GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE

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Let G be a discrete group with group algebra $C[G]$ over the complex numbers C. In **(5)** Kaplansky essentially proves that if G has a normal abelian subgroup of finite index n, then all irreducible representations of $C[G]$ have degree $\leqslant n$. Our main theorem is a converse of Kaplansky's result. In fact we show that if all irreducible representations of $C[G]$ have degree $\leqslant n$, then G has an abelian subgroup of index not greater than some function of n. (The degree of a representation of $C[G]$ for arbitrary G is defined precisely in § 3.)

This result is closely related to Jordan's theorem **(9**, Satz 200**)** which states that there exists a function $J(n)$ such that if a finite group G has a faithful representation of degree n over C, then G has an abelian normal subgroup of index $\leqslant J(n)$.

It appears that assuming that all representations of G are of degree $\leqslant n$ is closely analogous to assuming some representation of degree $\leqslant n$ is faithful. In § 5 we give some other examples of this analogy.

1. The main tool of this paper is the character theory of finite groups. We record here our notation and nomenclature. In this section all groups are assumed finite.

Let χ be an irreducible character of a group G and ϕ an irreducible character of a subgroup H.

ϕ induces a character ϕ^* of G and χ restricts down to a character $\chi|H$ of H. The Frobenius Reciprocity Theorem **(2**, Theorem 38.8**)** then states that

$$[\chi, \phi^*]_G = [\chi|H, \phi]_H,$$

where [,] denotes the inner product over the appropriate group.

Suppose $H \triangle G$ (H is normal in G). Then G acts on the irreducible characters of H by conjugation. That is, for $g \in G$ and all $x \in H$, $\phi^g(x) = \phi(gxg^{-1})$. The subgroup T fixing a given irreducible character ϕ is called the inertia group of ϕ. Clearly $T \supseteq H$. If $t = [G : T]$ (the index of T in G), then ϕ has precisely t distinct conjugates $\phi = \phi_1, \phi_2, \ldots, \phi_t$. If χ is a constituent of ϕ^* of multiplicity a, then by **(2**, Theorem 49.7**)**

$$\chi|H = a(\phi_1 + \phi_2 + \ldots + \phi_t).$$

Let $H \triangle G$. Each character of G/H can be viewed in a natural way as a character of G with kernel containing H. Conversely every such character

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of G comes from a unique one of G/H in this manner. In general, we shall use the same symbol to denote the character whether viewed in G or G/H. The precise situation will always be clear from context.

We denote by $|G|$ the order of the group G.

Several results which will be needed are as follows.

1.1. Proposition. *Let $H \triangle G$ and let χ be an irreducible character of G such that $\chi|H$ is irreducible. If β is any irreducible character of G/H, then the product character $\chi\beta$ is irreducible.*

Proof. Let $d = [G:H]$ and $\phi = \chi|H$. Then

$$\begin{aligned} \phi^*(g) &= d\chi(g), \quad g \in H, \\ &= 0, \qquad\quad g \notin H. \end{aligned}$$

By Frobenius Reciprocity we have

$$[\phi^*, \phi^*]_G = [\phi^*|H, \phi] = [d\phi, \phi] = d,$$

the latter since ϕ is irreducible in H.

Let ρ be the character of the regular representation of G/H. Viewed in G, $\rho(g) = d$ for $g \in H$ and 0 otherwise. Hence $\phi^* = \chi\rho$.

Write $\rho = \sum_i b_i\beta_i$, where the β_i are the irreducible characters of G/H and $b_i = \deg \beta_i$. Hence $\phi^* = \sum_i b_i\chi\beta_i$ and

$$d = [\phi^*, \phi^*]_G = \sum_{ij} b_i b_j [\chi\beta_i, \chi\beta_j]_G \geqslant \sum_i b_i^2 \, [\chi\beta_i, \chi\beta_i]_G \geqslant \sum_i b_i^2 = d.$$

Thus we have equality throughout. This implies that, for all i, $[\chi\beta_i, \chi\beta_i] = 1$ and hence that the $\chi\beta_i$ are irreducible.

1.2. Proposition. *Let H be a normal subgroup of G of prime index p. If χ is an irreducible character of G, then $\chi|H$ is either irreducible or the sum of p distinct conjugate characters of H.*

Proof. If $\chi|H$ is not irreducible, then

$$\chi|H = a(\phi_1 + \phi_2 + \ldots + \phi_t),$$

where the ϕ_i are distinct conjugate irreducible characters of H. We have $[\chi|H, \chi|H]_H = a^2t$.

As in the preceding proof $(\chi|H)^* = \sum_i \alpha_i\chi$, where the α_i are the irreducible characters of G/H. We note that since G/H is abelian, the α_i are linear and hence the $\alpha_i\chi$ are irreducible. Now

$$[(\chi|H)^*, \alpha_i\chi]_G = [\chi|H, \alpha_i\chi|H]_H = [\chi|H, \chi|H]_H = a^2t.$$

Thus each distinct $\alpha_i\chi$ occurs with multiplicity a^2t. If there are s distinct ones, then $sa^2t = p$ by counting degrees. Thus $a = 1$. Since $\chi|H$ is reducible, $t \neq 1$, and hence $t = p$. Thus the result follows.

1.3. Proposition. *Let N be a normal abelian subgroup of G. If G/N is abelian* *and χ is any irreducible character of G, then there exists a subgroup K of G and a linear character ψ of K with $N \subseteq K \subseteq G$ and $\chi = \psi^*$.*

Proof. Let $H \supseteq N$ be minimal in G with $\chi|H$ irreducible; then $H > N$. Since H/N is abelian, we may choose $L \supseteq N$ of prime index p in H. We see that $\chi|L$ is reducible and thus by Proposition 1.2 is a sum of p distinct irreducible characters of L, $\phi_1, \phi_2, \ldots, \phi_p$. Let T be the inertia group of ϕ_1 in G. Then $[G : T] = p$ and $\chi|T$ is reducible, since the ϕ_i are not conjugate in T. Thus

$$\chi|T = \sum_{i=1}^{p} \theta_i.$$

By reciprocity, χ is a constituent of θ_1^*; but $\deg \chi = p \deg \theta_1 = \deg \theta_1^*$, and thus $\chi = \theta_1^*$. Applying induction on the order of the group, we conclude that θ_1 is induced from a linear character on a subgroup $K \supseteq N$. The result now follows by transitivity of induction.

1.4. Definition. *G has r.b. n (representation bound n) if all the irreducible representations of G have degree $\leqslant n$.*

In § 3 we shall extend this definition to arbitrary infinite groups.

1.5. Proposition. *Let G have r.b. n. Then any subgroup or quotient group of G also has r.b. n.*

Proof. The fact for quotient groups is obvious. Let H be a subgroup of G and ϕ an irreducible character of H. Choose a constituent χ of ϕ^*. Then by reciprocity ϕ is a constituent of $\chi|H$. Thus $\deg \phi \leqslant \deg \chi \leqslant n$.

1.6. Proposition. *Let $N \triangle G$ with G/N non-abelian. If G has r.b. n, then N has r.b. $n/2$.*

Proof. Let ϕ be an irreducible character of N and χ a constituent of ϕ^*. Then

$$\chi|N = a(\phi_1 + \phi_2 + \ldots + \phi_t),$$

where $\phi_1 = \phi$ and $at \cdot \deg \phi = \deg \chi$. If either a or t is >1, then

$$2 \deg \phi \leqslant \deg \chi \leqslant n.$$

Otherwise $\chi|N = \phi$ is irreducible. In this case let β be a non-linear character of G/N. Then by Proposition 1.1 the product $\chi\beta$ is irreducible. Hence

$$n \geqslant \deg \chi\beta = \deg \chi \deg \beta \geqslant 2 \deg \chi = 2 \deg \phi.$$

Therefore, $\deg \phi \leqslant n/2$ and the result follows.

1.7. Corollary. *Let $G = N_0 \supseteq N_1 \supseteq \ldots \supseteq N_k$ be a sub-invariant series. Suppose m of the quotients are non-abelian. Then G has an irreducible representation of degree $\geqslant 2^m$.*

Proof. If G has r.b. n, then by Propositions 1.5 and 1.6 and induction, N_k has r.b. $n/2^m$. Thus $n/2^m \geqslant 1$.

2. In this section we continue studying finite groups with r.b. n and prove the following theorem.

2.1. THEOREM I. *There exists a function f defined on the positive integers with the following property. If G is a finite group all of whose irreducible representations have degree $\leqslant n$, then G has an abelian subgroup of index $\leqslant f(n)$. We may take*

$$f(n) = \prod_{i=0}^{r} J_1([n/2^i]),$$

where $r = [\log_2 n]$ and $J_1(m) = \max\{J(m), m^{m+1}\}$.

We remark that n^{n+1} is smaller than all known upper bounds for $J(n)$. In particular (**9**, Satz 200) gives

$$J(n) \leqslant n!\, 12^{n(\pi(n+1)+1)},$$

where $\pi(n)$ is the number of primes $\leqslant n$.

It would be of interest to show that $J(n) \geqslant n^{n+1}$. If this were true, we could replace J_1 by J in the formula. In any case we have $J(n) \geqslant (n+1)!$ since the symmetric group S_{n+1} has a faithful representation of degree n.

2.2. LEMMA. *Let R be a group with the following properties:*

(i) *R has a non-trivial normal abelian subgroup,*

(ii) *all non-trivial normal subgroups of R contain R',*

(iii) *R has an irreducible character χ of degree $m > 1$.*

Then a maximal normal abelian subgroup of R has index m and is a direct product of $\leqslant m$ cyclic prime-power factors.

Proof. Let χ be the character given in (iii). If χ is not faithful, then its kernel contains R' and χ has degree 1, a contradiction.

Let N be a maximal normal abelian subgroup of R. Then $N \supseteq R'$ and R/N is abelian. It follows from Proposition 1.3 that χ is induced from a linear character ψ of a subgroup $K \supseteq N$ of index m in R. Since $K \supseteq N \supseteq R'$, $K \triangle R$. By reciprocity, ψ is a constituent of $\chi|K$. Since K is normal, all constituents of $\chi|K$ are linear, and since χ is faithful, K is abelian. Since $K \supseteq N$, we have $K = N$ by the maximality of N. Thus $[R : N] = m$.

The Sylow subgroups of N are characteristic in N and thus normal in R and hence each contains $R' \neq 1$. Thus N must be a p-group.

N has a faithful representation by diagonal $m \times m$ matrices and thus is a subgroup of an abelian group with m generators. Thus N has $\leqslant m$ generators and the result follows.

2.3. LEMMA. *Let $K \triangle L$, where L/K is a cyclic p-group. If K has r.b. m*

and L has r.b. n, then there exists a subgroup T having r.b. m with $K \subseteq T \subseteq L$ and $[L : T] \leqslant n$.

Proof. For each M with $K \subseteq M \triangle L$ and each irreducible character ϕ of M let $T(\phi, M)$ be the inertia group of ϕ in L. Put $T = \bigcap_{\phi, M} T(\phi, M)$. Since L/K is a cyclic p-group, the subgroups of L containing K are linearly ordered. Thus $T = T(\phi, M)$ for some M and ϕ.

If $[L : T] = r$, let $\phi = \phi_1, \phi_2, \ldots, \phi_r$ be the distinct conjugates of ϕ. Let χ be an irreducible constituent of ϕ^*. Then ϕ and hence all of the ϕ_i are constituents of $\chi|M$. Thus $n \geqslant \deg \chi \geqslant r \deg \phi \geqslant r$.

To show that T has r.b. m, let θ be an irreducible character of T. Let M be minimal with $T \supseteq M \supseteq K$ and $\theta|M$ irreducible. We claim $M = K$. If $M > K$ choose N with $M \supseteq N \supseteq K$ and $[M : N] = p$. By the minimality of M, $\theta|N$ is reducible and since $N \triangle M$ we have by Proposition 1.2

$$\theta|N = \sum_{i=1}^{p} \psi_i,$$

where the ψ_i are distinct conjugate characters of N. The inertia group of ψ_1 in M is $T(\psi_1, N) \cap M \supseteq T \cap M = M$, which is a contradiction since ψ_1 has p distinct conjugates.

Since $M = K$, $\theta|K$ is irreducible and thus $\deg \theta \leqslant m$.

2.4. Lemma. *Let $K \triangle E$, where E/K is the direct product of r cyclic prime-power groups. If E has r.b. n and K has r.b. m, then E has a subgroup F such that $[E : F] \leqslant n^r$ and F has r.b. m.*

Proof. Let E_i, $1 \leqslant i \leqslant r$, be the inverse images of the direct factors of E/K in E. We define the subgroups $T_0, T_1, \ldots, T_r$ of E inductively as follows.

Put $T_0 = K = E_0$. Given

$$T_i \subseteq \prod_0^i E_j, \qquad T_i \supseteq K, \qquad \left[\prod_0^i E_j : T_i\right] \leqslant n^i$$

such that T_i has r b. m, then

$$T_i E_{i+1}/T_i \cong E_{i+1}/T_i \cap E_{i+1} = E_{i+1}/K,$$

which is a cyclic p-group. $T_i E_{i+1}$ has r.b. n by Proposition 1.5. Applying Lemma 2.3, we can find T_{i+1} with $T_i \subseteq T_{i+1} \subseteq T_i E_{i+1}$ and $[T_i E_{i+1} : T_{i+1}] \leqslant n$ and such that T_{i+1} has r.b. m. We continue in this manner and take $F = T_r$.

Proof of Theorem I. The proof will be by induction on n. Take $f(1) = 1$ and assume that we have $f(m)$ for all $m < n$. Let G have r.b. n. Since we shall always take $f(n) \geqslant 1$, we may then assume that G is non-abelian. Choose $K \triangle G$ maximal such that G/K is non-abelian. By Proposition 1.6, K has r.b. $n/2$.

The non-trivial normal subgroups of G/K have inverse images in G properly bigger than K and thus have abelian quotients. Since G/K is non-abelian, it has a non-linear irreducible representation of degree $\leqslant n$. This representation is faithful since the quotient of its kernel is non-abelian.

If $|G/K| > J(n)$, the Jordan function, then G/K has a non-trivial normal abelian subgroup. In this case, G/K satisfies the hypotheses of Lemma 2.2. If E is the inverse image in G of the abelian subgroup whose existence Lemma 2.2. guarantees, then $[G:E] \leqslant n$. Moreover, E and K satisfy the hypotheses of Lemma 2.4 with $r = n$ and $m = [n/2]$. Thus, we can find a subgroup F of G with $[G:F] = [G:E][E:F] \leqslant n \cdot n^n$ such that F has r.b. $[n/2]$.

In case $|G/K| \leqslant J(n)$ we take $F = K$. Thus in either case

$$[G:F] \leqslant \max\{J(n), n^{n+1}\}$$

and F has r.b. $[n/2]$. By the inductive hypothesis F has an abelian subgroup of index $\leqslant f([n/2])$ in F and thus it is sufficient to take

$$f(n) = f([n/2]) \cdot \max\{J(n), n^{n+1}\}.$$

This leads to the function given in the statement of the theorem.

3. In this section we extend Theorem I to infinite groups. This extension will be done in two steps, first to finitely generated and then to arbitrary groups.

Let G be a group and $C[G]$ its group algebra over the complex numbers. Let $\mathfrak{T}$ be an irreducible representation of $C[G]$. Then the image $P = \mathfrak{T}(C[G])$ is primitive and hence is isomorphic to a dense set of linear transformations over D, the commuting ring of $\mathfrak{T}$ (**4**, p. 28). Let L be the centre of D. If $\dim_L P < \infty$, then we say that $\mathfrak{T}$ is *finite*. In this case P is central simple over L and hence $\dim_L P = m^2$ (**4**, Theorem 2, p. 122). We let $m = \deg \mathfrak{T}$, the degree of $\mathfrak{T}$.

Note that if G is finite, then C is always the commuting ring of every irreducible representation. Thus the degree as defined above agrees with the usual degree in this case.

We say G has *representation bound n* (r.b. n) if all irreducible representations $\mathfrak{T}$ of $C[G]$ are finite and, for each, $\deg \mathfrak{T} \leqslant n$.

We shall prove the following theorem.

3.1. Theorem II. *If G is an arbitrary group with r.b. n, then G has an abelian subgroup of index $\leqslant f(n)$, where f is the function of Theorem* I.

3.2. Proposition. *Let G be a finitely generated group. If all the representations of G are finite, then G is a subdirect product of finite groups.*

Proof. This is precisely Theorem V of (**7**).

3.3. Proposition. *Let G be finitely generated and have r.b. n. Then G has an abelian subgroup of index $\leqslant f(n)$.*

Proof. Since G is finitely generated, it follows by a theorem of M. Hall (**6**, p. 56) that there are only a finite number of subgroups of G of index $\leqslant f(n)$.

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Call them $A_1, A_2, \ldots, A_k$. We show that one of these must be abelian. If not, choose $a_i, b_i \in A_i$ with $c_i = a_i b_i a_i^{-1} b_i^{-1} \neq 1$ for all i. Since the c_i are finite in number and not the identity, by Proposition 3.2 we can find a normal subgroup N of finite index in G with all $c_i \notin N$.

Any irreducible representation of G/N is also one of G, so G/N has all its irreducible representations of degree $\leqslant n$. Since G/N is finite, we conclude by Theorem I that G/N has an abelian subgroup A/N of index $\leqslant f(n)$. Thus $[G : A] \leqslant f(n)$ and A must be one of the A_i, say A_1. But this implies that a_1 and b_1 commute (mod N), or $c_1 = a_1 b_1 a_1^{-1} b_1^{-1} \in N$, a contradiction.

Hence one of the A_i must be abelian and the result is proved.

Let G be any group. We let $\mathfrak{F} = \mathfrak{F}(G)$ be the set of all finitely generated subgroups of G.

3.4. Proposition. *G has r.b. n if and only if for all $F \in \mathfrak{F}$, F has r.b. n.*

Proof. We use the well-known result (**8**, Theorem 5.2) that for any group H, $C[H]$ is semi-simple. Thus by (**1**, Theorem 1) H has r.b. n if and only if $C[H]$ satisfies the polynomial identity $[x_1, x_2, \ldots, x_{2n}] = 0$.

If G has r.b. n and $F \subseteq G$, then $C[F] \subseteq C[G]$. But the latter satisfies $[x_1, \ldots, x_{2n}] = 0$ and hence so does the former. Thus F has r.b. n.

Conversely, to check that $C[G]$ satisfies that particular identity, we need only look at its finitely generated subalgebras. But each such is contained in $C[F]$ for some $F \in \mathfrak{F}$, so the result is clear.

Assume now that G is an arbitrary group with r.b. n. If $F, H \in \mathfrak{F}$, $A \subseteq F$, and $B \subseteq H$ we say that $(H, B) \geqslant (F, A)$ if $H \supseteq F$ and $B \cap F = A$. This relation is clearly transitive.

Suppose that $(H, B) \geqslant (F, A)$. If $g_1, g_2, \ldots, g_s$ are in different cosets of A in F, then they are in different cosets of B in H since if $g_i g_j^{-1} \in B$, we have $g_i g_j^{-1} \in B \cap F = A$. It follows from this that if $[H : B] = m < \infty$, then $[F : A] \leqslant m$.

Now let $\mathfrak{P}$ be the collection of ordered pairs (F, A) such that $F \in \mathfrak{F}$, $A \subseteq F$, A abelian, and $[F : A] \leqslant f(n) < \infty$. If $(H, B) \in \mathfrak{P}$ and $(H, B) \geqslant (F, A)$, then $(F, A) \in \mathfrak{P}$.

Let $\mathfrak{E} \subseteq \mathfrak{P}$ be the collection of pairs (F, A) with the property that for every $H \supseteq F$, $H \in \mathfrak{F}$, there exists a $B \subseteq H$ with $(H, B) \geqslant (F, A)$ and $(H, B) \in \mathfrak{P}$.

3.5. Lemma. *Let $(F, A) \in \mathfrak{E}$. If $H \supseteq F$ and $H \in \mathfrak{F}$, then there is a B such that $(H, B) \geqslant (F, A)$ and $(H, B) \in \mathfrak{E}$.*

Proof. Since $(F, A) \in \mathfrak{E}$, there exists some B with $(H, B) \in \mathfrak{P}$ and $(H, B) \supseteq (F, A)$. On the other hand, by a theorem of M. Hall (**6**, p. 56), there are only finitely many subgroups of H with index $\leqslant f(n)$ and thus only finitely many such B's, say $B_1, B_2, \ldots, B_r$. We must show that some $(H, B_i) \in \mathfrak{E}$ for $i = 1, 2, \ldots, r$. If this is not the case, we can find $K_i \in \mathfrak{F}$,

$K_i \supseteq H$ for each i, such that no C_i exists with $(K_i, C_i) \geqslant (H, B_i)$ and $(K_i, C_i) \in \mathfrak{P}$.

Let K be the group $\langle K_1, K_2, \ldots, K_r\rangle$. Then $K \in \mathfrak{F}$ and $K \supseteq F$. Since $(F, A) \in \mathfrak{C}$, we may choose C with $(K, C) \in \mathfrak{P}$, $(K, C) \geqslant (F, A)$.

$$(K, C) \geqslant (H, H \cap C)$$

and since $(K, C) \in \mathfrak{P}$ we conclude that $(H, H \cap C) \in \mathfrak{P}$. However,

$$(H, H \cap C) \geqslant (F, A)$$

and thus $H \cap C = B_i$ for some i. Now $(K, C) \geqslant (K_i, K_i \cap C)$ and thus $(K_i, K_i \cap C) \in \mathfrak{P}$. But $(K_i, K_i \cap C) \geqslant (H, B_i)$ and this contradicts the definition of K_i. The result follows.

Note that $\mathfrak{C}$ is non-empty since $(1, 1) \in \mathfrak{C}$, for by Propositions 3.3 and 3.4 for every $F \in \mathfrak{F}$, there is an abelian $A \subseteq F$ with $[F : A] \leqslant f(n)$. Thus $(F, A) \in \mathfrak{P}$ and $(F, A) \geqslant (1, 1)$.

In the following, $\mathfrak{C}(A)$ denotes the centralizer in G of the subgroup A of G.

Proof of Theorem II. Choose $(F_0, A_0) \in \mathfrak{C}$ with $[F_0 : A_0] = s \leqslant f(n)$ maximal. If $(F, A) \in \mathfrak{C}$ with $(F, A) \geqslant (F_0, A_0)$, then we claim that $[G : \mathfrak{C}(A)] \leqslant f(n)$. Choose $g \in G$ arbitrarily and put $H = \langle F, g\rangle \in \mathfrak{F}$. Choose B with $(H, B) \geqslant (F, A) \geqslant (F_0, A_0)$, where $(H, B) \in \mathfrak{C}$. Now let $g_1, g_2, \ldots, g_s$ be a full set of coset representatives of A_0 in F_0. Then the g_i are in different cosets of B in H. However, $[H : B] \leqslant s$ by the choice of (F_0, A_0) and thus the g_i form a full set of coset representatives of B in H. We have then $g \in g_i B$ for some i. Since B is abelian and $B \supseteq A$, $B \subseteq \mathfrak{C}(A)$ and thus $g \in g_i\mathfrak{C}(A)$. Thus

$$G = \bigcup_{i=1}^{s} g_i\mathfrak{C}(A) \quad \text{and} \quad [G : \mathfrak{C}(A)] \leqslant s \leqslant f(n).$$

Now among the (F, A) of $\mathfrak{C}$ with $(F, A) \geqslant (F_0, A_0)$ choose (F_1, A_1) with $[G : \mathfrak{C}(A_1)]$ maximal. Put

$$J = \langle A \mid \text{there exists } F \in \mathfrak{F} \text{ with } (F, A) \in \mathfrak{C} \text{ and } (F, A) \geqslant (F_1, A_1)\rangle.$$

We claim that J is abelian and $[G : J] \leqslant f(n)$.

To show that J is abelian, it suffices to show that any two generators commute. Thus let

$$\begin{aligned} a \in A \text{ with } (F, A) \geqslant (F_1, A_1), &\qquad (F, A) \in \mathfrak{C}; \\ b \in B \text{ with } (H, B) \geqslant (F_1, A_1), &\qquad (H, B) \in \mathfrak{C}. \end{aligned}$$

Since $A \supseteq A_1$, $\mathfrak{C}(A_1) \supseteq \mathfrak{C}(A)$; but by choice of (F_1, A_1),

$$[G : \mathfrak{C}(A)] \leqslant [G : \mathfrak{C}(A_1)];$$

thus we have $\mathfrak{C}(A_1) = \mathfrak{C}(A)$.

Since $B \supseteq A_1$ is abelian, $B \subseteq \mathfrak{C}(A_1) = \mathfrak{C}(A)$ and thus b commutes with a.

Finally suppose that $[G:J] > f(n)$. Choose $g_1, g_2, \ldots, g_t$ representatives of $t > f(n)$ cosets of J in G. Put

$$H = \langle F_1, g_1, g_2, \ldots, g_t \rangle \in \mathfrak{F}.$$

Choose B with $(H, B) \geqslant (F_1, A_1)$ and $(H, B) \in \mathfrak{C}$. Then $B \subseteq J$. Since $[H:B] \leqslant f(n)$, some $g_i g_j^{-1} \in B$, and this is a contradiction.

3.6. Corollary. *There exists a function g defined on the positive integers with the following property. If G is any group all of whose irreducible representations have degree $\leqslant n$, then G has an abelian normal subgroup N of index $\leqslant g(n)$.*

Proof. Choose A as in Theorem II with $[G:A] \leqslant f(n)$. If N is the intersection of the conjugates of A, then N is abelian, normal in G, and

$$[G:N] \leqslant f(n)!.$$

Thus set $g(n) = f(n)!$.

4. In this section we consider p-groups of r.b. n and get a significantly stronger bound than that given in Theorem I. This is analogous to the Jordan Theorem situation where $J(n)$ can be replaced by $n!$ for p-groups.

4.1. Proposition. *Let G be a finite group with a faithful irreducible representation of degree n. If the centre Z of G has an abelian quotient, then $[G:Z] = n^2$.*

Proof. Let χ be the character of the given faithful representation. If $z \neq 1$ is in Z, then z is represented by the scalar matrix λI, where $|\lambda| = 1$ and $\lambda \neq 1$. For any $g \in G$, $\chi(zg) = \lambda\chi(g)$.

If $g \notin Z$, choose h with $hgh^{-1}g^{-1} \neq 1$. Since G/Z is abelian, $hgh^{-1}g^{-1} = z \in Z$. Thus $hgh^{-1} = zg$ and we have

$$\chi(g) = \chi(hgh^{-1}) = \chi(zg) = \lambda\chi(g).$$

Since $z \neq 1$, we have $\lambda \neq 1$ and $\chi(g) = 0$.

Because χ is irreducible and $\chi(z) = \lambda n$, we have

$$1 = \frac{1}{|G|}\sum_{g \in G} \chi(g)\overline{\chi(g)} = \frac{1}{|G|}\sum_{z \in Z} \chi(z)\overline{\chi(z)} = \frac{1}{|G|}\cdot|Z|n^2$$

and the result follows.

Since the degree of an irreducible representation must divide the order of the group, we may assume for p-groups that the representation bound is a power of p.

4.2. Theorem III. *If P is a p-group having r.b. p^e, then P has an abelian subgroup of index $\leqslant p^{3e(e+1)/2}$.*

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Proof. Since the group $R = G/K$ of the proof of Theorem I is a p-group, it has a non-trivial centre Z. By our assumptions on R, we see immediately that Z is cyclic, being the centre of a group with a faithful irreducible representation and $Z \supseteq R'$. Thus R has a cyclic p-subgroup Z of index $\leqslant n^2 = p^{2e}$ by Proposition 4.1.

Returning again to the proof of Theorem I, we see that this implies that G has a subgroup L of index $\leqslant n \cdot n^2 = p^{3e}$ having r.b. $n/2$. Since

$$n/2 = p^e/2 < p^e,$$

L has r.b. p^{e-1}.

The result now follows by induction on n. L has an abelian subgroup A of index $\leqslant p^{3e(e-1)/2}$ and hence $[G:A] = [G:L]\,[L:A] \leqslant p^{3e(e+1)/2}$.

5. Many other results about linear groups carry over to groups having r.b. n. We mention a few here. For convenience we consider only finite groups.

5.1. THEOREM IV. *Let G be a finite group all of whose irreducible representations have degree $\leqslant n$. Then G has an abelian Hall subgroup for all the primes dividing $|G|$ and greater than $n + 1$.*

Proof. We first prove, by induction on $|G|$, that G has the appropriate Hall subgroup H.

If $|G|$ has no prime divisors $\leqslant n + 1$, then we take $H = G$. Otherwise, let g be an element of order $p \leqslant n + 1$ in G. Let $\mathfrak{X}$ be an irreducible representation of G with $\mathfrak{X}(g) \neq 1$. Now $\mathfrak{X}(G) = G_1$ is a linear group of degree $\leqslant n$ and hence, by (**9**, Satz 196), it has a Hall subgroup H_1. Since $\mathfrak{X}(g) \neq 1$, we have $G_1 \neq H_1$. Let K be the complete inverse image of H_1 under the homomorphism $\mathfrak{X}$. Then $[G:K] = [G_1 : H_1]$ contains no prime factors $> n + 1$.

By Proposition 1.5, K has r.b. n and since K is properly smaller than G, we have by induction a Hall subgroup H of K. But by the remarks above about $[G:K]$ we see that H is a full Hall subgroup of G.

Now let H be any Hall subgroup of G for primes $> n + 1$. Since H has r.b. n and the degrees of all irreducible representations of H must divide $|H|$ we see that all irreducible representations of H are linear. Hence H is abelian.

5.2. THEOREM V. *Let G be a finite group all of whose irreducible representations have degree $\leqslant n$. If p is a prime dividing $|G|$ with $p > 2n + 1$, then the Sylow p-subgroup of G is normal.*

Proof. We prove this by induction on $|G|$. If G is a p-group, then the result is trivial. Thus, let $g \in G$ be an element of prime order $q \neq p$. Let $\mathfrak{X}$ be an irreducible representation of G with $\mathfrak{X}(g) \neq 1$. Now $\mathfrak{X}(G) = G_1$ is a linear group of degree $\leqslant n$ and hence by (**3**, Theorem 1) it has a normal Sylow

p-subgroup P_1. Since $\mathfrak{X}(g) \neq 1$, we have $P_1 \neq G_1$. Let K be the complete inverse image of P_1 under the homomorphism $\mathfrak{X}$. Then $K \triangle G$ and

$$[G : K] = [G_1 : P_1]$$

is relatively prime to p.

By Proposition 1.5, K has r.b. n and since K is properly smaller than G we have by induction a normal Sylow p-subgroup P of K. Then clearly P is a full Sylow subgroup of G. Since P is a characteristic subgroup of $K \triangle G$, we have $P \triangle G$.

Added in proof.

1. Using slightly different techniques, we can replace $J_1(m)$ in Theorem I by $\max\{J(m), m^3\}$. However, since $J(m) \geqslant (m+1)!$ for $m \geqslant 4$ and $J(3) \geqslant J(2) = 60$, we see that for all m, $J(m) \geqslant m^3$. Moreover, $J(s)J(t) \leqslant J(s+t)$ implies that

$$\prod_{i=0}^{r} J\left(\left[\frac{n}{2^i}\right]\right) \leqslant J\left(\sum_{i=0}^{r}\left[\frac{n}{2^i}\right]\right) \leqslant J(2n).$$

Therefore, we can take $f(n) = J(2n)$ in Theorem I and in Theorem II as well.

2. A recent paper of E. Thomas, *Über unitäre Darstellungen abzählbarer, diskreter Gruppen* (to appear), discusses another condition on the irreducible representations of a discrete group which is equivalent to the group having an abelian subgroup of finite index.

3. A more detailed study of p-groups with r.b. p^e can be found in our paper *Groups whose irreducible representations have degrees dividing p^e*, which will shortly appear in the Illinois Journal of Mathematics.

REFERENCES

1. S. A. Amitsur, *Groups with representations of bounded degree* II, Illinois J. Math., *5* (1961), 198–205.
2. C. W. Curtis and I. Reiner, *Representation theory of finite groups* (New York, 1962).
3. W. Feit and J. G. Thompson, *Groups which have a faithful representation of degree less than* $(p-1)/2$, Pac. J. Math., *11* (1961), 1257–1262.
4. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Pub. vol. 37.
5. I. Kaplansky. *Groups with representations of bounded degree*, Can. J. Math., *1* (1949), 105–112.
6. A. G. Kurosh, *The theory of groups*, Vol. II (New York, 1960).
7. D. S. Passman, *On groups with enough finite representations*, Proc. Amer. Math. Soc. *14* (1963), 782–787.
8. C. Rickart, *Uniqueness of norm in Banach algebras*, Ann. of Math., *51* (1950), 615–628.
9. A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung* (Basel, 1956).

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