Convective Cores in Stellar Models

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Abstract: The condition for homogeneous radiative stellar models to be marginally stable to convection at the centre is investigated for the family of models where the opacity κ and energy generation ε are given by power laws in temperature and density $\kappa = \kappa_0 \rho^{\alpha} T^{-\beta}$, $\varepsilon = \varepsilon_0 \rho T^{\eta}$. The Naur-Osterbrock (1953) condition $6\eta > 6 + 10\beta - 15\alpha$ is a necessary but not sufficient condition. A better estimate is obtained by taking the effective polytropic index n = dlogP/dlogT - 1 to be a linear function of temperature T throughout the model. This gives the condition

$$6\eta = 10\beta - 15\alpha + \frac{12 + 4\beta}{1+\alpha}$$

The predictions of this condition agree well with results for a set of stellar models $0 \le \alpha \le 1$, $0 \le \beta \le 5$.

1 Analysis.

The equations of stellar structure for homogeneous radiative stellar models with an ideal gas equation of state and the opacity and energy generation given by power laws are:

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho \qquad \frac{dT}{dr} = -\frac{3\kappa\rho L_r}{16\pi acT^3 r^2} \qquad \frac{dM_r}{dr} = 4\pi r^2\rho \qquad \frac{dL_r}{dr} = 4\pi r^2\varepsilon\rho$$

$$P = \frac{\Re}{\mu}\rho T \qquad \kappa = \kappa_0\rho^{\alpha}T^{-\beta} \qquad \varepsilon = \varepsilon_0\rho T^{\eta}$$

Dimensionless variables θ, π, ϕ, λ, ζ are defined by $T = T_c\theta$, $P = P_0\pi$, $M_r = M_0\phi$, $L = L_0\lambda$, $r = R_0\,\zeta$ where T_c is the central value of T, and P_0, M_0, L_0, R_0 are suitably chosen scaling constants, the equations of stellar structure reduce to:

$$\frac{d\pi}{d\zeta} = -\frac{\pi\,\phi}{\theta\,\zeta^2} \qquad \frac{d\theta}{d\zeta} = -\frac{\pi^{1+\alpha}\lambda}{\theta^{4+\alpha+\beta}\zeta^2} \qquad \frac{d\phi}{d\zeta} = \frac{\pi\,\zeta^2}{\theta} \qquad \frac{d\lambda}{d\zeta} = \pi^2\,\theta^{\eta-2}\,\zeta^2$$

The central boundary conditions take the form $\theta = 1$, $\pi = \pi_0$, $\phi = 0$, $\lambda = 0$, $\zeta = 0$. With $g_0 = \pi_0^{2+\alpha}$ the solution for θ and π can be developed as a series in ζ as

$$\theta = 1 - \pi_0 g_0\zeta^2 + \frac{\pi_0^2 g_0}{360}[\,11 + 5\alpha + g_0(3\eta - 26 - 5\alpha - 5\beta)]\,\zeta^4 + \;....$$

$$\pi = \pi_0 - \frac{\pi_0^2}{6}\zeta^2 + \frac{\pi_0^3(1 - g_0)}{45}\zeta^4 + \;...$$

The polytropic index n is defined by

$$n+1 = \frac{d\log P}{d\log T} = \frac{d\log\pi}{d\log\theta} = \frac{\theta}{\pi}\frac{d\pi}{d\zeta}\frac{1}{d\theta/d\zeta} = \frac{\theta^{4+\alpha+\beta}\,\phi}{\pi^{1+\alpha}\lambda}$$

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which therefore has the series expansion

$$n+1 = \frac{1}{g_0}\{ 1 + \frac{p_0}{60}[22 + 10\alpha + 2g_0(3\eta - 26 - 5\alpha - 5\beta) - 6(1 - g_0)]\zeta^2 + ...$$

If the centre is marginally unstable to convection then $n+1 = 2.5$ at $\zeta = 0$, $g_0 = 2/5$ and the series expansion for $(n+1)$ in powers of $(1-\theta)$ and the value of $d(n+1)/d\theta$ are

$$n+1 = \frac{5}{2} - \frac{1}{4}(6 + 10\beta - 15\alpha - 6\eta)(1 - \theta) + O((1-\theta)^2)$$

$$\frac{d(n+1)}{d\theta} = \frac{1}{4}(6 + 10\beta - 15\alpha - 6\eta) \quad \text{at} \quad \theta = 1,\ \zeta = 0.$$

The Naur-Osterbrock (1954) condition follows from requiring that $d(n+1)/d\theta < 0$ at $\zeta = 0$

$$6\eta > 6 + 10\beta - 15\alpha \qquad \text{(N-O)}$$

This is only a necessary condition. A more accurate condition is obtained by requiring that $d(n+1)/d\theta$ is such that $(n+1)$ increases from 2.5 at the $\theta = 1$ to the surface value at $\theta = 0$ (cf Roxburgh 1985). The value of n+1 in the surface layers is readily determined since as $\theta \to 0$, $\lambda \to \lambda_0$, $\phi \to \phi_0$ where λ_0 and ϕ_0 are constants, hence

$$\frac{d\pi}{d\theta} \to \frac{\theta^{3+\alpha+\beta}\phi_0}{\pi^{\alpha}\lambda_0}, \qquad \pi \to \theta^{(4+\beta+\alpha)/(1+\alpha)}\frac{(1+\alpha)\phi_0}{(4+\beta+\alpha)\lambda_0}, \qquad \text{as } \theta \to 0$$

$$n+1 \to \frac{4+\beta+\alpha}{1+\alpha} \qquad \text{as } \theta \to 0$$

If the centre is marginally stable to convection then $(n+1) = 2.5$ at $\theta = 1$; hence the average value of $d(n+1)/d\theta$ throughout the model is

$$\frac{d\overline{(n+1)}}{d\theta} = \frac{5}{2} - \frac{4+\beta+\alpha}{1+\alpha}$$

If we now take $d(n+1)/d\theta$ to be constant throughout the model, this mean value is equal to the value at the centre, hence

$$6\eta = 10\beta - 15\alpha + \frac{12 + 4\beta}{1+\alpha} \qquad \text{(R-M)}$$

This is the new criterion to estimate whether stellar models have convective cores.

2 Results.

The equations of stellar structure in the dimensionless form given above were solved to determine η such that $n+1 = 2.5$ at $\zeta = 0$, given α and β. The solutions are very sensitive to the assumed value of η; if η is too small $(n+1)$ rapidly decreases, whereas if η is too large $(n+1)$ rapidly increases, the requirement that n+1 should tend to

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$(4+\alpha+\beta)/(1+\alpha)$ for large ζ (small θ) determines η without too much difficulty. The results are shown in Figure 1. In Figure 2 these numerical results are compared with the predictions of the Naur-Osterbrock condition and the new condition. The predictions of the new condition are in satisfactory agreement whereas the Naur-Osterbrock condition underestimates the value of η required for the existence of a convective core.

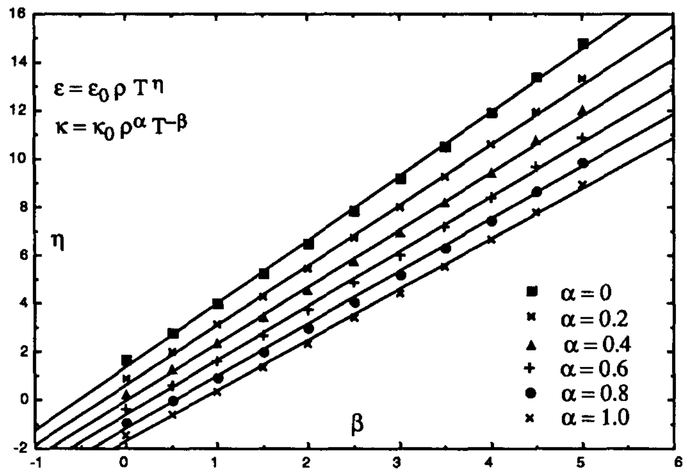

Figure 1. Values of η as a function of α and β, such that radiative stellar models are marginally unstable to convection at the centre.

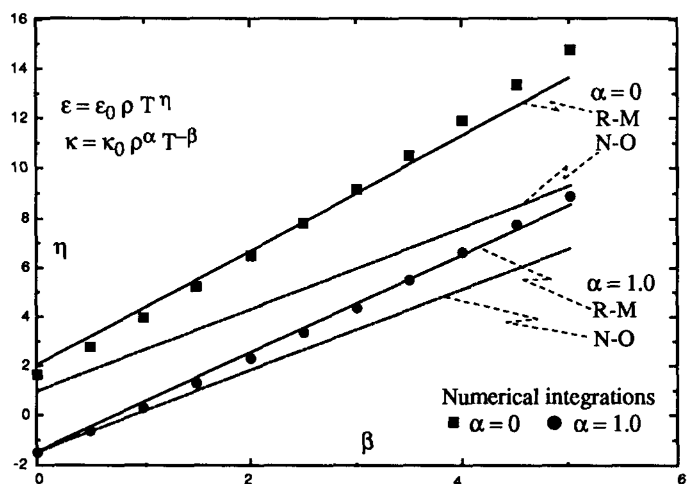

Figure 2. Values of η from numerical integrations, the new (R-M) condition and old Naur Osterbrock (N-O) condition.

References

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