations induces an exact sequence of abelian groups:

$$0 \rightarrow Z^1(G_F, \text{ad}^0 \bar{\rho}) \rightarrow Z^1(G_F, \text{ad} \bar{\rho}) \xrightarrow{Z^1(\text{tr})} Z^1(G_F, k),$$

and hence, $r = \dim_k \ker(Z^1(\text{tr}))$. The map $Z^1(\text{tr}) : Z^1(G_F, \text{ad} \bar{\rho}) \rightarrow Z^1(G_F, k)$ is the induced map on Zariski tangent spaces of the map of deformation rings $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$, and thus lifts to a surjection

$$\tilde{\phi} : \tilde{R} := R_{\det \bar{\rho}} \llbracket x_1, \dots, x_r \rrbracket \twoheadrightarrow R_{\bar{\rho}}^{\square}.$$

We set $J := \ker \tilde{\phi}$. By Nakayama’s lemma, we need to show that $\dim_k J/\tilde{\mathfrak{m}}J \leq t$.

The module $J/\tilde{\mathfrak{m}}J$ appears as the kernel in the sequence

$$0 \rightarrow J/\tilde{\mathfrak{m}}J \rightarrow \tilde{R}/\tilde{\mathfrak{m}}J \rightarrow \tilde{R}/J \cong R_{\bar{\rho}}^{\square} \rightarrow 0. \tag{26}$$

In view of the above sequence, we shall construct a homomorphism

$$\alpha : \text{Hom}_k(J/\tilde{\mathfrak{m}}J, k) \rightarrow \ker(H^2(\text{tr}) : H^2(G_F, \text{ad} \bar{\rho}) \rightarrow H^2(G_F, k))$$

and show that the kernel of α injects into $\text{coker}(H^1(\text{tr}))$. This will imply the existence of the presentation in the statement of the Proposition, since then

$$\dim_k J/\tilde{\mathfrak{m}}J \leq \dim_k \ker(H^2(\text{tr})) + \dim_k \text{coker}(H^1(\text{tr})) = \dim_k H^2(G_F, \text{ad}^0 \bar{\rho}), \tag{27}$$

where the last equality comes from the long exact cohomology sequence that arises from $0 \rightarrow \text{ad}^0 \bar{\rho} \rightarrow \text{ad} \bar{\rho} \rightarrow k \rightarrow 0$.

Fix $u \in \text{Hom}_k(J/\tilde{\mathfrak{m}}J, k)$. The pushout under u of the sequence (26) yields

$$0 \rightarrow I_u \rightarrow R_u \xrightarrow{\phi_u} R_{\bar{\rho}}^{\square} \rightarrow 0,$$

where $I_u = k$. The surjection of profinite groups $\text{GL}_d(R_u) \twoheadrightarrow \text{GL}_d(R_{\bar{\rho}}^{\square})$ has a continuous section by [45, Proposition 2.2.2] (which is not necessarily a group homomorphism). Thus, there is a continuous function $\tilde{\rho}_u : G_F \rightarrow \text{GL}_d(R_u)$ such that the diagram of sets

$$\begin{array}{ccc}
 G_F & \xrightarrow{\tilde{\rho}_u} & \mathrm{GL}_d(R_u) \\
 & \searrow^{\rho^\square} & \downarrow \mathrm{GL}_d(\phi_u) \\
 & & \mathrm{GL}_d(R_\rho^\square)
 \end{array}$$

commutes. The kernel $1 + M_d(I_u)$ of $\mathrm{GL}_d(\phi_u)$ can be identified with $\mathrm{ad} \bar{\rho} \otimes_k I_u$, and so the set-theoretic lift yields a continuous 2-cocycle

$$c_u \in Z^2(G_F, \mathrm{ad} \bar{\rho}) \otimes_k I_u$$

given by $1 + c_u(g_1, g_2) = \tilde{\rho}_u(g_1 g_2) \tilde{\rho}_u(g_2)^{-1} \tilde{\rho}_u(g_1)^{-1}$. The class

$$[c_u] \in H^2(G_F, \mathrm{ad} \bar{\rho}) \otimes_k I_u$$

is independent of the chosen lifting since any other lift $\tilde{\rho}'_u$ gives rise to a class $c'_u \in Z^2(G_F, \mathrm{ad} \bar{\rho}) \otimes_k I_u$, which differs from c_u by a coboundary in $B^2(G_F, \mathrm{ad} \bar{\rho}) \otimes_k I_u$, so the representation ρ^\square can be lifted to a homomorphism $G_F \rightarrow \mathrm{GL}_d(R_u)$ if and only if $[c_u] = 0$. The existence of the homomorphisms $R_{\det \bar{\rho}} \rightarrow R_u \rightarrow R_\rho^\square$ together with the universality of $R_{\det \bar{\rho}}$ imply that the image of $[c_u]$ in $H^2(G_F, k)$ is zero. We define α as the homomorphism $u \mapsto [c_u]$.

To analyze the kernel of α , let u be such that $[c_u] = 0$, so that ρ^\square can be lifted to R_u . By the universality of R_ρ^\square , we obtain a splitting s_u of ϕ_u . One deduces that the map from I_u to the kernel of the surjective map

$$t_u : \mathfrak{m}_{R_u} / (\mathfrak{m}_{R_u}^2 + \varpi R_u) \rightarrow \mathfrak{m}^\square / ((\mathfrak{m}^\square)^2 + \varpi R_\rho^\square)$$

of mod ϖ cotangent spaces is an isomorphism.

The map t_u , in turn, is induced from the homomorphism $\tilde{R} / \tilde{\mathfrak{m}} J \rightarrow R_\rho^\square$ by pushout and from the analogous surjection

$$\tilde{t} : \tilde{\mathfrak{m}} / (\tilde{\mathfrak{m}}^2 + \varpi \tilde{R}) \rightarrow \mathfrak{m}^\square / ((\mathfrak{m}^\square)^2 + \varpi R_\rho^\square).$$

Via our identification $I_u \cong \ker t_u$, the pushout along u induces a surjective homomorphism $\gamma_u : \ker(\tilde{t}) \rightarrow I_u \cong k$ of k -vector spaces. One easily verifies that $u \mapsto \gamma_u$ induces an injective k -linear map

$$\ker(\alpha) \hookrightarrow \mathrm{Hom}_k(\ker(\tilde{t}), k).$$

Upon identifying $\ker(\tilde{t})^*$ with $\mathrm{coker}(H^1(\mathrm{tr}))$, the proof of the bound (27) is complete.

It remains to show that f_1, \dots, f_t is a regular sequence. We may write $\mathcal{O}[\mu] = \mathcal{O}[[z]] / ((1+z)^m - 1)$, where m is the order of μ . By Lemma 4.1, we get a presentation

$$\frac{\mathcal{O}[[z, y_1, \dots, y_{[F:\mathbb{Q}_p]+1}, x_1, \dots, x_r]]}{((1+z)^m - 1, f_1, \dots, f_t)} \cong R_\rho^\square.$$

By the same argument as in (22), the Euler–Poincaré characteristic formula implies that $r - t = \dim_k(\mathrm{ad}^0 \bar{\rho})([F : \mathbb{Q}_p] + 1) = (d^2 - 1)([F : \mathbb{Q}_p] + 1)$, and thus it follows from Corollary 3.38 that

$$\dim R_\rho^\square = [F : \mathbb{Q}_p] + 2 + r - t. \tag{28}$$

This implies that $(1+z)^m - 1, f_1, \dots, f_t$ can be extended to a system of parameters in a regular ring $S := \mathcal{O}[[z, y_1, \dots, y_{[F:\mathbb{Q}_p]+1}, x_1, \dots, x_r]]$. Thus, $(1+z)^m - 1, f_1, \dots, f_t$ is a regular sequence in S and so f_1, \dots, f_t is a regular sequence in $R_{\det \bar{\rho}}[[x_1, \dots, x_r]] = S / ((1+z)^m - 1)$. \square

Remark 4.4. The Proposition also holds for continuous representations $\bar{\rho} : G_F \rightarrow \text{GL}_d(\kappa)$, where κ is a local field, with essentially the same proof. The only difference is that one has to work harder to show the existence of the continuous section $\tilde{\rho}_u$, as the groups $\text{GL}_d(R_u)$ and $\text{GL}_d(R_u^\square)$ are not profinite anymore. The existence of such a section is well explained in [21, Lecture 6].

Corollary 4.5. *For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ and each closed point $x \in X^{\text{gen}}$ above $\mathfrak{m}_{R^{\text{ps}}}$, the following hold:*

- (1) $R_{\rho_x}^{\square, \chi}$ is \mathcal{O} -torsion free of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is complete intersection;
- (2) $R_{\rho_x}^{\square, \chi} / \varpi$ is complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p]$.

Proof. Without loss of generality, we may assume that the residue field of x is equal to k . Proposition 4.3 gives the presentation

$$R_{\det \rho_x}^\chi \llbracket x_1, \dots, x_r \rrbracket / (f_1, \dots, f_t) \xrightarrow{\cong} R_{\rho_x}^{\square, \chi},$$

where $R_{\det \rho_x}^\chi := R_{\det \rho_x} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$. Since $R_{\det \rho_x}^\chi$ is formally smooth over \mathcal{O} of dimension $[F : \mathbb{Q}_p] + 2$ by Lemma 4.1, it is enough to show that

$$\dim R_{\rho_x}^{\square, \chi} / \varpi \leq [F : \mathbb{Q}_p] + 1 + r - t.$$

Then the same argument as in the proof of Proposition 4.3 shows that the sequence ϖ, f_1, \dots, f_t is regular in $R_{\det \rho_x}^\chi \llbracket x_1, \dots, x_r \rrbracket$. Since $R_{\rho_x}^{\square, \chi}$ is a quotient of $R_{\rho_x}^\square$ and $R_{\rho_x}^\square$ is \mathcal{O} -torsion free by Corollary 3.38, we have $\dim R_{\rho_x}^{\square, \chi} / \varpi \leq \dim R_{\rho_x}^\square / \varpi = \dim R_{\rho_x}^\square - 1$, and the desired inequality follows from (28). \square

The restriction of a pseudo-character $D : A[G_F] \rightarrow A$ to G defines a continuous group homomorphism $\det D : G_F \rightarrow A^\times$; see [9, Definition 4.1.5]. Moreover, if D is associated to a representation $\rho : G_F \rightarrow \text{GL}_d(A)$, then $\det D = \det \rho$. This induces a map of deformation rings $R_{\det \bar{D}} \rightarrow R^{\text{ps}}$ and makes R^{ps} into an $\mathcal{O}[\mu]$ -algebra.

Since A^{gen} is an R^{ps} -algebra, we may define

$$A^{\text{gen}, \chi} := A^{\text{gen}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}, \quad X^{\text{gen}, \chi} := \text{Spec } A^{\text{gen}, \chi},$$

and we let $\bar{X}^{\text{gen}, \chi}$ denote its special fibre. Note that since a character of G_F^{ab} valued in a characteristic p field is trivial after pulling back to $\mu_{p^\infty}(F)$, we have that $\bar{X}^{\text{gen}, \chi} = \bar{X}^{\text{gen}, \mathbf{1}}$ for all χ , where $\mathbf{1}$ is the trivial character. Moreover, the reduced subschemes of \bar{X}^{gen} and $\bar{X}^{\text{gen}, \chi}$ coincide and so

$$\dim \bar{X}^{\text{gen}, \chi} = \dim \bar{X}^{\text{gen}} = d^2 + d^2[F : \mathbb{Q}_p],$$

where the last equality is given by Corollary 3.40.

Corollary 4.6. *For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, the following hold:*

- (1) $A^{\text{gen}, \chi}$ is \mathcal{O} -torsion free, equi-dimensional of dimension $1 + d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection;
- (2) $A^{\text{gen}, \chi} / \varpi$ is equi-dimensional of dimension $d^2 + d^2[F : \mathbb{Q}_p]$ and is locally complete intersection.

Proof. We claim that the local rings at closed points of $X^{\text{gen}, \chi}$ are \mathcal{O} -torsion free and complete intersection. Given the claim, the proof is the same as in Corollary 3.40.

We will prove the claim using the strategy outlined in Remark 3.43. We already know from Corollary 4.5 that $R_{\rho_x}^{\square, \chi}$ is \mathcal{O} -torsion free and complete intersection of dimension $d^2 + d^2[F : \mathbb{Q}_p] + 1$ whenever $x \in X^{\text{gen}, \chi}$ is a closed point with $\kappa(x)/k$ a finite extension. By applying $\otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$, we obtain the χ -versions of Propositions 3.34 and 3.41 and Corollary 3.42.

Let x be a closed point of $X^{\text{gen}, \chi}$. If $\kappa(x)$ is a finite extension of k , then $\widehat{\mathcal{O}}_{X^{\text{gen}, \chi}, x} \cong R_{\rho_x}^{\square, \chi}$ by Proposition 3.34, and hence, $\mathcal{O}_{X^{\text{gen}, \chi}, x}$ is complete intersection. Otherwise, let x' and z be as in Corollary 3.42.

In particular, z is a closed point of $X^{\text{gen},\chi}$, and $\kappa(z)$ is a finite extension of k . It follows from the argument explained in Remark 3.43 that if $\kappa(x)$ is a local field of characteristic p , then

$$\widehat{\mathcal{O}}_{X^{\text{gen},\chi},x}[[T]] \cong R_{\rho_x}^{\square,\chi} \cong R_{\rho_{x'}}^{\square,\chi} \cong \widehat{(R_{\rho_z}^{\square,\chi})}_{x'}[[T]],$$

and if $\kappa(x)$ is a finite extension of L , then

$$\widehat{\mathcal{O}}_{X^{\text{gen},\chi},x} \cong R_{\rho_x}^{\square,\chi} \cong R_{\rho_{x'}}^{\square,\chi} \cong \widehat{(R_{\rho_z}^{\square,\chi})}_{x'}.$$

Since $R_{\rho_z}^{\square,\chi}$ is complete intersection, it follows from [48, Tag 09Q4] that the local ring $(R_{\rho_z}^{\square,\chi})_{x'}$ (and hence its completion) is also complete intersection. The isomorphisms above imply that $\widehat{\mathcal{O}}_{X^{\text{gen},\chi},x}$ is complete intersection. Hence, $\mathcal{O}_{X^{\text{gen},\chi},x}$ is complete intersection; see [48, Tag 09Q3]. \square

Remark 4.7. Alternatively, one could first prove a version of Proposition 4.3 for deformation rings of $\bar{\rho} : G_F \rightarrow \text{GL}_d(\kappa(x))$ to Artinian Λ -algebra as in Section 3.5 for any closed point of $x \in X^{\text{gen}}$ by changing \mathcal{O} to Λ and k to $\kappa(x)$ everywhere. The Euler–Poincaré characteristic formula still holds in this setting; see [9, Theorem 3.4.1(c)]. Then deduce Corollary 4.5 in this more general setting using the same proof and then obtain Corollary 4.6 by repeating verbatim the proof of Corollary 3.40.

In the Lemmas below, κ is either a finite extension of k , a finite extension of L or a local field of characteristic p containing k . The ring Λ is defined exactly as in the beginning of Section 3.5. If $\text{char}(\kappa) = 0$, then $\Lambda = \kappa$, and if $\text{char}(\kappa) = p$, then Λ is an \mathcal{O} -algebra, which is a complete DVR with uniformiser ϖ and residue field κ . As in Section 3.5, we consider deformation problems of $\rho : G_F \rightarrow \text{GL}_d(\kappa)$ to local Artinian Λ -algebras with residue field κ .

Lemma 4.8. *Let $\rho : G_F \rightarrow \text{GL}_d(\kappa)$ be a continuous representation, where κ is either a finite extension of k , a local field of characteristic p or a finite extension of L . If $H^2(G_F, \text{ad}^0 \rho) = 0$, then for all characters $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, the ring $R_{\rho}^{\square,\chi}$ is formally smooth over Λ .*

Proof. It follows from the proof of [9, Lemma 3.4.2], where an analogous statement is proved for the deformation functors without the framing and for Artinian κ -algebras, that the map

$$R_{\det \rho} \rightarrow R_{\rho}^{\square},$$

induced by sending a deformation of ρ to an Artinian Λ -algebra to its determinant, is formally smooth. By applying $\otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$, we deduce that the map

$$R_{\det \rho}^{\chi} \rightarrow R_{\rho}^{\square,\chi}$$

is formally smooth.

Since the group $G_F^{\text{ab}}/\text{Art}_F(\mu_{p^\infty}(F))$ is p -torsion free, the ring $R_{\det \rho}^{\chi}$ is formally smooth over Λ . Hence, $R_{\rho}^{\square,\chi}$ is formally smooth over Λ . (Alternatively, one could prove Proposition 4.3 for ρ – see Remark 4.4 – and then obtain the Lemma as a Corollary.) \square

Recall that in Section 3.7 we have defined an open subscheme $U^{\text{n-spcl}}$ of $\overline{X}^{\text{ps}} \setminus \{\mathfrak{m}_{R^{\text{ps}}}\}$ and defined $V^{\text{n-spcl}}$ to be a preimage of $U^{\text{n-spcl}}$ in $\overline{X}^{\text{gen}}$. We will refer to $V^{\text{n-spcl}}$ as the *absolutely irreducible non-special locus*.

Proposition 4.9. *For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, the absolutely irreducible non-special locus in $\overline{X}^{\text{gen},\chi}$ is regular.*

Proof. It is enough to show that localization of $A^{\text{gen},\chi}/\varpi$ at x is a regular ring for every closed point x in $V^{\text{n-spcl}} \cap \overline{X}^{\text{gen},\chi}$. It follows from Lemma 3.35 applied with $R = R^{\text{ps},\chi}/\varpi$ and $A = A^{\text{gen},\chi}/\varpi$

that it is enough to show that the completion of $\kappa(x) \otimes_{\mathcal{O}} A^{\text{gen},\chi}$ at the kernel of the map of $\kappa(x)$ -algebras $\kappa(x) \otimes_{\mathcal{O}} A^{\text{gen},\chi} \rightarrow \kappa(x)$ is regular. Proposition 3.34 implies that we may identify this ring with deformation ring $R_{\rho_x}^{\square,\chi}/\varpi$. If $\zeta_p \in F$, then since x is non-special $H^2(G_F, \text{ad}^0 \rho_x) = 0$ (see [9, Lemma 5.1.1]), Lemma 4.8 implies that $R_{\rho_x}^{\square,\chi}/\varpi$ is formally smooth over $\kappa(x)$. If $\zeta_p \notin F$, then μ is trivial, so that $R_{\rho_x}^{\square,\chi} = R_{\rho_x}^{\square}$, and $H^2(G_F, \text{ad} \rho_x) = 0$; see [9, Lemma 5.1.1]. It follows from (21) that $R_{\rho_x}^{\square}/\varpi$ is formally smooth over $\kappa(x)$. \square

Proposition 4.10. *For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, the absolutely irreducible locus in $X^{\text{gen},\chi}[1/p]$ is regular.*

Proof. Let x be a closed point in $X^{\text{gen},\chi}[1/p]$ and let $\rho_x : G_F \rightarrow \text{GL}_d(\kappa(x))$ be the corresponding Galois representation. We claim that if ρ_x is absolutely irreducible, then $H^2(G_F, \text{ad}^0 \rho_x) = 0$. Since $\kappa(x)$ is a finite extension of L , $\text{ad}^0 \rho_x$ is a direct summand of $\text{ad} \rho_x$, and thus it is enough to show that $H^2(G_F, \text{ad} \rho_x) = 0$. By local Tate duality, it is enough to show that $H^0(G_F, \text{ad} \rho_x(1)) = 0$. Since ρ_x is absolutely irreducible, non-vanishing of this group is equivalent to $\rho_x \cong \rho_x(1)$. By considering determinants, we would obtain that the d -th power of the cyclotomic character is trivial, yielding a contradiction.

Given the claim, the rest of the proof is the same as the proof of Proposition 4.9 since Lemma 3.37 implies that $\widehat{\mathcal{O}}_{X^{\text{gen},\chi},x} \cong R_{\rho_x}^{\square,\chi}$. \square

Lemma 4.11. *Assume that $F = \mathbb{Q}_p$ and $d = 2$. Let κ be either a finite extension of L or a finite or local field of characteristic p . If $\text{char}(\kappa) = p$, then we further assume that $p > 2$. Let $\rho : G_{\mathbb{Q}_p} \rightarrow \text{GL}_2(\kappa)$ be a continuous representation with semi-simplification isomorphic to a direct sum of characters $\psi_1 \oplus \psi_2$ satisfying $\psi_1 \neq \psi_2(1)$ and $\psi_2 \neq \psi_1(1)$. Then*

$$H^2(G_{\mathbb{Q}_p}, \text{ad} \rho) = H^2(G_F, \text{ad}^0 \rho) = 0.$$

In particular, $R_\rho^{\square,\chi}$ is formally smooth over Λ .

Proof. Since $\text{char}(\kappa) \neq 2$, $\text{ad}^0 \rho$ is a direct summand $\text{ad} \rho$, and thus it is enough to show that $H^2(G_{\mathbb{Q}_p}, \text{ad} \rho) = 0$. By local Tate duality (see [9, Theorem 3.4.1(b)]), it is enough to show that $H^0(G_{\mathbb{Q}_p}, \text{ad} \rho(1)) = 0$. Non-vanishing of this group would imply that $\psi_i \psi_j^{-1}(1)$ is a trivial character for some $i, j \in \{1, 2\}$. If $i = j$, then this would imply $\chi_{\text{cyc}} \otimes_{\mathbb{Z}_p} \kappa$ is trivial, which is not the case as $\text{char}(\kappa) \neq 2$. If $i \neq j$, then this does not hold by assumption.

The last assertion follows from Lemma 4.8. \square

Lemma 4.12. *Assume that $p = 2$, $F = \mathbb{Q}_2$ and $d = 2$. Let κ be a finite or local field of characteristic 2 and let $\rho : G_{\mathbb{Q}_2} \rightarrow \text{GL}_2(\kappa)$ be a continuous representation, which is a non-split extension of distinct characters.*

Then $H^2(G_{\mathbb{Q}_2}, \text{ad}^0 \rho) = 0$. In particular, $R_\rho^{\square,\chi}$ is formally smooth over Λ .

Proof. After twisting, we may assume that we can choose a basis of the underlying vector space of ρ , such that with respect to that basis,

$$\rho(g) = \begin{pmatrix} 1 & b(g) \\ 0 & \psi(g) \end{pmatrix}, \quad \forall g \in G_{\mathbb{Q}_2},$$

where $\psi : G_{\mathbb{Q}_2} \rightarrow \kappa^\times$ is a nontrivial character. We use the same basis to identify $\text{ad} \rho$ with $M_2(k)$ with the $G_{\mathbb{Q}_2}$ -action given by

$$g \cdot M := \rho(g)M\rho(g)^{-1}.$$

For $i, j \in \{1, 2\}$, let $e_{ij} \in M_2(k)$ be the matrix with the ij -entry equal to 1 and all the other entries equal to zero. Let $\overline{\text{ad}} \rho$ be the quotient $\text{ad} \rho$ by the scalar matrices and let \bar{e}_{ij} be the image of e_{ij} in $\overline{\text{ad}} \rho$. A direct computation shows that

$$g \cdot \bar{e}_{12} = \psi(g)^{-1} \bar{e}_{12}, \quad g \cdot \bar{e}_{11} = \bar{e}_{11} - \psi(g)^{-1} b(g) \bar{e}_{12}, \quad g \cdot \bar{e}_{21} = \psi(g) \bar{e}_{21} - \psi(g)^{-1} b(g)^2 \bar{e}_{12}.$$

Since ρ is non-split, $b(g) \neq 0$ for some $g \in G_{\mathbb{Q}_2}$. Thus, $\kappa \bar{e}_{12}$ is the unique irreducible subrepresentation of $\overline{\text{ad}}\rho$. Since $G_{\mathbb{Q}_2}$ acts on \bar{e}_{12} by a nontrivial character, we deduce that $H^0(G_{\mathbb{Q}_2}, \overline{\text{ad}}\rho) = 0$.

It follows from local Tate duality (see [9, Theorem 3.4.1(b)]) that $H^2(G_{\mathbb{Q}_2}, \text{ad}^0 \rho) = 0$. Note that the cyclotomic character is trivial modulo 2.

The last assertion follows from Lemma 4.8. □

Proposition 4.13. *There is an open subscheme $V^{0,\chi} \subset \overline{X}^{\text{gen},\chi}$ such that*

- (1) $H^2(G_F, \text{ad}^0 \rho_x) = 0$ for all closed points $x \in V^{0,\chi}$;
- (2) $\dim \overline{X}^{\text{gen},\chi} - \dim(\overline{X}^{\text{gen},\chi} \setminus V^{0,\chi}) \geq 2$.

In particular, $\overline{X}^{\text{gen},\chi}$ is regular in codimension 1.

Proof. We first note that if $V \subset \overline{X}^{\text{gen},\chi}$ is open and satisfies part (1), then V is regular by the argument explained in the proof of Proposition 4.9. Thus if (1) and (2) hold then $\overline{X}^{\text{gen},\chi}$ is regular in codimension 1. We also note that Lemma A.2 implies that part (1) holds for $V^{\text{Kirr},\chi} := V^{\text{Kirr}} \cap \overline{X}^{\text{gen},\chi}$. We consider three separate cases.

Case 1: $d > 2$ or $F \neq \mathbb{Q}_p$ or \overline{D} is (absolutely) irreducible. These three conditions correspond to parts (1), (2) and (3) of Proposition 3.53, respectively, and indeed Proposition 3.53 implies that the complement of $V^{\text{Kirr},\chi}$ in $\overline{X}^{\text{gen},\chi}$ has dimension at most $\dim \overline{X}^{\text{gen},\chi} - 2$. Hence, we may take $V^{0,\chi} = V^{\text{Kirr},\chi}$.

Case 2: $d = 2$ and $F = \mathbb{Q}_p$ and $p > 2$ and \overline{D} is reducible. In this case, $\mu = \{1\}$ so $\chi = \mathbf{1}$, and thus, $\overline{X}^{\text{gen},\mathbf{1}} = \overline{X}^{\text{gen}}$. It follows from Proposition 3.26, Lemma 3.30 and Lemma 3.52 that

$$V^{0,\chi} := \overline{X}^{\text{gen}} \setminus (Y \cup Z_{\mathcal{P}_{\max}}^{12} \cup Z_{\mathcal{P}_{\max}}^{21} \cup Z^{\text{Kred}})$$

satisfies part (2). We may also write $V^{0,\chi} = V^{\text{Kirr}} \cup V'_{\mathcal{P}_{\max}}$, where we use the notation introduced in the proof of Proposition 3.27. Since part (1) holds for V^{Kirr} , it is enough to consider closed points $x \in V'_{\mathcal{P}_{\max}}$. The definition of $V'_{\mathcal{P}_{\max}}$ implies firstly that ρ_x is reducible and secondly that if we let ψ_1 and ψ_2 denote its irreducible Jordan-Hölder constituents, then $\psi_1 \neq \psi_2(1)$ and $\psi_2 \neq \psi_1(1)$. Therefore, $H^2(G_F, \text{ad}^0 \rho_x) = 0$ by Lemma 4.11.

Case 3: $d = 2$ and $F = \mathbb{Q}_2$ and \overline{D} is reducible. The proof is the same as in Case 2, using Lemma 4.12 instead of Lemma 4.11. However, one additionally has to remove the reducible semi-simple locus in $\overline{X}^{\text{gen},\chi}$. Its dimension is at most $4 + 2 = 6$ by Corollary 3.49 and the dimension of $\overline{X}^{\text{gen},\chi}$ is 8. Thus, the codimension is at least 2. □

Proposition 4.14. $\overline{X}^{\text{gen},\chi}$ is normal.

Proof. Since $\overline{X}^{\text{gen},\chi}$ is a local complete intersection by Corollary 4.6, it is Cohen–Macaulay and satisfies property (S2), and Proposition 4.13 says that it satisfies property (R1). Hence, $\overline{X}^{\text{gen},\chi}$ is normal by Serre’s criterion for normality. □

Corollary 4.15. For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ and $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$, the ring $R_{\bar{\rho}}^{\square,\chi}/\varpi$ is a normal integral domain.

Proof. Since $\overline{X}^{\text{gen},\chi}$ is normal and excellent the completions of its local rings are normal by [36, Theorem 32.2 (i)]. So after formally completing along the maximal ideal corresponding to $\bar{\rho}$, Proposition 3.34 (after applying $\otimes_{\mathcal{O}[\mu],\chi} \mathcal{O}$) and Proposition 3.37 tell us that $R_{\bar{\rho}}^{\square,\chi}/\varpi$ is normal, and thus an integral domain since it is a local ring. □

Lemma 4.16. Let \hat{Y} be the preimage of $\mathfrak{m}_{R_{\bar{\rho}}}$ in $\text{Spec } R_{\bar{\rho}}^{\square,\chi}/\varpi$. Let W be a closed subscheme of $\text{Spec } R_{\bar{\rho}}^{\square,\chi}/\varpi$ such that $H^2(G_F, \text{ad}^0 \rho_x) \neq 0$ for all closed points $x \in W \setminus \hat{Y}$. Then $\dim R_{\bar{\rho}}^{\square,\chi}/\varpi - \dim W \geq 2$.

Proof. The assumptions imply that W is contained in $\hat{Z} \cup \hat{Y}$, where $Z = \overline{X}^{\text{gen},\chi} \setminus V^{0,\chi}$ and \hat{Z} is a formal completion of Z at the point corresponding to $\bar{\rho}$. In terms of commutative algebra, the ring of functions of \hat{Z} corresponds to the completion of the ring of functions of Z with respect to the maximal ideal corresponding to $\bar{\rho}$. Hence, $\dim \hat{Z} \leq \dim Z$, and Proposition 4.13 implies that \hat{Z} has codimension at least 2 in $\text{Spec } R_{\bar{\rho}}^{\square,\chi}/\varpi$. Similarly, \hat{Y} is a formal completion of Y (the preimage of $\{\mathfrak{m}_{R^{\text{ps}}}\}$ in X^{gen}) at the point corresponding to $\bar{\rho}$, and using Lemma 3.30 we conclude that \hat{Y} also has codimension of at least 2 in $\text{Spec } R_{\bar{\rho}}^{\square,\chi}/\varpi$. \square

Proposition 4.17. $X^{\text{gen},\chi}[1/p]$ is normal.

Proof. The proof is essentially the same as the proof of Proposition 4.14. It follows from Corollary 4.6 that $X^{\text{gen},\chi}[1/p]$ is Cohen–Macaulay, and we have to check that the codimension of the singular locus is at least 2. Since $X^{\text{gen},\chi}[1/p]$ is a preimage of $\text{Spec } R^{\text{ps},\chi}[1/p]$ in X^{gen} , Lemma 3.18 implies that $X^{\text{gen},\chi}[1/p]$ is Jacobson and we may argue with closed points.

We have already shown in Proposition 4.10 that the absolutely irreducible locus $V^{\text{irr},\chi}$ in $X^{\text{gen},\chi}[1/p]$ is regular. Thus, the singular locus is contained in

$$\bigcup_{\mathcal{P}_{\min} < \mathcal{P}} X_{\mathcal{P}}^{\text{gen},\chi}[1/p],$$

where $X_{\mathcal{P}}^{\text{gen},\chi} := X^{\text{gen},\chi} \cap X_{\mathcal{P}}^{\text{gen}}$.

If either $\bar{\rho}$ is absolutely irreducible, $F \neq \mathbb{Q}_p$ or $d > 2$, then it follows from Proposition 3.58 that $X^{\text{gen},\chi}[1/p]$ is regular in codimension 1.

If $\bar{\rho}$ is reducible, $F = \mathbb{Q}_p$ and $d = 2$, then there are two partitions \mathcal{P}_{\min} and \mathcal{P}_{\max} and $\dim X^{\text{gen},\chi}[1/p] - \dim X_{\mathcal{P}_{\max}}^{\text{gen},\chi}[1/p] = 1$, so the previous argument does not work. If $x \in X^{\text{gen},\chi}[1/p]$ is a closed singular point, then it follows from Proposition 4.10 and Lemma 4.11 that ρ_x is reducible and its semi-simplification has the form $\psi \oplus \psi(1)$ for some character $\psi : G_F \rightarrow \kappa(x)^\times$. Thus, we may assume that the pseudo-character \overline{D} associated to $\bar{\rho}$ is equal to $\bar{\psi} + \bar{\psi}(1)$. We will now recall a construction, carried out after Lemma 3.24, in this special case. Let $R_{\bar{\psi}}$ be the universal deformation ring of $\bar{\psi}$. Mapping a deformation ψ_A to the pseudo-character $\psi_A + \psi_A(1)$ induces a map of local \mathcal{O} -algebras $R^{\text{ps}} \rightarrow R_{\bar{\psi}}$. Let $X_{\mathcal{P}_{\max}}^{\text{ps},12}$ be the schematic image of $\text{Spec } R_{\bar{\psi}} \rightarrow X^{\text{ps}}$ induced by this map. Let $W := X^{\text{gen},\chi} \times_{X^{\text{ps}}} X_{\mathcal{P}_{\max}}^{\text{ps},12}$. The generic fibre $W[1/p]$ contains all the singular closed points, and since $X^{\text{gen}}[1/p]$ is Jacobson, $W[1/p]$ contains the singular locus of $X^{\text{gen}}[1/p]$. The special fibre \overline{W} is a union of $Z_{\mathcal{P}_{\max}}^{12}$ and Y (as underlying topological spaces). Thus, $\dim \overline{W} \leq 6$ as $\dim Y \leq 4$ by Lemma 3.30 and $\dim Z_{\mathcal{P}_{\max}}^{12} \leq 6$ by Proposition 3.26. Since W is a GL_d -invariant subscheme of X^{gen} , Lemma 3.23 implies that $\dim W[1/p] \leq 6$. It follows from Corollary 4.6 that $\dim X^{\text{gen},\chi}[1/p] = \dim \overline{X}^{\text{gen},\chi} = 8$. Thus, the codimension of the singular locus in $X^{\text{gen},\chi}[1/p]$ is at least 2. \square

Corollary 4.18. $X^{\text{gen},\chi}$ is normal.

Proof. Since $A^{\text{gen},\chi}$ is \mathcal{O} -torsion free by Corollary 4.6, the map $\mathcal{O} \rightarrow A^{\text{gen},\chi}$ is flat. We have shown in Propositions 4.14 and 4.17 that the fibre rings $L \otimes_{\mathcal{O}} A^{\text{gen},\chi}$ and $k \otimes_{\mathcal{O}} A^{\text{gen},\chi}$ are normal. Since \mathcal{O} is a regular ring, [11, Corollary 2.2.23] implies that $A^{\text{gen},\chi}$ is normal. \square

Corollary 4.19. For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ and $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$, the ring $R_{\bar{\rho}}^{\square,\chi}$ is a normal integral domain.

Proof. The proof is essentially the same as the proof of Corollary 4.15. To see this, note that Corollary 4.18 implies that $X^{\text{gen},\chi}$ is normal, and that the formal completion of $X^{\text{gen},\chi}$ along the maximal ideal corresponding to $\bar{\rho}$ is $R_{\bar{\rho}}^{\square,\chi}$ by the χ -version of Proposition 3.34 as explained in the proof of Corollary 4.6. \square

Lemma 4.20. *Let W be a closed subscheme of $\text{Spec } R_{\bar{\rho}}^{\square, \chi}[1/p]$ with the property that $H^2(G_F, \text{ad}^0 \rho_x) \neq 0$ for all closed points $x \in W$. Then $\dim R_{\bar{\rho}}^{\square, \chi}[1/p] - \dim W \geq 2$.*

Proof. Since in characteristic zero $\text{ad}^0 \rho_x$ is a direct summand of $\text{ad } \rho_x$ we obtain that $H^2(G_F, \text{ad } \rho_x) \neq 0$ for all $x \in W$. This implies that W is contained in the singular locus of $\text{Spec } R_{\bar{\rho}}^{\square, \chi}[1/p]$. Since $R_{\bar{\rho}}^{\square, \chi}[1/p]$ is normal, the singular locus has codimension of at least 2. \square

The next result answers affirmatively a question raised by GB–Juschka in [8, Question 1.10].

Corollary 4.21. *The map $R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square}$ induces a bijection between the sets of irreducible components.*

Proof. Since $R_{\bar{\rho}}^{\square}$ is \mathcal{O} -torsion free by Corollary 3.38, the irreducible components of $R_{\bar{\rho}}^{\square}$ and $R_{\bar{\rho}}^{\square}[1/p]$ coincide. Since the algebra $\mathcal{O}[\mu_{p^\infty}(F)][1/p]$ is semi-simple, we have

$$R_{\bar{\rho}}^{\square}[1/p] \cong \prod_{\chi: \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times} R_{\bar{\rho}}^{\square, \chi}[1/p]. \tag{29}$$

It follows from Corollaries 4.5 and 4.19 that $R_{\bar{\rho}}^{\square, \chi}$ is an \mathcal{O} -torsion free integral domain. We note that the special fibres of these rings are non-zero, thus the rings themselves are non-zero. Hence, the localization $R_{\bar{\rho}}^{\square, \chi}[1/p]$ is non-zero and is an integral domain. \square

Corollary 4.22. *$R_{\bar{\rho}}^{\square}[1/p]$ is normal and $R_{\bar{\rho}}^{\square}$ is reduced.*

Proof. The first assertion follows from (29) and Corollary 4.19. Since $R_{\bar{\rho}}^{\square}$ is \mathcal{O} -torsion free by Corollary 3.38, it is a subring of $R_{\bar{\rho}}^{\square}[1/p]$. This implies the second assertion as normal rings are reduced. \square

Corollary 4.23. *If either $d = 2$ and $[F : \mathbb{Q}_p] \geq 4$ or $d \geq 3$ and $[F : \mathbb{Q}_p] \geq 3$, then for each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ and $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$, the rings $R_{\bar{\rho}}^{\square, \chi}$, $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are regular in codimension 3. In particular, $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are factorial.*

Proof. The assumptions together with the lower bound on the codimension of the Kummer-irreducible locus in Proposition 3.53 and the containment $V^{\text{Kirr}} \subset V^{\text{n-spcl}}$ (resp. Proposition 3.58) imply that the complement of the absolutely irreducible non-special locus in $\overline{X}^{\text{gen}, \chi}$ (resp. absolutely irreducible locus in $X^{\text{gen}, \chi}[1/p]$) has codimension at least 4. It follows from Proposition 4.9 (resp. Proposition 4.10) that it contains the singular locus in $\overline{X}^{\text{gen}, \chi}$ (resp. $X^{\text{gen}, \chi}[1/p]$). Hence, $X^{\text{gen}, \chi}$ and $\overline{X}^{\text{gen}, \chi}$ are regular in codimension 3, which implies that $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are regular in codimension 3. Since both rings are also complete intersection by Corollary 4.5, they are factorial by a theorem of Grothendieck; see [13] for a short proof. \square

Remark 4.24. The assumptions in Corollary 4.23 are not optimal as the next Corollary shows. To find the optimal assumptions, one would have to further study the reducible locus, and we do not want to pursue this here. We note that if $F = \mathbb{Q}_p$, $p \geq 5$ and $\bar{\rho} = \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix}$ is non-split, where ω is the cyclotomic character modulo p , then it follows from [42, Corollary B.5] that $R_{\bar{\rho}}^{\square, \chi} \cong \mathcal{O}[[x_1, \dots, x_9]]/(x_1 x_2 - x_3 x_4)$ and hence is not factorial. Therefore, some assumptions in Corollary 4.23 have to be made.

Corollary 4.25. *If $\bar{\rho}$ is absolutely irreducible, then $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are factorial, except in the case $d = 2$, $F = \mathbb{Q}_3$ and $\bar{\rho} \cong \bar{\rho}(1)$.*

Proof. Since $\bar{\rho}$ is absolutely irreducible, $X^{\text{gen}, \chi}[1/p]$ is regular by Proposition 4.10, and the singular locus of $\overline{X}^{\text{gen}, \chi}$ is contained in Z^{spcl} , which has codimension at least $\frac{1}{2}[F : \mathbb{Q}_p]d^2$ by Lemma 3.52. Thus, if either $d > 2$ or $F \neq \mathbb{Q}_p$, then we can conclude that $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are regular in codimension 3 and hence factorial.

If $\bar{\rho} \not\cong \bar{\rho}(1)$, then $H^2(G_{\mathbb{Q}_p}, \text{ad}^0 \bar{\rho}) = 0$, and it follows from Lemma 4.8 that $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are formally smooth, hence regular and hence factorial.

If $d = 2$, then $\bar{\rho} \cong \bar{\rho}(1)$ implies that $\det \bar{\rho} = (\det \bar{\rho})\omega^2$. This leaves us with two cases: $F = \mathbb{Q}_2$ or $F = \mathbb{Q}_3$. If $p = 2$, then $R_{\bar{\rho}}^{\square}$ is formally smooth over $\mathcal{O}[\mu]$ by [16, Proposition 4.5], and thus $R_{\bar{\rho}}^{\square, \chi}$ and $R_{\bar{\rho}}^{\square, \chi}/\varpi$ are regular.

We claim that if $F = \mathbb{Q}_3$, $d = 2$ and $\bar{\rho} \cong \bar{\rho}(1)$, then the ring $R_{\bar{\rho}}^{\square, \chi}$ is not factorial. It follows from [6, Theorem 5.1] that, in this case, $R_{\bar{\rho}}^{\square, \chi}$ is formally smooth over $\mathcal{O}[[b, c, d]]/(r)$, where $r = (1 + d)^6(1 + bcu) - (1 + bcv)$ and u, v are units in $\mathcal{O}[[b, c]]$. The ideal $\mathfrak{p} = (b, d)$ is prime of height 1. If $R_{\bar{\rho}}^{\square, \chi}$ were factorial, then \mathfrak{p} would have to be principal, [48, Tag 0AFT], and thus there would exist $\pi \in \mathcal{O}[[b, c, d]]$ such that we have an equality of ideals $(b, d) = (r, \pi)$ in $\mathcal{O}[[b, c, d]]$. By considering this modulo (ϖ, c) , we would conclude that $(d^3 - d^6, \bar{\pi})$ is the maximal ideal in $k[[b, d]]$. Since $(d^3 - d^6, \bar{\pi}) \rightarrow (b, d)/(b, d)^2$ is not surjective, we obtain a contradiction. The same argument shows that $R_{\bar{\rho}}^{\square, \chi}/\varpi$ is also not factorial. \square

Proposition 4.26. *For each character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, the rings $A^{\text{gen}, \chi}$ and $A^{\text{gen}, \chi}/\varpi$ are integral domains.*

Proof. Since $A^{\text{gen}, \chi}$ is normal by Corollary 4.18, it is a product of normal domains $A^{\text{gen}, \chi} \cong A_1 \times \dots \times A_m$. The action of G on $X^{\text{gen}, \chi}$ leaves the connected components invariant by Lemma 2.1. It follows from Lemma 3.21 that each $\text{Spec } A_i$ contains a closed point over the closed point X^{ps} . Thus, $A_i \otimes_{R^{\text{ps}}} k$ are non-zero for $1 \leq i \leq m$. If $m > 1$, then this would imply that the fibre at the closed point of X^{ps} is not connected, contradicting Lemma 3.7. The same proof works also for the special fibre. \square

Define $R^{\text{ps}, \chi} := R^{\text{ps}} \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$ for a character $\chi : \mu \rightarrow \mathcal{O}^\times$ and using the isomorphism from Lemma 4.1. We let $X^{\text{ps}, \chi} = \text{Spec } R^{\text{ps}, \chi}$ and let $\bar{X}^{\text{ps}, \chi}$ be its special fibre.

Corollary 4.27. *The rings $R^{\text{ps}}[1/p]$, $R^{\text{ps}, \chi}[1/p]$ and the rigid spaces $(\text{Spf } R^{\text{ps}})^{\text{rig}}$, $(\text{Spf } R^{\text{ps}, \chi})^{\text{rig}}$ are normal. Moreover, $R^{\text{ps}, \chi}[1/p]$ is an integral domain, and thus the map $R_{\det \bar{\rho}}[1/p] \rightarrow R^{\text{ps}}[1/p]$ induces a bijection between the sets of irreducible components.*

Proof. The assertion follows from [43, Theorem A.1] using Corollary 4.22. As part of the proof, one obtains $R^{\text{ps}}[1/p] = (A^{\text{gen}}[1/p])^G$. This yields $R^{\text{ps}, \chi}[1/p] = (A^{\text{gen}, \chi}[1/p])^G$. Proposition 4.26 implies that $A^{\text{gen}, \chi}[1/p]$ is an integral domain. Hence, $R^{\text{ps}, \chi}[1/p]$ is an integral domain, and the assertion about irreducible components is proved in the same manner as Corollary 4.21. \square

Corollary 4.28. *The image of R^{ps} in A^{gen} is the maximal \mathcal{O} -torsion free quotient of R^{ps} and is also the maximal reduced quotient of R^{ps} . In particular, the map $R_{\det \bar{D}} \rightarrow R^{\text{ps}} \rightarrow R^{\text{ps}}[1/p]$ induces a bijection between the sets of irreducible components. Moreover, if \bar{D} is multiplicity free, then R^{ps} is reduced and \mathcal{O} -torsion free.*

Proof. By [50, Theorem 2.20], the map $X^{\text{gen}} // \text{GL}_d \rightarrow X^{\text{ps}}$ is an adequate homeomorphism. It follows from [1, Proposition 3.3.5] that the kernel of $R^{\text{ps}} \rightarrow (A^{\text{gen}})^{\text{GL}_d}$ is nilpotent and vanishes after inverting p . Since A^{gen} is \mathcal{O} -torsion free and reduced, this implies that both quotients coincide and are equal to the image of R^{ps} in A^{gen} . This together with the last part of Corollary 4.27 implies the assertion about the irreducible components.

If \bar{D} is multiplicity free, then E is a generalized matrix algebra by [18, Theorem 2.22], and it follows from [50, Theorem 3.8 (4)] that $R^{\text{ps}} = (A^{\text{gen}})^{\text{GL}_d}$, and so R^{ps} is \mathcal{O} -torsion free and reduced. \square

Corollary 4.29. *The image of $R^{\text{ps}, \chi}/\varpi$ in $A^{\text{gen}, \chi}/\varpi$ is the maximal reduced quotient of $R^{\text{ps}, \chi}/\varpi$. The image of $R^{\text{ps}, \chi}$ in $A^{\text{gen}, \chi}$ is the maximal reduced quotient of $R^{\text{ps}, \chi}$ and is also the maximal \mathcal{O} -torsion free quotient of $R^{\text{ps}, \chi}$. Moreover, if \bar{D} is multiplicity free, then both $R^{\text{ps}, \chi}/\varpi$ and $R^{\text{ps}, \chi}$ are integral domains.*

Proof. If we work with the algebra $E^\chi := E \otimes_{\mathcal{O}[\mu], \chi} \mathcal{O}$ instead of E , then the argument in the proof of Corollary 4.28 gives adequate homeomorphisms

$$X^{\text{gen}, \chi} // \text{GL}_d \rightarrow X^{\text{ps}, \chi}, \quad \bar{X}^{\text{gen}, \chi} // \text{GL}_d \rightarrow \bar{X}^{\text{ps}, \chi}.$$

In particular, the kernel of $R^{\text{ps},\chi}/\varpi \rightarrow A^{\text{gen},\chi}/\varpi$ is nilpotent. Since $A^{\text{gen},\chi}/\varpi$ is an integral domain by Proposition 4.26, we obtain the first assertion. The argument with $R^{\text{ps},\chi}$ is the same as in Corollary 4.28 using that $A^{\text{gen},\chi}$ is an integral domain.

If \bar{D} is multiplicity free, then E^χ and E^χ/ϖ are generalized matrix algebras, and the argument in Corollary 4.28 carries over. □

Lemma 4.30. *If $R^{\text{ps},\chi}/\varpi$ satisfies Serre’s condition (S1), then $R^{\text{ps},\chi}/\varpi$ and $R^{\text{ps},\chi}$ are integral domains.*

Proof. We first note that $R^{\text{ps},\chi}/\varpi$ satisfies Serre’s condition (R0). Since the underlying reduced subschemes of \bar{X}^{ps} and $\bar{X}^{\text{ps},\chi}$ coincide, Proposition A.9 implies that the Kummer-irreducible locus $(\bar{X}^{\text{ps},\chi})^{\text{Kirr}}$ in $\bar{X}^{\text{ps},\chi}$ is open dense. If $x \in (\bar{X}^{\text{ps},\chi})^{\text{Kirr}}$ is a closed point, then the pseudo-character D_x is absolutely irreducible and hence is associated to an absolutely irreducible representation which we denote by ρ_x . Let R_{ρ_x} be the universal deformation ring of ρ_x and let $R_{\rho_x}^\chi$ be the quotient of R_{ρ_x} parameterizing deformations such that the restriction of the determinant to $\text{Art}_F(\mu) \subset G_F^{\text{ab}}$ is equal to χ . Since $R_{\rho_x}^{\square,\chi}$ is formally smooth over $R_{\rho_x}^\chi$, the Kummer-irreducibility of x implies that $R_{\rho_x}^\chi$ is regular. The proof of [9, Lemma 5.1.6] shows that x is a regular point in $\bar{X}^{\text{ps},\chi}$. Hence, $\bar{X}^{\text{ps},\chi}$ contains an open dense regular subscheme, which implies that $R^{\text{ps},\chi}/\varpi$ satisfies (R0). Since $R^{\text{ps},\chi}/\varpi$ satisfies (S1), by assumption we conclude that $R^{\text{ps},\chi}/\varpi$ is reduced. It follows from Lemma 4.29 and Proposition 4.26 that $R^{\text{ps},\chi}/\varpi$ is an integral domain.

Let $R^{\text{ps},\chi} \twoheadrightarrow R_{\text{tf}}^{\text{ps},\chi}$ be the maximal \mathcal{O} -torsion free quotient and let \mathfrak{a} be the kernel of this map. We have an exact sequence $0 \rightarrow \mathfrak{a}/\varpi \rightarrow R^{\text{ps},\chi}/\varpi \rightarrow R_{\text{tf}}^{\text{ps},\chi}/\varpi \rightarrow 0$. It follows from Corollary 4.29 that \mathfrak{a} is nilpotent. Since $R^{\text{ps},\chi}/\varpi$ is reduced, we deduce from the exact sequence that \mathfrak{a}/ϖ is zero. Nakayama’s lemma implies that $\mathfrak{a} = 0$. Thus, $R^{\text{ps},\chi}$ is \mathcal{O} -torsion free and hence is a subring of $A^{\text{gen},\chi}$ by Corollary 4.29. Since $A^{\text{gen},\chi}$ is domain, we conclude that $R^{\text{ps},\chi}$ is an integral domain. □

Remark 4.31. We expect that the rings $R^{\text{ps},\chi}$ and $R^{\text{ps},\chi}/\varpi$ are integral domains. Although we know the dimension of $R^{\text{ps},\chi}/\varpi$ by [9, Theorem 5.5.1], we cannot conclude that the ring is complete intersection (which would imply that (S1) holds) as we lack a presentation analogous to (21). Since $A^{\text{gen},\chi}$ and $A^{\text{gen},\chi}/\varpi$ are integral domains, this question is closely related to the embedding problem discussed in [5, Section 1.3.4].

5. Deformation rings with fixed determinant

Let $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$ be a representation with pseudo-character \bar{D} and let $\psi : G_F \rightarrow \mathcal{O}^\times$ be a character lifting $\det \bar{\rho} = \det \bar{D}$. Let

$$R_{\bar{\rho}}^{\square,\psi} := R_{\bar{\rho}}^{\square} \otimes_{R_{\det \bar{\rho}, \psi}} \mathcal{O}.$$

Let $\mu := \mu_{p^\infty}(F)$ and let $\chi : \mu \rightarrow \mathcal{O}^\times$ be a character such that the restriction of ψ to μ under the Artin map $\mu \rightarrow G_F^{\text{ab}}$ from local class field theory is equal to χ . Then $R_{\bar{\rho}}^{\square,\psi}$ is a quotient of the ring $R_{\bar{\rho}}^{\square,\chi}$ considered in the previous section. We let $X^{\square,\chi} = \text{Spec } R_{\bar{\rho}}^{\square,\chi}$, $X^{\square,\psi} = \text{Spec } R_{\bar{\rho}}^{\square,\psi}$ and denote by $\bar{X}^{\square,\chi}$ and $\bar{X}^{\square,\psi}$ their special fibres.

Let $\mathcal{X} : \mathfrak{A}_{\mathcal{O}} \rightarrow \text{Sets}$ be the functor, which sends (A, \mathfrak{m}_A) to the group $\mathcal{X}(A)$ of continuous characters $\theta : G_F \rightarrow 1 + \mathfrak{m}_A$ whose restriction to μ under the Artin map is trivial. It follows from Lemma 4.1 that the functor \mathcal{X} is pro-represented by

$$\mathcal{O}(\mathcal{X}) \cong R_1 \otimes_{\mathcal{O}[\mu]} \mathcal{O} \cong \mathcal{O}[[y_1, \dots, y_{[F:\mathbb{Q}_p]+1}]] \tag{30}$$

For $e \in \mathbb{N}$, let $\varphi_e : \mathcal{X} \rightarrow \mathcal{X}$ be the natural transformation that sends $\theta \in \mathcal{X}(A)$ to θ^e . We also write φ_e for the induced maps $\mathcal{O}(\mathcal{X}) \rightarrow \mathcal{O}(\mathcal{X})$ and $\text{Spec } \mathcal{O}(\mathcal{X}) \rightarrow \text{Spec } \mathcal{O}(\mathcal{X})$. The natural transformation

$D_{\bar{\rho}}^{\square, \chi} \rightarrow \mathcal{X}, \rho \mapsto (\det \rho)\psi^{-1}$ induces a homomorphism of local \mathcal{O} -algebras $\mathcal{O}(\mathcal{X}) \rightarrow R_{\bar{\rho}}^{\square, \chi}$; we will consider $R_{\bar{\rho}}^{\square, \chi}$ as $\mathcal{O}(\mathcal{X})$ -algebra via this map in the statements below.

Proposition 5.1. *One has a natural isomorphism of functors*

$$D_{\bar{\rho}}^{\square, \chi} \times_{\mathcal{X}, \varphi_d} \mathcal{X} \cong D_{\bar{\rho}}^{\square, \psi} \times \mathcal{X}.$$

Proof. Let (A, \mathfrak{m}_A) be in $\mathfrak{A}_{\mathcal{O}}$. An element in $(D_{\bar{\rho}}^{\square, \chi} \times_{\mathcal{X}, \varphi_d} \mathcal{X})(A)$ is a pair (ρ, θ) such that $\theta : G_F \rightarrow 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on μ , $\rho : G_F \rightarrow \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho$ and χ agree when restricted to μ , and one has $(\det \rho)\psi^{-1} = \theta^d$. An element in $(D_{\bar{\rho}}^{\square, \psi} \times \mathcal{X})(A)$ is a pair (ρ_1, θ_1) , where $\theta_1 : G_F \rightarrow 1 + \mathfrak{m}_A$ is a continuous homomorphism that is trivial on μ and $\rho_1 : G_F \rightarrow \mathrm{GL}_d(A)$ is a continuous homomorphism such that $\det \rho_1 = \psi$. One verifies that the map

$$(\rho, \theta) \mapsto (\rho \otimes \theta^{-1}, \theta)$$

defines a bijection that is natural in A . □

Corollary 5.2. *Proposition 5.1 induces a natural isomorphism*

$$R_{\bar{\rho}}^{\square, \chi} \otimes_{\mathcal{O}(\mathcal{X}), \varphi_d} \mathcal{O}(\mathcal{X}) \cong R_{\bar{\rho}}^{\square, \psi} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X}).$$

We now clarify some properties of the map $\varphi_d : \mathcal{O}(\mathcal{X}) \rightarrow \mathcal{O}(\mathcal{X})$.

Lemma 5.3. *The map φ_d is finite and flat and becomes étale after inverting p . Moreover, it induces a universal homeomorphism on the special fibres.*

Proof. We may write $d = ep^m$ such that p does not divide e . Then $\varphi_d = \varphi_{p^m} \circ \varphi_e$. Since e is prime to p , elements in $1 + \mathfrak{m}_A$ for (A, \mathfrak{m}_A) in $\mathfrak{A}_{\mathcal{O}}$ possess a unique e -th root in $1 + \mathfrak{m}_A$ by the binomial theorem, and it follows that φ_e is an isomorphism. We thus may assume that d is a power of p .

The map $\varphi_d : \mathcal{O}(\mathcal{X}) \rightarrow \mathcal{O}(\mathcal{X})$ sends y_i to $(1+y_i)^d - 1$. One checks that the monomials $\prod_{i=1}^{[F:\mathbb{Q}_p]+1} y_i^{m_i}$ with $0 \leq m_i \leq d - 1$ form a basis of $\mathcal{O}(\mathcal{X})$ as $\mathcal{O}(\mathcal{X})$ -module via φ_d by checking the assertion modulo ϖ and using Nakayama’s lemma. A (standard) calculation shows that the discriminant is a power of p up to a sign. Thus, φ_d becomes étale after inverting p .

The map $\bar{\varphi}_d : \mathcal{O}(\mathcal{X})/\varpi \rightarrow \mathcal{O}(\mathcal{X})/\varpi$ is a power of the relative Frobenius of $\mathrm{Spec}(\mathcal{O}(\mathcal{X})/\varpi)/\mathrm{Spec} k$. The last assertion follows from [48, Tag 0CCB]. □

In the following results, we deduce properties of the ring $R_{\bar{\rho}}^{\square, \psi}$.

Corollary 5.4. *The following hold:*

- (1) $R_{\bar{\rho}}^{\square, \psi}$ is a local complete intersection, flat over \mathcal{O} and of relative dimension $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$.
- (2) $R_{\bar{\rho}}^{\square, \psi}/\varpi$ is a local complete intersection of dimension $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$.

Proof. The pushout of the isomorphism from Proposition 4.3 under $R_{\det \bar{\rho}} \rightarrow \mathcal{O}$, which corresponds to ψ , gives an isomorphism

$$\mathcal{O}[[x_1, \dots, x_r]]/(f_1, \dots, f_t) \xrightarrow{\cong} R_{\bar{\rho}}^{\square, \psi}$$

with $r - t = (d^2 - 1)([F : \mathbb{Q}_p] + 1)$. To prove (1) and (2), it thus suffices to show that the dimension of $R_{\bar{\rho}}^{\square, \psi}/\varpi$ is at most $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$, or equivalently (see (30)), it suffices to show that

$$\dim((R_{\bar{\rho}}^{\square, \psi} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}(\mathcal{X}))/\varpi) \leq d^2([F : \mathbb{Q}_p] + 1). \tag{31}$$

Let us write $\overline{\mathcal{X}} := \text{Spec } \mathcal{O}(\mathcal{X})/\varpi$. Since $\overline{\varphi}_d : \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}$ is a universal homeomorphism, by Lemma 5.3, the map

$$\overline{X}^{\square, \chi} \times_{\overline{\mathcal{X}}, \overline{\varphi}_d} \overline{\mathcal{X}} \rightarrow \overline{X}^{\square, \chi} \tag{32}$$

is a homeomorphism. In particular, the spaces have the same dimension, which is equal to $d^2([F : \mathbb{Q}_p] + 1)$ by Corollary 4.5. We conclude using Corollary 5.2 that (31) is an equality. \square

Lemma 5.5. *Let $\rho : G_F \rightarrow \text{GL}_d(\kappa)$ be a representation, such that $\det \rho = \psi$, where κ is either a finite or local field of characteristic p or a finite extension of L . If $H^2(G_F, \text{ad}^0 \rho) = 0$, then the ring $R_{\rho}^{\square, \psi}$ is formally smooth over Λ with Λ as in Subsection 3.5.*

Proof. This is the same proof as the proof of Lemma 4.8. \square

Theorem 5.6. *The rings $R_{\rho}^{\square, \psi}$ and $R_{\rho}^{\square, \psi} / \varpi$ are normal integral domains.*

Proof. We will first prove that $R_{\rho}^{\square, \psi} / \varpi$ is normal. Since $R_{\rho}^{\square, \psi} / \varpi$ is complete intersection by Corollary 5.4, it suffices to show that $R_{\rho}^{\square, \psi} / \varpi$ satisfies Serre’s condition (R1). Let $\mathfrak{p} \in \overline{X}^{\square, \psi} := \text{Spec } R_{\rho}^{\square, \psi} / \varpi$ be a point of height at most 1 and assume that the local ring at \mathfrak{p} is not regular. Then by Lemma 5.5 there is a closed irreducible subset Z of $\overline{X}^{\square, \psi}$ of codimension at most 1, the closure of \mathfrak{p} , such that for all $z \in Z$ with finite or local residue field the space $H^2(G_F, \text{ad}^0 \rho_z)$ is non-zero. Using the explicit bijection from the proof of Proposition 5.1, and the isomorphism of Corollary 5.2 modulo ϖ it follows that there is a closed irreducible subset $W \subset \overline{X}^{\square, \chi} \times_{\overline{\mathcal{X}}, \overline{\varphi}_d} \overline{\mathcal{X}}$ of codimension at most 1, such that for all $w \in W$ with finite or local residue field, the space $H^2(G_F, \text{ad}^0 \rho_w)$ is non-zero, where, as in the proof of Proposition 5.1, the point w corresponds to a pair (ρ_w, θ_w) . Since the map (32) is a homeomorphism and sends (ρ_w, θ_w) to ρ_w , the image of W in $\overline{X}^{\square, \chi}$, which we denote by W' , is closed irreducible of codimension at most 1 in $\overline{X}^{\square, \chi}$, and all $x \in W'$ with finite or local residue field have non-vanishing $H^2(G_F, \text{ad}^0 \rho_x)$. Lemma 4.16 implies that the codimension of W' is at least 2, yielding a contradiction.

Let us prove that $R_{\rho}^{\square, \psi}$ is normal. Since $R_{\rho}^{\square, \psi}$ is \mathcal{O} -torsion free by Corollary 5.4 and we know that the special fibre is normal, it is enough to prove that $R_{\rho}^{\square, \psi} [1/p]$ is normal; see the proof of Proposition 4.18. Lemma 5.3 implies that the map

$$X^{\square, \chi} [1/p] \times_{\mathcal{X}[1/p], \varphi_d} \mathcal{X} [1/p] \rightarrow X^{\square, \chi} [1/p] \tag{33}$$

is finite étale. We proceed exactly as in the proof for the special fibre, using (33) instead of (32) and Lemma 4.20 instead of Lemma 4.16. \square

Corollary 5.7. *The absolutely irreducible locus is dense in $\text{Spec } R_{\rho}^{\square, \psi} [1/p]$ and the Kummer-irreducible locus is dense in $\text{Spec } R_{\rho}^{\square, \psi} / \varpi$.*

Proof. By Proposition 3.55 and Corollary 3.59, the absolutely irreducible locus is dense open in $\text{Spec } R_{\rho}^{\square, \chi} / \varpi$ and in $\text{Spec } R_{\rho}^{\square, \chi} [1/p]$. Arguing as in the proof of Theorem 5.6, one deduces that the absolutely irreducible locus is dense open in the spaces $\text{Spec } R_{\rho}^{\square, \psi} / \varpi$ and $\text{Spec } R_{\rho}^{\square, \psi} [1/p]$. For absolutely irreducible $x \in \text{Spec } R_{\rho}^{\square, \chi} / \varpi$, Kummer-irreducibility implies $H^2(G_F, \text{ad}^0 \rho_x) = 0$, so the assertion on the density of the Kummer-irreducible locus in $\text{Spec } R_{\rho}^{\square, \chi} / \varpi$ follows from the proof of Theorem 5.6. \square

As explained in Section 4, both R^{ps} and A^{gen} are naturally $R_{\det \overline{D}}$ -algebras. Moreover, $\det \overline{D} = \det \overline{\rho}$. We let

$$R^{\text{ps}, \psi} := R^{\text{ps}} \otimes_{R_{\det \overline{D}, \psi}} \mathcal{O}, \quad A^{\text{gen}, \psi} := A^{\text{gen}} \otimes_{R_{\det \overline{D}, \psi}} \mathcal{O}.$$

Corollary 5.8. *The following hold:*

- (1) $A^{\text{gen},\psi}$ is \mathcal{O} -flat, equi-dimensional of dimension $1 + (d^2 - 1)([F : \mathbb{Q}_p] + 1)$, normal and is locally complete intersection;
- (2) $A^{\text{gen},\psi}/\varpi$ is equi-dimensional of dimension $(d^2 - 1)([F : \mathbb{Q}_p] + 1)$, normal and is locally complete intersection.

Proof. The ring A^{gen} is excellent, since it is finitely generated over a complete local Noetherian ring. Thus, its local rings are also excellent. An excellent local ring is normal if and only if its completion with respect to the maximal ideal is normal, [36, Theorem 32.2 (i)]. Given this, the proof is the same as the proof of Corollary 4.6 using Theorem 5.6. □

Corollary 5.9. *The rings $A^{\text{gen},\psi}$ and $A^{\text{gen},\psi}/\varpi$ are integral domains.*

Proof. The proof is the same as the proof of Proposition 4.26. □

Corollary 5.10. *The ring $R^{\text{ps},\psi}[1/p]$ and the rigid space $(\text{Spf } R^{\text{ps},\psi})^{\text{rig}}$ are normal. The ring $R^{\text{ps},\psi}[1/p]$ is an integral domain.*

Proof. This follows from [43, Corollary A.10]. The last part is proved in the same way as Corollary 4.27 using Corollary 5.9. □

Corollary 5.11. *The maximal reduced quotient of $R^{\text{ps},\psi}$ is equal to the maximal \mathcal{O} -torsion free quotient of $R^{\text{ps},\psi}$ and is an integral domain. Moreover, if \overline{D} is multiplicity free, then $R^{\text{ps},\psi}$ is an \mathcal{O} -torsion free integral domain.*

Proof. This is proved in the same way as Corollary 4.28. □

Proposition 5.12. *The map*

$$R_{\det \bar{\rho}} \rightarrow R_{\bar{\rho}}^{\square} \tag{34}$$

is flat.

Proof. Let $S := \mathcal{O}[[z, y_1, \dots, y_{1+[F:\mathbb{Q}_p]}]]$. By arguing as in the proof of Proposition 4.3, we may choose presentations

$$R_{\det \bar{\rho}} \cong S/((1+z)^m - 1), \quad R_{\bar{\rho}}^{\square} \cong S[[x_1, \dots, x_r]]/((1+z)^m - 1, f_1, \dots, f_t),$$

such that (34) is a map of S -algebras and $(1+z)^m - 1, f_1, \dots, f_t$ is a regular sequence in $S[[x_1, \dots, x_r]]$. Let $S' := S[[x_1, \dots, x_r]]/(f_1, \dots, f_t)$. Then S' is complete intersection, and hence Cohen–Macaulay, and the fibre ring $k \otimes_S S'$ is isomorphic to $R_{\bar{\rho}}^{\square,\psi}/\varpi$, which has dimension equal to $\dim R_{\bar{\rho}}^{\square} - \dim R_{\det \bar{\rho}} = \dim S' - \dim S$, by Corollary 5.4. Since S is regular, the fibre-wise criterion for flatness, [36, Theorem 23.1], implies that S' is flat over S . Hence, $R_{\bar{\rho}}^{\square} \cong S'/((1+z)^m - 1)$ is flat over $R_{\det \bar{\rho}} \cong S/((1+z)^m - 1)$. □

6. Density of points with prescribed p -adic Hodge theoretic properties

We fix a continuous representation $\bar{\rho} : G_F \rightarrow \text{GL}_d(k)$. Let $R_{\bar{\rho}}^{\square}$ be its universal framed deformation ring and let $X^{\square} = \text{Spec } R_{\bar{\rho}}^{\square}$. If $x : R_{\bar{\rho}}^{\square} \rightarrow \overline{\mathbb{Q}}_p$ is an \mathcal{O} -algebra homomorphism, then we denote by $\rho_x^{\square} : G_F \rightarrow \text{GL}_d(\overline{\mathbb{Q}}_p)$ the specialization of the universal framed deformation $\rho^{\square} : G_F \rightarrow \text{GL}_d(R_{\bar{\rho}}^{\square})$ at x . In this section, we will study Zariski closures of subsets $\Sigma \subset X^{\square}(\overline{\mathbb{Q}}_p)$, such that ρ_x^{\square} is potentially semi-stable for all $x \in \Sigma$ and satisfies additional conditions imposed on either the Hodge–Tate weights or the inertial type of ρ_x^{\square} . Recall that the Hodge–Tate weights $\text{HT}(\rho)$ of a potentially semi-stable

representation ρ is a collection \underline{k} of d -tuples of integers $\underline{k}_\sigma = (k_{\sigma,1} \geq k_{\sigma,2} \geq \dots \geq k_{\sigma,d})$ for each embedding $\sigma : F \hookrightarrow \overline{\mathbb{Q}}_p$, and we say that \underline{k} is *regular* if all the inequalities are strict. Let

$$\Sigma^{\text{cris}} := \{x \in X^\square(\overline{\mathbb{Q}}_p) : \rho_x^\square \text{ is crystalline with regular Hodge–Tate weights}\}.$$

For a fixed regular Hodge–Tate weight \underline{k} , we let

$$\Sigma_{\underline{k}} := \{x \in X^\square(\overline{\mathbb{Q}}_p) : \rho_x^\square \text{ is potentially semi-stable with } \text{HT}(\rho_x^\square) = \underline{k}\}.$$

If ρ is potentially semi-stable, then to it we may attach a Weil–Deligne representation $\text{WD}(\rho)$; we denote by $\text{WD}(\rho)^{F\text{-ss}}$ its Frobenius semi-simplification. We may attach a smooth irreducible representation of $\text{GL}_d(F)$, which we denote by $\text{LL}(\text{WD}(\rho))$, to $\text{WD}(\rho)^{F\text{-ss}}$ via the classical Langlands correspondence; see [14, Section 1.8] for more details and further references.

Let $\Sigma_{\underline{k}}^{\text{prnc}}$ be the subset of $\Sigma_{\underline{k}}$, such that $x \in \Sigma_{\underline{k}}$ lies in $\Sigma_{\underline{k}}^{\text{prnc}}$ if and only if $\text{LL}(\text{WD}(\rho_x^\square))$ is a principal series representation. In terms of the Galois side $\Sigma_{\underline{k}}^{\text{prnc}}$ may be characterised as the set of $x \in \Sigma_{\underline{k}}$ such that the restriction of ρ_x^\square to the Galois group of some finite abelian extension of F is crystalline.

Let $\Sigma_{\underline{k}}^{\text{spcd}}$ be the subset of $\Sigma_{\underline{k}}$ such that x lies in $\Sigma_{\underline{k}}^{\text{spcd}}$ if and only if $\text{WD}(\rho_x)$ is irreducible as a representation of the Weil group W_F of F and is induced from a 1-dimensional representation of W_E , where E is an unramified extension of F of degree d . In this case, $\text{LL}(\text{WD}(\rho_x))$ is a supercuspidal representation of $\text{GL}_d(F)$.

The goal of this section is the following theorem.

Theorem 6.1. *Assume that $p \nmid 2d$. Let Σ be any of the sets Σ^{cris} , $\Sigma_{\underline{k}}^{\text{prnc}}$, $\Sigma_{\underline{k}}^{\text{spcd}}$ defined above. Then Σ is Zariski dense in X^\square .*

Remark 6.2. We could additionally require the representations in Σ^{cris} to be *benign* in the sense of [25, Definition 6.8], or instead of considering crystalline representations, fix an inertial type.

One could also change the definition of $\Sigma_{\underline{k}}^{\text{spcd}}$ to allow E to be a ramified extension of F ; see [25, Section 5.3].

The problem for Σ^{cris} has been studied by Colmez [19], Kisin [34], Chenevier [17], Nakamura [39], [40]. Hellmann and Schraen have studied the problem for $\Sigma_{\underline{k}}^{\text{prnc}}$ and Σ^{cris} in [28]. Emerton and VP have studied the problem for Σ^{cris} , $\Sigma_{\underline{k}}^{\text{prnc}}$ and $\Sigma_{\underline{k}}^{\text{spcd}}$ in [25]. A common feature of these papers is that they show that the closure of Σ is a union of irreducible components of X^\square and density is equivalent to showing that Σ meets each irreducible component. If one knows the irreducible components, then one might hope to show density this way. This strategy has been carried out for Σ^{cris} by Colmez–Dospinescu–VP in [20] for $p = d = 2$ and $F = \mathbb{Q}_p$ and by AI in [30] for $p > d$ and F arbitrary, when $\overline{\rho}$ is the trivial representation, where after determining irreducible components, one can write down the lifts explicitly. We note that using Corollary 4.21, one may remove the assumption $p > d$ in [30, Theorem 5.11]. It seems impossible to carry this out for arbitrary \underline{k} and $\overline{\rho}$ directly, even if one knows that the irreducible components of X^\square are in bijection with irreducible components of $\text{Spec } R_{\det \overline{\rho}}$. Instead, we combine our knowledge of irreducible components with results of [25].

The paper [25] builds on the global patching arguments carried out in [14], which assumes that $p \nmid 2d$ and $\overline{\rho}$ has a potentially diagonalisable lift. This last condition can be easily verified if $\overline{\rho}$ is semi-simple (see [14, Lemma 2.2]); it has been shown to always be satisfied in [24, Theorem 1.2.2]. The output of [14] is a complete local Noetherian \mathcal{O} -algebra R_∞ with residue field k and a linearly compact R_∞ -module M_∞ , which carries a continuous R_∞ -linear action of $G := \text{GL}_d(F)$. Moreover, the action of $R_\infty[K]$ on M_∞ extends (uniquely) to a continuous action of the completed group algebra $R_\infty[[K]]$, where $K := \text{GL}_d(\mathcal{O}_F)$, so that M_∞ is a finitely generated $R_\infty[[K]]$ -module.

Lemma 6.3. *We have an isomorphism of R_ρ^\square -algebras:*

$$R_\infty \cong R_\rho^\square \widehat{\otimes}_{\mathcal{O}} A,$$

where A is a complete local Noetherian \mathcal{O} -algebra, which is \mathcal{O} -torsion free, reduced and equi-dimensional. Thus, the ring R_∞ is a reduced, \mathcal{O} -torsion free and flat R_ρ^\square -algebra.

After replacing L by a finite extension, the irreducible components of $\text{Spec } R_\infty$ are of the form $\text{Spec}(R_\rho^{\square, \chi} \widehat{\otimes}_{\mathcal{O}} A/\mathfrak{p})$, for a character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ and a minimal prime \mathfrak{p} of A . Moreover, distinct pairs (χ, \mathfrak{p}) give rise to distinct irreducible components of $\text{Spec } R_\infty$.

Proof. The ring R_∞ is defined in [14, Section 2.8] and is formally smooth over the ring denoted by R^{loc} in [14, Section 2.6]. The ring R^{loc} is a completed tensor product over \mathcal{O} of R_ρ^\square , the ring $R_{\mathfrak{v}_1}^\square$, which is formally smooth over \mathcal{O} by [14, Lemma 2.5], and potentially semi-stable rings at other places above p , denoted by $R_{\mathfrak{v}}^{\square, \xi, \tau}$ in [14, Section 2.4]. These are \mathcal{O} -torsion free, reduced and equi-dimensional by [33, Theorem 3.3.8]. Thus, $R_\infty \cong R_\rho^\square \widehat{\otimes}_{\mathcal{O}} A$, where A is formally smooth over the ring $\widehat{\otimes}_{\mathfrak{v} \in S_p \setminus \mathfrak{p}} R_{\mathfrak{v}}^{\square, \xi, \tau}$ in the notation of [14]. Since the rings $R_{\mathfrak{v}}^{\square, \xi, \tau}$ are \mathcal{O} -torsion free, reduced and equi-dimensional, so is the ring A by [14, Corollary A.2] and [29, Lemma A.1]. Since R_ρ^\square is also \mathcal{O} -torsion free, reduced and equi-dimensional, we obtain that the same holds for R_∞ . Since A is \mathcal{O} -torsion free, R_∞ is a flat R_ρ^\square -algebra.

It follows from [29, Lemma A.5] that after replacing L with a finite extension, we may assume that for all minimal primes \mathfrak{p} of A , the quotient A/\mathfrak{p} is geometrically integral, by which we mean that $(A/\mathfrak{p}) \otimes_{\mathcal{O}} \mathcal{O}_L$ is integral domain for all finite extensions L'/L . If \mathfrak{p}' is a minimal prime of R_ρ^\square , then $R_\rho^\square/\mathfrak{p}' = R_\rho^{\square, \chi}$ for a unique character $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$ by Corollary 4.21. The moduli interpretation of $R_\rho^{\square, \chi}$ together with Corollary 4.19 shows that the ring is geometrically integral. It follows from [4, Lemma 3.3 (5)] that the minimal primes \mathfrak{q} of R_∞ are of the form $\mathfrak{p}'(R_\rho^\square \widehat{\otimes}_{\mathcal{O}} A) + \mathfrak{p}(R_\rho^\square \widehat{\otimes}_{\mathcal{O}} A)$, where \mathfrak{p}' is the image of \mathfrak{q} in $\text{Spec } R_\rho^\square$ and \mathfrak{p} is the image of \mathfrak{q} in $\text{Spec } A$. This implies the last assertion. \square

In our arguments, we will not invoke the assumption $p \nmid 2d$, since eventually this restriction used in construction of M_∞ should become redundant. In particular, the next two Lemmas do not use this assumption.

Lemma 6.4. *Let $\psi : G_F \rightarrow \mathcal{O}^\times$ be a character such that ψ is trivial on the torsion subgroup of G_F^{ab} . Then after replacing L by a finite extension, we may find a character $\eta : G_F \rightarrow \mathcal{O}^\times$ such that $\eta^d = \psi$.*

Proof. It follows from local class field theory that the maximal torsion-free quotient of G_F^{ab} is isomorphic to $\widehat{\mathbb{Z}} \times \mathbb{Z}_p^m$, where $m = [F : \mathbb{Q}_p]$. We choose topological generators $\gamma_1, \dots, \gamma_{m+1}$, where γ_1 is a generator of $\widehat{\mathbb{Z}}$. Let $\overline{\psi}(\gamma_1)$ be the image of $\psi(\gamma_1)$ in k . If it is not equal to 1, then choose $\lambda \in \overline{k}$ such that $\lambda^d = \overline{\psi}(\gamma_1)$. We enlarge L , so that the residue field contains λ and let $\mu : \widehat{\mathbb{Z}} \times \mathbb{Z}_p^m \rightarrow \widehat{\mathbb{Z}} \rightarrow \mathcal{O}^\times$ be the unramified character, such that $\mu(\gamma_1)$ is equal to the Teichmüller lift of λ . After replacing ψ with $\psi\mu^{-d}$, we may assume that $\psi(\gamma_1) \equiv 1 \pmod{\varpi}$. Thus, we may view ψ as a character on \mathbb{Z}_p^{m+1} and $\psi(\gamma_i) \equiv 1 \pmod{\varpi}$ for $1 \leq i \leq m+1$. After enlarging L , we may find $y_i \in (\varpi)$ such that $(1 + y_i)^d = \psi(\gamma_i)$. Since the series $(1 + y_i)^x := \sum_{n=0}^\infty \binom{x}{n} y_i^n$ converges for all $x \in \mathbb{Z}_p$, we may define η on \mathbb{Z}_p^{m+1} by sending γ_i to $1 + y_i$ and then inflate it to G_F . \square

Lemma 6.5. *Let $\kappa : G_F \rightarrow \mathcal{O}^\times$ be a character. Then there is a crystalline character $\psi : G_F \rightarrow \mathcal{O}^\times$ such that $\psi\kappa^{-1}$ is trivial on the torsion part of G_F^{ab} .*

In particular, given characters $\bar{\kappa} : G_F \rightarrow k^\times$ and $\chi : \mu_{p^\infty}(F) \rightarrow \mathcal{O}^\times$, there exists a crystalline character $\psi : G_F \rightarrow \mathcal{O}^\times$ lifting $\bar{\kappa}$ such that $\psi(\text{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$.

Proof. The Artin map $\text{Art}_F : F^\times \rightarrow G_F^{\text{ab}}$ of local class field theory allows us to identify characters $\psi : G_F \rightarrow \mathcal{O}^\times$ with characters $\psi : F^\times \rightarrow \mathcal{O}^\times$. Under this identification, ψ is crystalline if and only if $\psi(x) = \prod_{\sigma: F \hookrightarrow L} \sigma(x)^{n_\sigma}$ for some integers n_σ and for all $x \in \mathcal{O}_F^\times$ by [22, Proposition B.4].

Let ζ be a generator of the torsion subgroup of F^\times and let m be the multiplicative order of ζ . Let ξ be a primitive m -th root of unity in L . Then $\kappa(\zeta) = \xi^a$ for some integer a . Let $\sigma : F \hookrightarrow L$ be an embedding such that $\sigma(\zeta) = \xi$. Let $\psi : F^\times \rightarrow \mathcal{O}^\times$ be the character $\psi(x) = \sigma(x\varpi_F^{-v(x)})^a$ for all $x \in F^\times$, where v is a valuation on F normalized so that $v(\varpi_F) = 1$. Then $\psi\kappa^{-1}(\zeta) = 1$, and hence, $\psi\kappa^{-1}$ is trivial on the torsion subgroup of F^\times . Moreover, ψ is crystalline by the above. Note that $\psi\kappa^{-1} \equiv 1 \pmod{\varpi}$.

For the last part, we may choose any $\kappa : G_F \rightarrow \mathcal{O}^\times$ lifting $\bar{\kappa}$ and satisfying $\kappa(\text{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$ and apply the previous part. □

Lemma 6.6. *Let $\psi : G_F \rightarrow \mathcal{O}^\times$ be a character lifting $\det \bar{\rho}$ and let $x : R_{\det \bar{\rho}} \rightarrow \mathcal{O}$ be the corresponding \mathcal{O} -algebra homomorphism. Then the centre Z of G acts on $M_\infty \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O}$ via the character δ^{-1} , where $\delta : Z \rightarrow \mathcal{O}^\times$ is the composition*

$$Z \xrightarrow{\cong} F^\times \xrightarrow{\text{Art}_F} G_F^{\text{ab}} \xrightarrow{\varepsilon^{d(d-1)/2}\psi} \mathcal{O}^\times,$$

where ε is the p -adic cyclotomic character.

Moreover, $M_\infty \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O}$ is non-zero and projective in the category of linearly compact $\mathcal{O}[[K]]$ -modules on which $Z \cap K$ acts by δ^{-1} .

Further, if ψ is crystalline, then there is an algebraic character $\theta : \text{Res}_{\mathbb{Q}_p}^F \mathbb{G}_m \rightarrow \mathbb{G}_m$ defined over L such that $\delta|_{K \cap Z}$ is equal to the composition

$$\mathcal{O}_F^\times \hookrightarrow (\text{Res}_{\mathbb{Q}_p}^F \mathbb{G}_m)(\mathbb{Q}_p) \rightarrow (\text{Res}_{\mathbb{Q}_p}^F \mathbb{G}_m)(L) \xrightarrow{\theta} \mathbb{G}_m(L),$$

where $\text{Res}_{\mathbb{Q}_p}^F$ denotes the restriction of scalars.

Proof. It follows from the discussion at the beginning of [14, Section 4.22] that Z acts via δ on the Pontryagin dual of $M_\infty \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O}$. Hence, it acts on $M_\infty \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O}$ via δ^{-1} . The second part follows from [14, Corollary 4.26]. The last part follows from [22, Proposition B.4] as explained in the proof of Lemma 6.5. □

If V is a continuous representation of K on a finite dimensional L -vector space, then we define a finitely generated $R_\infty[1/p]$ -module $M_\infty(V)$ as follows. Since K is compact, it stabilizes a bounded \mathcal{O} -lattice Θ in V . Let

$$M_\infty(\Theta) := (\text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_\infty, \Theta^d))^d,$$

where $(\cdot)^d := \text{Hom}_{\mathcal{O}}^{\text{cont}}(\cdot, \mathcal{O})$. Then $M_\infty(\Theta)$ is a finitely generated R_∞ -module. The module $M_\infty(V) := M_\infty(\Theta) \otimes_{\mathcal{O}} L$ does not depend on the choice of a lattice Θ .

We will denote by $\text{Irr}(G)$ the set of equivalence classes of irreducible algebraic representation of $(\text{Res}_{\mathbb{Q}_p}^F \text{GL}_d)_L$ defined over L . If $\xi \in \text{Irr}(G)$, then we will consider it as a representation of K by evaluating at L and letting K act via the composition

$$K \hookrightarrow (\text{Res}_{\mathbb{Q}_p}^F \text{GL}_d)(\mathbb{Q}_p) \rightarrow (\text{Res}_{\mathbb{Q}_p}^F \text{GL}_d)(L).$$

If M is a compact \mathcal{O} -module, then we define an L -Banach space

$$\Pi(M) := \text{Hom}_{\mathcal{O}}^{\text{cont}}(M, L),$$

equipped with supremum norm. If M is a compact $\mathcal{O}[[K]]$ -module, then the action of K on M makes $\Pi(M)$ into a unitary L -Banach space representation of K . For example, the map $K \rightarrow \mathcal{O}[[K]]$ induces an isomorphism of unitary L -Banach space representations $\Pi(\mathcal{O}[[K]]) \cong \mathcal{C}(K, L)$, the space of continuous functions from K to L , with K -action given by left translations; [46, Corollary 2.2].

Lemma 6.7. *Let $\theta : \text{Res}_{\mathbb{Q}_p}^F \mathbb{G}_m \rightarrow \mathbb{G}_m$ be an algebraic character defined over L and let $\delta : Z \cap K \rightarrow \mathcal{O}^\times$ be the character associated to θ in Lemma 6.6. Let M be non-zero and projective in the category of linearly compact $\mathcal{O}[[K]]$ -modules on which $Z \cap K$ acts by δ^{-1} . Then there is $\xi \in \text{Irr}(G)$ such that $\text{Hom}_K^{\text{cont}}(M, \xi^*) \neq 0$.*

Proof. We may assume that M is a direct summand of $\mathcal{O}[[K]] \widehat{\otimes}_{\mathcal{O}[[K \cap Z]]} \mathcal{O}$ since an arbitrary projective module is isomorphic to a product of indecomposable projectives, and these are direct summands of $\mathcal{O}[[K]] \widehat{\otimes}_{\mathcal{O}[[K \cap Z]]} \mathcal{O}$. Then the Banach space $\Pi(M)$ is a non-zero direct summand $\mathcal{C}_\delta(K, L)$, the subspace of $\mathcal{C}(K, L)$ on which $Z \cap K$ acts by δ .

Using the theory of highest weight, we may find $\tau \in \text{Irr}(G)$ such that the central character of τ is equal to θ . It follows from [23, Corollary 7.8] that the evaluation map

$$\bigoplus_{\xi' \in \text{Irr}(G/Z)} \tau \otimes \xi' \otimes \text{Hom}_K^{\text{cont}}(\tau \otimes \xi', \Pi(M)) \rightarrow \Pi(M)$$

has dense image. Thus, there is $\xi' \in \text{Irr}(G/Z)$ and an irreducible summand ξ of $\tau \otimes \xi'$ such that $\text{Hom}_K^{\text{cont}}(\xi, \Pi(M)) \neq 0$. Dually, this implies $\text{Hom}_K^{\text{cont}}(M, \xi^*) \neq 0$. □

Theorem 6.8. *The action of R_ρ^\square on M_∞ is faithful.*

Proof. Let \mathfrak{p} be a minimal prime of R_ρ^\square . We have shown in Corollary 4.21 that there is a character $\chi : \mu_{p^\infty}(F) \rightarrow L^\times$ such that $R_\rho^\square/\mathfrak{p} = R_\rho^{\square, \chi}$. It follows from Lemma 6.5 that there is a crystalline character $\psi : G_F \rightarrow \mathcal{O}^\times$ lifting $\det \bar{\rho}$ such that $\psi(\text{Art}_F(z)) = \chi(z)$ for all $z \in \mu_{p^\infty}(F)$. Let $x : R_{\det \bar{\rho}} \rightarrow \mathcal{O}$ be the corresponding \mathcal{O} -algebra homomorphism. It follows from Lemmas 6.6, 6.7 that there is $\xi \in \text{Irr}(G)$ such that

$$\text{Hom}_K^{\text{cont}}(M_\infty \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O}, \xi^*) \neq 0.$$

This implies that $M_\infty(\xi) \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O} \neq 0$.

Let \mathfrak{a} be the R_∞ annihilator of M_∞ . In [25, Theorem 6.12], it is shown, following the approach of Chenevier [17] and Nakamura [40], that the closure in $\text{Spec } R_\infty$ of the union of the supports of $M_\infty(\xi')$ for all $\xi' \in \text{Irr}(G)$ is a union of irreducible components of $\text{Spec } R_\infty$. Thus, there is a minimal prime \mathfrak{q} of R_∞ such that $\text{Supp } M_\infty(\xi) \subset V(\mathfrak{q}) \subset V(\mathfrak{a})$.

Since $M_\infty(\xi) \otimes_{R_{\det \bar{\rho}, x}} \mathcal{O} \neq 0$, Lemma 6.3 implies that the image of \mathfrak{q} in $\text{Spec } R_\rho^\square$ is equal to \mathfrak{p} . Thus, \mathfrak{p} contains $\mathfrak{a} \cap R_\rho^\square$, which is the R_ρ^\square -annihilator of M_∞ . Since R_ρ^\square is reduced by Corollary 4.22, the intersection of all minimal prime ideals is zero, and hence, R_ρ^\square acts faithfully on M_∞ . □

Proof of Theorem 6.1. This is proved in the same way as [25, Theorems 5.1, 5.3]. Let us sketch the proof in the case of Σ^{cris} for the convenience of the reader. For each $\xi \in \text{Irr}(G)$, let \mathfrak{a}_ξ be the R_ρ^\square -annihilator of $M_\infty(\xi)$. It follows from [14, Lemma 4.18] that $R_\rho^\square/\mathfrak{a}_\xi$ is a quotient of the crystalline deformation ring of $\bar{\rho}$ with Hodge–Tate weights corresponding to the highest weight of ξ ; see [14, Section 1.8], [23, Remark 5.14]. Moreover, $R_\rho^\square/\mathfrak{a}_\xi$ is a union of irreducible components of that ring. This implies that $R_\rho^\square/\mathfrak{a}_\xi$ is reduced and \mathcal{O} -torsion free. The set Σ^{cris} contains the set of maximal ideals of $(R_\rho^\square/\mathfrak{a}_\xi)[1/p]$. Since $(R_\rho^\square/\mathfrak{a}_\xi)[1/p]$ is Jacobson, if $a \in R_\rho^\square$ is contained in the intersection of all maximal ideals in Σ^{cris} , then a will annihilate $M_\infty(\xi)$ for all $\xi \in \text{Irr}(G)$. The continuous L -linear dual of $M_\infty(\xi)$ can be identified with $\text{Hom}_K(\xi, \Pi(M_\infty))$. The key point is that the image of the evaluation map

$$\bigoplus_{\xi \in \text{Irr}(G)} \xi \otimes_L \text{Hom}_K(\xi, \Pi(M_\infty)) \rightarrow \Pi(M_\infty) \tag{35}$$

is dense. Thus, a will annihilate the left-hand side of (35), and by density it will annihilate $\Pi(M_\infty)$. The continuous L -linear dual of $\Pi(M_\infty)$ can be identified with $M_\infty[1/p]$. Since R_ρ^\square is \mathcal{O} -torsion free and R_ρ^\square acts faithfully on M_∞ , by Theorem 6.8, we deduce that $a = 0$.

If $\Sigma = \Sigma_{\underline{k}}^{\text{prnc}}$ or $\Sigma_{\underline{k}}^{\text{spcd}}$, then the argument is the same, except that instead of considering all $\xi \in \text{Irr}(G)$, one fixes $\xi \in \text{Irr}(G)$, such that the highest weight of ξ corresponds to the Hodge–Tate weights \underline{k} and one considers the family $\xi \otimes_L V$, where V are principal series or appropriate supercuspidal types; see the proof of Theorems 5.1, 5.3 in [25] for more details. \square

Remark 6.9. It is natural to ask whether the ring R_∞ acts faithfully on M_∞ . We cannot answer this question in general since it boils down to whether every irreducible component of the potentially semi-stable rings $R_v^{\square, \xi, \tau}$ (see the proof of Lemma 6.3, where $v \in S_p \setminus \mathfrak{p}$ is a place above p , different from the place at which the patching construction is carried out) has a point corresponding to an automorphic Galois representation. These questions are connected with modularity lifting theorems and the Fontaine–Mazur conjecture; see [14, Remark 4.20].

However, if all $R_v^{\square, \xi, \tau}$ were integral domains, then the ring A in Lemma 6.3 would also be an integral domain, and we would deduce from the proof of Theorem 6.8 that R_∞ acts faithfully on M_∞ . A further possibility is to avoid the modularity lifting related issues by carrying out the patching construction of [14] at all places above p at once. Then the proof of Theorem 6.8 would carry over in this new setting to show that R_∞ acts faithfully on M_∞ .

Remark 6.10. In [7], which is a sequel to this paper, we have proved the density of Σ^{cris} in the rigid analytic space $(\text{Spf } R_\rho^{\square})^{\text{rig}}$ associated to the formal scheme $\text{Spf } R_\rho^{\square}$ by making a strong use of [24] to show that Σ^{cris} is non-empty. This, in turn, implies Theorem 6.1 for Σ^{cris} without any restrictions on the prime p ; see [7, Corollary 5.2]. However, the sets $\Sigma_{\underline{k}}^{\text{prnc}}$ and $\Sigma_{\underline{k}}^{\text{spcd}}$ for a fixed regular \underline{k} are not Zariski dense in $(\text{Spf } R_\rho^{\square})^{\text{rig}}$, and the argument explained in this section is the only known method to approach the density result in Theorem 6.1 in these cases. We also find Theorem 6.8 to be an interesting result in its own right: if one believes the expectation in [14, Section 6] that M_∞ should realize the conjectural p -adic Langlands correspondence, then Theorem 6.8 has to hold.

A. Kummer-irreducible points

The purpose of the appendix is to slightly generalize the notion of non-special points in $\overline{X}^{\text{ps}} = \text{Spec } R^{\text{ps}}/\varpi$ in [9, Definition 5.1.2]. We use the notation of the main text. In particular, ζ_p is a primitive p -th root of unity in a fixed algebraic closure \overline{F} of F . If $x \in \overline{X}^{\text{ps}}$, then we let $D_x = D^u \otimes_{R^{\text{ps}}} \overline{\kappa(x)}$, where $\overline{\kappa(x)}$ is an algebraic closure of the residue field at x , and we let $\rho_x : G_F \rightarrow \text{GL}_d(\overline{\kappa(x)})$ be the semisimple representation whose pseudo-character is D_x .

Definition A.1. We say that $x \in P_1(R^{\text{ps}}/\varpi)$ is *Kummer-irreducible* if the restriction of D_x to $G_{F'}$ is absolutely irreducible for all degree p Galois extensions F' of $F(\zeta_p)$. Otherwise, we say that x is *Kummer-reducible*.

Thus, x is Kummer-irreducible if and only if $\rho_x|_{G_{F(\zeta_p)}}$ is non-special in the sense of [9, Definition 5.1.2]. In particular, if $\zeta_p \in F$, then both notions coincide. Our main interest in this notion is the following Lemma.

Lemma A.2. *If x is Kummer-irreducible, then $H^2(G_F, \text{ad}^0 \rho_x) = 0$.*

Proof. Since the order of $\text{Gal}(F(\zeta_p)/F)$ is prime to p , we have

$$H^2(G_F, \text{ad}^0 \rho_x) \cong H^2(G_{F(\zeta_p)}, \text{ad}^0 \rho_x)^{\text{Gal}(F(\zeta_p)/F)}.$$

Since x is Kummer-irreducible, the restriction of ρ_x to $G_{F(\zeta_p)}$ is non-special, and it follows from [9, Lemma 5.1.1] that $H^2(G_{F(\zeta_p)}, \text{ad}^0 \rho_x) = 0$. \square

If $E \subset \overline{F}$ is a finite extension of F , then we denote by R_E^{ps} the universal ring for deformations of the pseudo-character $\overline{D}|_{G_E}$. We let $\overline{X}_E^{\text{ps}} = \text{Spec } R_E^{\text{ps}}$, U_E^{irr} the absolute irreducible locus in $\overline{X}_E^{\text{ps}}$ and $U_E^{\text{n-spl}}$

the non-special locus in U_E^{irr} . These are open subschemes of $\overline{X}_E^{\text{ps}}$. Let U_E^{spcl} be the complement of $U_E^{\text{n-spcl}}$ of U_E^{irr} . We drop the subscript E , when $E = F$.

Lemma A.3. *If $E \subset \overline{F}$ is a finite extension of F , then the morphism $r : X^{\text{ps}} \rightarrow X_E^{\text{ps}}$, induced by restriction of pseudo-characters of G_F to G_E , is finite.*

Proof. The proof is similar to the proof of Proposition 3.24. The map $R_E^{\text{ps}} \rightarrow R^{\text{ps}}$ is a local homomorphism of complete local rings with residue field k . Topological Nakayama’s lemma implies that it is enough to show that the fibre ring $S := R^{\text{ps}}/\mathfrak{m}_{R_E^{\text{ps}}}R^{\text{ps}}$ is a finite dimensional k -vector space, which amounts to showing $\text{Spec } S = \{\mathfrak{m}_S\}$. We note that S represents the functor of deformations $D_A : A[G_F] \rightarrow A$ of \overline{D} to local artinian k -algebras A such that $D_A|_{G_E} = \overline{D}|_{G_E} \otimes_k A$.

Let y be any point of $\text{Spec } S$ with associated pseudo-character D_y and semisimple representation $\rho_y : G_F \rightarrow \text{GL}_d(\overline{\kappa}(y))$. The restriction $\rho_y|_{G_E}$ is semisimple, cf. [9, Lemma 2.1.4], and its associated pseudo-character is $\overline{D}|_{G_E} \otimes_k \overline{\kappa}(y)$, so that $\rho_y(G_E)$ is finite. Hence, $\rho_y(G_F)$ is finite, and therefore, D_y is defined over a finite field $k' \supset k$. This shows that the corresponding ring map $S \rightarrow \overline{\kappa}(y)$ factors via k' , and thus its kernel y is the maximal ideal \mathfrak{m}_S . □

We define the *Kummer-reducible* locus in U^{irr} as

$$U^{\text{Kred}} := U^{\text{irr}} \cap \left(\bigcup_{F'} r^{-1}(\overline{X}_{F'}^{\text{ps}} \setminus U_{F'}^{\text{irr}}) \right),$$

where the union is taken over all degree p Galois extensions F' of $F(\zeta_p)$. Since there are only finitely many such extensions, U^{Kred} is closed in U^{irr} . We define the *cyclotomic-reducible* locus in U^{irr} as

$$U^{\text{Cred}} := U^{\text{irr}} \cap r^{-1}(\overline{X}_{F(\zeta_p)}^{\text{ps}} \setminus U_{F(\zeta_p)}^{\text{irr}}).$$

This is also a closed subset of U^{irr} and is contained in U^{Kred} . If F does not contain a primitive p -th root of unity, then $U^{\text{Cred}} = U^{\text{spcl}}$, and $U^{\text{Cred}} = \emptyset$ otherwise.

Lemma A.4. *We have $U^{\text{spcl}} \subset U^{\text{Kred}}$. Moreover, the inclusion is an equality if F contains a primitive p -th root of unity.*

Proof. If $\zeta_p \in F$, then the definitions of U^{Kred} and U^{spcl} coincide. If $\zeta_p \notin F$, then $y \in U^{\text{spcl}}$ if and only if D_y is irreducible and the restriction of D_y to $G_{F(\zeta_p)}$ is reducible. If we further restrict D_y to $G_{F'}$, where F' is any degree p Galois extension of $F(\zeta_p)$, then the pseudocharacter remains reducible. Thus, $y \in U^{\text{Kred}}$. □

Lemma A.5. *Let T be a locally closed subset of U^{irr} , let \overline{T} be its closure in U^{irr} and let Z be its closure in \overline{X}^{ps} . Then $\dim T = \dim \overline{T}$ and $\dim Z = \dim T + 1$.*

Proof. Since U^{irr} is open in \overline{X}^{ps} , T is locally closed in \overline{X}^{ps} . Thus, T is open in Z . Lemma 3.18 (5) applied with $\text{Spec } R = \text{Spec } S = Z$ and $U = T$ implies that $\dim Z = \dim T + 1$. Since \overline{T} is contained in U^{irr} , it does not contain the closed point of \overline{X}^{ps} . Thus $\overline{T} \subset Z \setminus \{\mathfrak{m}_{R^{\text{ps}}}\}$. Since Z is the spectrum of a local ring, $\dim(Z \setminus \{\mathfrak{m}_{R^{\text{ps}}}\}) = \dim Z - 1$. We conclude that $\dim \overline{T} \leq \dim Z - 1 = \dim T$. Since \overline{T} contains T , $\dim T \leq \dim \overline{T}$. □

Remark A.6. The equality $\dim T = \dim \overline{T}$ in Lemma A.5 may also be deduced from [48, Tag 0DRT], which applies in a more general context.

Lemma A.7. *We have*

$$\dim U^{\text{irr}} - \dim U^{\text{Cred}} \geq \frac{1}{2}d^2[F : \mathbb{Q}_p] \geq 2.$$

Proof. It follows from [9, Theorem 5.5.1] and Lemma A.5 that

$$\dim U^{\text{irr}} = d^2[F : \mathbb{Q}_p]. \tag{36}$$

If $\zeta_p \in F$, then U^{Cred} is empty and the required bound follows. If $\zeta_p \notin F$, then $U^{\text{Cred}} = U^{\text{spl}}$, and it follows from [9, Theorem 5.4.1 (a)] and Lemma A.5 that $\dim U^{\text{spl}} \leq \frac{1}{2}d^2[F : \mathbb{Q}_p]$. \square

Lemma A.8. *We have*

$$\dim U^{\text{irr}} - \dim U^{\text{Kred}} \geq d[F : \mathbb{Q}_p] \geq 2.$$

Proof. It follows from [9, Lemma 5.1.1] that the preimage of $U_{F(\zeta_p)}^{\text{spl}}$ in \overline{X}^{ps} under the morphism $r : \overline{X}^{\text{ps}} \rightarrow \overline{X}_{F(\zeta_p)}^{\text{ps}}$ from Lemma A.3 with $E = F(\zeta_p)$ is equal to $U^{\text{Kred}} \setminus U^{\text{Cred}}$. Thus, the induced morphism $r : U^{\text{Kred}} \setminus U^{\text{Cred}} \rightarrow U_{F(\zeta_p)}^{\text{spl}}$ is also finite. We deduce

$$\dim(U^{\text{Kred}} \setminus U^{\text{Cred}}) \leq \dim U_{F(\zeta_p)}^{\text{spl}}$$

from [48, Tag 01WG].

Since $F(\zeta_p)$ contains a primitive p -th root of unity, [9, Lemma 5.1.1] implies that if p does not divide d , then $U_{F(\zeta_p)}^{\text{spl}}$ is empty; thus, $U^{\text{Kred}} = U^{\text{Cred}}$, and the required bound follows from Lemma A.7.

Let us assume that p divides d . Part (a) of [9, Theorem 5.4.1] applied with $K = F(\zeta_p)$ bounds the dimension of $U_{F(\zeta_p)}^{\text{spl}}$ by $\frac{1}{2}d^2[F(\zeta_p) : \mathbb{Q}_p]$ from above. The $\frac{1}{2}$ in this estimate appears by estimating $[K' : K] \geq 2$ (see the proof of [9, Theorem 5.4.1] for the notation; K' corresponds to our F'). If K contains a p -th root of unity, then it follows from Case II of [9, Lemma 5.1.1] that $[K' : K] = p$. Since $F(\zeta_p)$ contains a p -th root of unity, the argument in the proof of [9, Theorem 5.4.1] gives us

$$\dim U_{F(\zeta_p)}^{\text{spl}} \leq (d/p)^2[F' : \mathbb{Q}_p] = \frac{[F(\zeta_p) : F]}{p} d^2[F : \mathbb{Q}_p].$$

Since $[F(\zeta_p) : F] \leq p - 1$, we conclude that

$$\dim(U^{\text{Kred}} \setminus U^{\text{Cred}}) \leq \frac{p-1}{p} d^2[F : \mathbb{Q}_p].$$

Lemma A.5 implies that the same bound holds for the dimension of the closure of $U^{\text{Kred}} \setminus U^{\text{Cred}}$ in U^{irr} . Lemma A.7 gives the bound

$$\dim U^{\text{Cred}} \leq \frac{1}{2}d^2[F : \mathbb{Q}_p].$$

Thus,

$$\dim U^{\text{Kred}} \leq \frac{p-1}{p} d^2[F : \mathbb{Q}_p].$$

Since $\dim U^{\text{irr}} = d^2[F : \mathbb{Q}_p]$ by (36), we obtain

$$\dim U^{\text{irr}} - \dim U^{\text{Kred}} \geq \frac{d}{p} d^2[F : \mathbb{Q}_p] \geq d[F : \mathbb{Q}_p] \geq 2,$$

where we have used that p divides d in the second inequality. \square

Proposition A.9. *There exists an open dense subscheme $U^{\text{Kirr}} \subset U^{\text{irr}}$ such that $x \in P_1(R^{\text{ps}}/\varpi)$ is Kummer-irreducible if and only if x is a closed point in U^{Kirr} . Moreover,*

$$\dim U^{\text{irr}} - \dim(U^{\text{irr}} \setminus U^{\text{Kirr}}) \geq d[F : \mathbb{Q}_p] \geq 2.$$

Proof. Let U^{Kred} be the complement of U^{Kred} in U^{irr} . Since U^{Kred} is closed in U^{irr} , U^{Kirr} is open in U^{irr} . If $y \in U^{\text{irr}}$, then y lies in U^{Kred} if and only if there exists a degree p Galois extension F' of $F(\zeta_p)$ such that $D_y|_{G_{F'}}$ is reducible. If $y \in \overline{X}^{\text{ps}}$ is not the closed point and $D_y|_{G_{F'}}$ is irreducible for all such F' , then D_y is irreducible, and hence $y \in U^{\text{Kirr}}$. It follows from Lemma 3.18 (4) that $U^{\text{Kirr}} \cap P_1(R^{\text{ps}}/\varpi)$ is the set of closed points in U^{Kirr} . The bound for the difference of dimensions follows from Lemma A.8. Since U^{irr} is equi-dimensional by [9, Theorem 5.5.1], this implies density. \square

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