COUNTING PATHS, CIRCUITS, CHAINS, AND CYCLES IN GRAPHS: A UNIFIED APPROACH

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1. Introduction. The problem of counting partial subgraphs (or patterns, for short) in a given graph has been approached by several mathematicians from various points of view (see, e.g., [**1**; **3**; **5**; **13–15**; **17–23**; **26–29**]; applications may also be found in [**2**; **8**; **9**; **16**]). Specific algorithms have been presented and almost all of them are essentially based upon a careful analysis of the graph under consideration. In these cases, we say that a direct approach has been followed. Unfortunately, when large graphs are considered, all direct counting methods require rather cumbersome computations. For this reason, during the last few years many efforts have been made in finding suitable indirect counting methods. First, Biondi [**5**] faced the problem of counting cycles in non-oriented graphs by inspection of the complementary graph. More recently, a number of papers [**1**; **3**; **21**; **22**; **28**] have been concerned with counting trees in classes of non-oriented graphs having complementary graphs with special structural properties. However, to the best of our knowledge, no general indirect counting method is available in the literature. It is our aim in the present work to point out a quite general indirect method, the major quality of which is perhaps its complementary nature with respect to more usual direct methods, from a computational point of view. This method may be considered as an extension of the idea proposed in [**5**] and is readily adaptable for digital computation.

Both oriented and non-oriented graphs will be considered in this paper, following a unified approach. Thus, it has been found necessary to coin a number of terms and to provide some unusual definitions; on the other hand, Berge's basic terminology [**4**] has been adopted everywhere possible. In particular, with open routes we refer to Berge's elementary paths or chains, according to whether they are oriented or not; likewise, with closed routes we refer to Berge's elementary circuits or cycles.

The paper has been organized in sections as follows. A basic theorem (Theorem 1), derived from the classic Principle of Inclusion and Exclusion, is reported in § 2 together with a straightforward corollary. The idea of counting patterns in graphs by means of the Principle of Inclusion and Exclusion is not completely new. A slightly less general formulation of Theorem 1 is given in [**10**], where Hamilton circuits, complete subgraphs,

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and regular partial subgraphs of order 6 and degree 3 are counted. In a recent paper [**29**], by means of a different rearrangement of the above-mentioned principle, Wilf obtained formulas which enable counting Hamilton cycles and 1-factors in graphs. However, Wilf's method is a direct one, in the previously specified sense, and this is a crucial difference from our approach here. On the other hand, one of the main features of Wilf's counting method is that "it suggests uses in which one does not need to have in mind a specific set of objects and properties, but rather one discovers what the objects and properties were after the fact". This is not the case for the present method, in which objects and properties have less magic but more direct connections with the problem in question.

2. A basic theorem. Let G be a given oriented or non-oriented graph. X will denote the finite and non-empty set of vertices of G while U will indicate the finite set of the links of G. Formally:

$$G = (X, U); \qquad X = V(G); \qquad U = W(G).$$

In the following, oriented links are called arcs while non-oriented links are called edges.

Every graph G_0 such that G is a partial graph of G_0 is herein referred to as an overgraph of G. Let U_0 be the set of the links of a given overgraph G_0 of G; i.e., $U_0 = W(G_0)$. The graph $\tilde{G} = (X, \tilde{U})$, where $\tilde{U} = U_0 - U$, is called the partial complementary graph of G with respect to G_0.

Let Γ be a set of distinct patterns defined on an overgraph G_0 of G; e.g., complete graphs, regular graphs, routes, and so on. Every graph of Γ is a prime graph if Γ does not contain two graphs γ_i and γ_j such that γ_i is a partial subgraph of γ_j. Γ_G will denote the set of the γ-patterns of G; i.e.,

$$\Gamma_G = \{\gamma | \gamma \in \Gamma, W(\gamma) \subseteq U\}.$$

Specifically, $\Gamma_G{}^{\mathrm{e}}$ and $\Gamma_G{}^{\mathrm{o}}$ will denote the subsets of Γ_G consisting of those γ-patterns of G which have an even and an odd number of links, respectively. Furthermore, let T be a subset of U_0; then the subset of Γ consisting of the γ-patterns of G_0 such that T is contained in $W(\gamma)$ will be denoted by Γ^T; formally

$$\Gamma^T = \{\gamma | \gamma \in \Gamma, W(\gamma) \supseteq T\}.$$

The power of a set S is the set of all possible subsets of S; it will be denoted by $\mathscr{P}(S)$. In order to simplify many statements of this paper, we call the link-power of G, denoted by $\mathscr{P}_W(G)$, the power of $U = W(G)$. Then, for any set Γ of graphs, the link-power of Γ can be defined as follows:

$$\mathscr{P}_W(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathscr{P}_W(\gamma).$$

In quite a similar way, the proper link-power of G and Γ can also be defined. They will be denoted by $\mathscr{P}_W'(G)$ and $\mathscr{P}_W'(\Gamma)$, respectively.

Finally, a k-subset of a set S is a subset of S consisting of k elements. The set of all possible k-subsets of S will be denoted by $\mathscr{P}^k(S)$ and will be called the k-power of S. It follows that $\mathscr{P}^0(S) = \{\emptyset\}$.

The following theorem may now be stated.

THEOREM 1. *The number of γ-patterns contained in G is given by:*

$$|\Gamma_G| = \sum_{k=0}^{K(\tilde{U},\Gamma)} (-1)^k \sum_{S\in\mathscr{R}_k(\tilde{U},\Gamma)} |\Gamma^S|, \tag{1}$$

where

$$\mathscr{R}_k(\tilde{U}, \Gamma) = \mathscr{P}^k(\tilde{U}) \cap \mathscr{P}_W(\Gamma)$$

and $K(\tilde{U}, \Gamma)$ is the minimum value of k such that $\mathscr{R}_{k+1}(\tilde{U}, \Gamma) = \emptyset$.

COROLLARY 1. *If Γ is a set of prime graphs,* (1) *may be rewritten as follows:*

$$|\Gamma_G| = |\Gamma_{\tilde{G}}{}^{e}| - |\Gamma_{\tilde{G}}{}^{o}| + \sum_{k=0}^{K'(\tilde{U},\Gamma)} (-1)^k \sum_{S\in\mathscr{R}_k'(\tilde{U},\Gamma)} |\Gamma^S|, \tag{2}$$

where

$$\mathscr{R}_k'(\tilde{U}, \Gamma) = \mathscr{P}^k(\tilde{U}) \cap \mathscr{P}_W'(\Gamma)$$

and $K'(\tilde{U}, \Gamma)$ is the minimum value of k such that $\mathscr{R}_{k+1}'(\tilde{U}, \Gamma) = \emptyset$.

This theorem and its corollary are proved in Appendix 1. It is worth noting that (1) and (2) relate a single counting problem, defined in G, to the solution of a number of different counting problems defined in G_0. However, it must be pointed out that for large families of graphs a common overgraph G_0 may often be chosen with special symmetry properties in order to simplify the computation. Then, several results can be listed once and for all in general purpose tables (as it will be seen later, with reference to specific examples).

3. Counting open and closed routes. As an application of the preceding theory, the problem of counting open and closed routes, both in oriented and non-oriented graphs, is considered. (Further application may be found in [**10**; **11**].) All graphs to be considered here are assumed to have neither loops nor multiple links. Such graphs will be referred to in what follows as flow diagrams or networks, according to whether they are oriented or not. In this section, suitable formulas will be derived for counting elementary circuits and cycles or elementary paths and chains between two fixed vertices of a given graph G. As overgraph of G, the complete (symmetric) graph G_n defined on X will be chosen ($n = |X|$). The partial complementary graph of G with respect to G_n will be denoted by $\bar{G}$ and for the sake of brevity referred to as the complementary graph of G.

Any pattern of a given route σ will be called a subroute of σ. A subroute is always a collection of fragments (open routes or disconnected vertices) which are called components of the subroute. Now, let ω be an open route. The components of a subroute of ω are said to be free if they do not contain any extremity of ω.

It is worth noting that both closed routes and open routes between two fixed vertices are prime graphs; therefore, only (2) will be employed in the remainder of this section.

Open routes. Let Ψ be the set of all possible open routes (paths or chains) between two fixed vertices of G_n. Then, from (2), we have:

$$(3) \qquad |\Psi_G| = |\Psi_{\bar{G}}^{e}| - |\Psi_{\bar{G}}^{o}| + \sum_{k=0}^{K'(\bar{U},\Psi)} (-1)^k \sum_{S \in \mathscr{R}_k'(\bar{U},\Psi)} |\Psi^S|.$$

Before stating Theorem 2, which will enable us to compute $|\Psi^S|$ immediately, it is important to note that, from definitions, every graph $G_S = (X, S)$, $S \in \mathscr{R}_k'(\bar{U}, \Psi)$, is a subroute of some open route of G_n.

THEOREM 2. *For any $S \in \mathscr{R}_k'(\bar{U}, \Psi)$, let p be the number of free components of G_S; then*

$$(4) \qquad |\Psi^S| = D(p)\, H(p, n - k - p - 2),$$

where $n = |X|$, while D and H are two integer functions defined as follows:

$$(5) \qquad D(i) = \begin{cases} 1 & \textit{for flow diagram}, \\ 2^i & \textit{for networks}, \end{cases}$$

$$(6) \qquad H(i, j) = \sum_{k=0}^{j} (k + i)! \binom{j}{k}.$$

A few remarks about some computational properties of the H-function may be found in Appendix 2. Table I gives $H(i, j)$, for $i \geqq -1$, $j \geqq 0$, $i + j \leqq 10$. Note that, when $S = \emptyset$, then $k = p = 0$ and $\Psi^{\emptyset} = \Psi$; hence, (4) yields:

$$(7) \qquad |\Psi| = H(0, n - 2)$$

for all $n > 1$; i.e., the number of open routes between two fixed vertices of G_n is given by $H(0, n - 2)$.

COROLLARY 2. *Let $A(k, p)$ be the number of sets $S \in \mathscr{R}_k'(\bar{U}, \Psi)$ such that G_S has p free components. Then, in view of* (4), (3) *can be rewritten as follows:*

$$(8) \qquad |\Psi_G| = |\Psi_{\bar{G}}^{e}| - |\Psi_{\bar{G}}^{o}| + H(0, n - 2) + \sum_{k=1}^{K'(\bar{U},\Psi)} (-1)^k \sum_{p=1}^{P(k)} A(k, p) D(p) H(p, n - k - p - 2),$$

where $P(k)$ is the maximum of p, for all $S \in \mathscr{R}_k'(\bar{U}, \Psi)$.

The proof of Theorem 2 and Corollary 2 is given in Appendix 3.

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TABLE I

The H-function

$H(i,j)$	0	1	2	3	4	5	6	7	8	9	10
−1	−1	0	2	7	23	88	414	2371	16071	125672	1112082
0	1	2	5	16	65	326	1957	13700	109601	986410	9864101
1	1	3	11	49	261	1631	11743	95901	876809	8877691	
2	2	8	38	212	1370	10112	84158	780908	8000882		
3	6	30	174	1158	8742	74046	696750	7219974			
4	24	144	984	7584	65304	622704	6523224				
5	120	840	6600	57720	557400	5900520					
6	720	5760	51120	499680	5343120						
7	5040	45360	448560	4843440							
8	40320	403200	4394880								
9	362880	3991680									
10	3628800										

Closed routes. Let Φ be the set of all possible closed routes (circuits or cycles) of G_n. Then, from (2), we have:

$$|\Phi_G| = |\Phi_{\bar{G}}{}^{e}| - |\Phi_{\bar{G}}{}^{o}| + \sum_{k=0}^{K'(\bar{U},\Phi)} (-1)^k \sum_{S \in \mathscr{R}_k'(\bar{U},\Phi)} |\Phi^S|, \tag{9}$$

where $|\Phi^S|$ can be computed easily, due to the following result.

THEOREM 3. *For any $S \in \mathscr{R}_k'(\bar{U}, \Phi)$, let q be the number of components of G_S; then*

$$|\Phi^S| = \delta_k{}^0 F_0(n) + \delta_k{}^1 F_1(n) + (1 - \delta_k{}^0 - \delta_k{}^1) D(q-1) H(q-1, n-k-q), \tag{10}$$

where $\delta_i{}^j$ is the Kronecker delta while $F_0(n)$ and $F_1(n)$ are integer functions defined in Table II. *Specifically, $F_0(n) = |\Phi|$ while $F_1(n) = |\Phi^S|$, $|S| = 1$.*

TABLE II

Number of closed routes in G_n

	Flow-diagrams	Networks
$F_0(n)$	$H(-1, n) + 1 - n$	$\frac{1}{2}\left[H(-1, n) + 1 - n - \binom{n}{2}\right]$
$F_1(n)$	$H(0, n-2)$	$H(0, n-2) - 1$

It is worth noting that the function $F_0(n)$ may also be calculated by means of suitable recurrence relationships.

For oriented G_n (complete symmetric flow diagrams):

$$F_0(n) = nF_0(n-1) - (n-1)[F_0(n-2) - 1], \qquad F_0(0) = F_0(1) = 0.$$

For non-oriented G_n (complete networks):

$$F_0(n) = nF_0(n-1) - (n-1)F_0(n-2) + \tfrac{1}{2}n(n-3) + 1, \qquad F_0(1) = F_0(2) = 0.$$

Proofs may be found in [**6**]; see also [**19**].

COROLLARY 3. *Let $B(k, q)$ be the number of sets $S \in \mathscr{R}_k'(\bar{U}, \Phi)$ such that G_S has q components. Then, in view of* (10), (9) *can be rewritten as follows:*

$$|\Phi_G| = |\Phi_{\bar{G}}{}^{e}| - |\Phi_{\bar{G}}{}^{o}| + F_0(n) - B(1,1)F_1(n) + \sum_{k=2}^{K'(\bar{U},\Phi)} (-1)^k \sum_{q=1}^{Q(k)} B(k,q) D(q-1) H(q-1, n-k-q), \tag{11}$$

where $B(1, 1) = \bar{U}$ and $Q(k)$ is the maximum of q for all $S \in \mathscr{R}_k'(\bar{U}, \Phi)$.

The proof of Theorem 3 and its related Corollary 3 is given in Appendix 3.

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4. Examples. Consider the flow diagram η shown in Figure 1 together with its complementary graph $\bar{\eta}$. Vertices are numbered 1, 2, 3, 4. Suppose that elementary paths from 2 to 3 are to be counted. First of all, we determine $|\Psi_{\bar{\eta}}^{e}| - |\Psi_{\bar{\eta}}^{o}|$ and $A(k, p)$ for all possible k and p. With this aim, the following detailed analysis of $\bar{\eta}$ must be performed.

$$W(\bar{\eta}) = \{(1, 2), (2, 3), (3, 2), (4, 1)\} = \{t_1, t_2, t_3, t_4\} = T,$$

$$\Psi_{\bar{\eta}}^{e} = \emptyset, \qquad \Psi_{\bar{\eta}}^{o} = \{\{t_2\}\}, \qquad |\Psi_{\bar{\eta}}^{e}| - |\Psi_{\bar{\eta}}^{o}| = -1,$$

$$\mathscr{P}^1(T) = \{\{t_1\}, \{t_2\}, \{t_3\}, \{t_4\}\}, \quad \mathscr{R}_1'(T, \Psi) = \{\{t_4\}\}, A(1, 1) = 1,$$

$$\mathscr{P}^2(T) = \{\{t_1, t_2\}, \{t_1, t_3\}, \{t_1, t_4\}, \{t_2, t_3\}, \{t_2, t_4\}, \{t_3, t_4\}\},$$

$$\mathscr{R}_2'(T, \Psi) = \emptyset, \quad A(k, p) = 0 \quad \text{for all } k > 1 \text{ and } p > 0.$$

Thus, in view of Corollary 2, we have:

$$|\Psi_\eta| = -1 + 5 - 1 = 3.$$

This is a trivial result, as the three elementary paths $(2 - 1 - 3)$, $(2 - 4 - 3)$, and $(2 - 1 - 4 - 3)$ were evident in η. However, consider the graph σ of Figure 2. It is easy to see that

$$W(\bar{\sigma}) = W(\bar{\eta}) = T;$$

therefore, the analysis to count elementary paths from 2 to 3 in σ is the same as the preceding one. Thus, from (8), the result is:

$$|\Psi_\sigma| = -1 + 326 - 49 = 276.$$

In a similar way, elementary circuits can be counted in η and in σ. Specifically, the resulting steps are:

$$\Phi_{\bar{\eta}}^{e} = \{\{t_2, t_3\}\}; \qquad \Phi_{\bar{\eta}}^{o} = \emptyset, \qquad |\Phi_{\bar{\eta}}^{e}| - |\Phi_{\bar{\eta}}^{o}| = 1,$$

$$\mathscr{R}_2'(T, \Phi) = \{\{t_1, t_2\}, \{t_1, t_4\}, \{t_3, t_4\}\},$$

$$B(2, 1) = 2; \qquad B(2, 2) = 2,$$

$$\mathscr{R}_3'(T, \Phi) = \{\{t_1, t_2, t_4\}\},$$

$$B(3, 1) = 1; \qquad B(3, 2) = B(3, 3) = 0.$$

Therefore, from (11), it follows that:

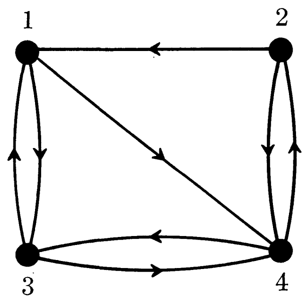
$$|\Phi_\eta| = 1 + 20 - 4 \cdot 5 + 2 \cdot 2 + 2 - 1 = 6,$$

while

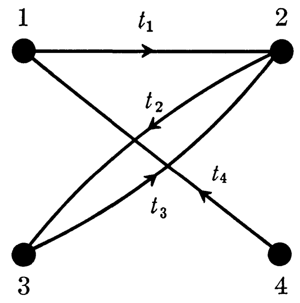
$$|\Phi_\sigma| = 1 + 2365 - 4 \cdot 326 + 2 \cdot 65 + 2 \cdot 49 - 16 = 1274.$$

Finally, it is worth noting that every set $\mathscr{R}_k'(T, \Phi)$, and $\mathscr{R}_k'(T, \Psi)$, can also be derived directly from $\mathscr{R}_2'(T, \Phi)$, and $\mathscr{R}_2'(T, \Psi)$, respectively, by a number of suitable tests. Specifically, the following theorem can be proved easily (see [7]): *For any S, $|S| = k > 2$, we have:*

$$\mathscr{P}^2(S) \subseteq \mathscr{R}_2'(T, \Phi) \Leftrightarrow S \in \mathscr{R}_k(T, \Phi).$$



(a)
The graph η



(b)
The complementary graph $\bar{\eta}$

FIGURE 1

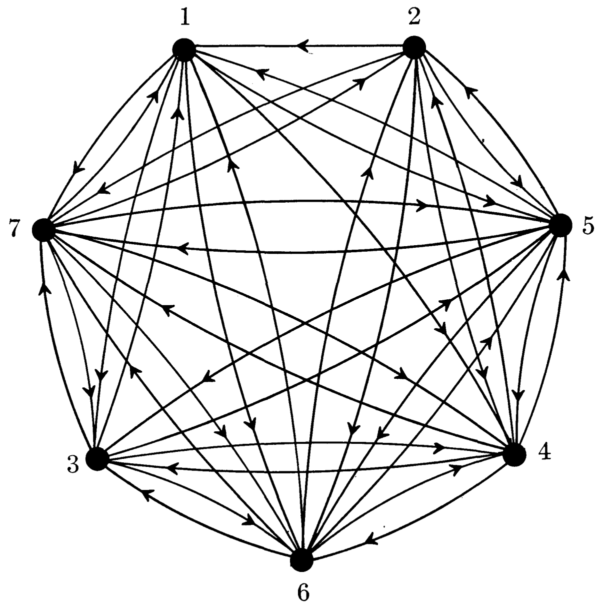


FIGURE 2

5. Conclusions. A new method for counting patterns in graphs has been presented in this paper. As an application, paths, chains, circuits, and cycles have been counted both in oriented and non-oriented graphs.

One of the main aspects of the present approach is to lead the original counting problem to a somewhat different one which basically consists of two distinct subproblems. The first one is that of counting constrained patterns in complete or highly symmetric graphs (see (1) and (2)). Such a problem may often be solved by formulas or general tables, as shown in the present paper and in [**10**]. The second one is mainly a collection of counting problems which are, however, concerned with the partial complementary graph $\tilde{G}$ of G. These are also very easy to solve when $\tilde{G}$ is sufficiently "sparse", as it has been shown in this paper by means of examples. More specifically, this indirect method is attractive indeed, from a computational point of view, when the partial complementary graph $\tilde{G}$ has a very small number of links with respect to G. Otherwise, more customary direct methods may prove to be preferable.

Appendix 1. Let A be a given set of objects and B a set of properties defined on A. For any $S \subseteq B$, $g(S)$ denotes the number of objects having all the properties contained in S. Conversely, $g^*(S)$ denotes the number of objects having none of the properties contained in S. Then, the following equation holds:

$$g^*(B) = \sum_{k=0}^{K} (-1)^k \sum_{S \in \mathscr{P}^k(B)} g(S), \tag{12}$$

where $K = |B|$ and $\mathscr{P}^k(B)$ is the k-power of B (see § 2). The proof of this important equation of combinatorial analysis follows from the Principle of Inclusion and Exclusion and may be obtained by mathematical induction [**25**, Chapter 3].

In order to prove Theorem 1, we now choose A and B in such a way that (12) yields (1). With this aim, let Γ (see § 2) be the collection of objects under consideration. For these objects, a set B of properties can be defined as follows. For any $u_i \in \tilde{U}$, we say that a γ-pattern of G_0 has property b_i if and only if $u_i \in W(\gamma)$. Thus, the number of γ-patterns of G_0 having none of the properties b_i (i.e. containing no one of the links of $\tilde{G}$) is equal, of course, to the cardinality of Γ_G; formally:

$$g^*(B) = |\Gamma_G|. \tag{13}$$

Moreover, we have established a one-to-one correspondence between the elements of B and $\tilde{U}$; then, we can write:

$$\sum_{S \in \mathscr{P}^k(B)} g(S) = \sum_{S \in \mathscr{P}^k(\tilde{U})} |\Gamma^S|. \tag{14}$$

However, in order to avoid useless redundancies, it must be noted that

$$S \notin \mathscr{P}(\Gamma) \Rightarrow \Gamma^S = \emptyset;$$

hence, it has been found convenient to define

$$\mathscr{R}_k(\tilde{U}, \Gamma) = \mathscr{P}^k(\tilde{U}) \cap \mathscr{P}(\Gamma)$$

so that, in view of (13) and (14), (12) can be rewritten as follows:

$$|\Gamma_G| = \sum_{k=0}^{K(\tilde{U},\Gamma)} (-1)^k \sum_{S \in \mathscr{R}_k(\tilde{U},\Gamma)} |\Gamma^S|, \tag{15}$$

where $K(\tilde{U}, \Gamma)$ is the minimum value (it is quite obvious that

$$\mathscr{R}_k(\tilde{U}, \Gamma) = \emptyset \Rightarrow \mathscr{R}_{k+i}(\tilde{U}, \Gamma) = \emptyset$$

for all positive integers i) of k such that $\mathscr{R}_{k+1}(\tilde{U}, \Gamma) = \emptyset$. Therefore, Theorem 1 is proved.

Let Γ be a set of prime graphs. Since, in view of the definition of proper link-power of Γ (see § 2), we have:

$$\mathscr{P}_W'(\Gamma) \subseteq \mathscr{P}_W(\Gamma),$$

we can define

$$\mathscr{P}_W''(\Gamma) = \mathscr{P}_W(\Gamma) - \mathscr{P}_W'(\Gamma).$$

Then

$$\begin{aligned}\mathscr{R}_k(\tilde{U}, \Gamma) = \mathscr{P}^k(\tilde{U}) \cap \mathscr{P}_W(\Gamma) &= (\mathscr{P}^k(\tilde{U}) \cap \mathscr{P}_W'(\Gamma)) \cup (\mathscr{P}^k(\tilde{U}) \cap \mathscr{P}_W''(\Gamma)) \\ &= \mathscr{R}_k'(\tilde{U}, \Gamma) \cup \mathscr{R}_k''(\tilde{U}, \Gamma),\end{aligned}$$

where $\mathscr{R}_k'(\tilde{U}, \Gamma)$ and $\mathscr{R}_k''(\tilde{U}, \Gamma)$ as well as $\mathscr{P}_W'(\Gamma)$ and $\mathscr{P}_W''(\Gamma)$ are disjoint sets. Therefore, we can write:

$$(16) \qquad \sum_{S \in \mathscr{R}_k(\tilde{U}, \Gamma)} |\Gamma^S| = \sum_{S' \in \mathscr{R}_k'(\tilde{U}, \Gamma)} |\Gamma^{S'}| + \sum_{S'' \in \mathscr{R}_k''(\tilde{U}, \Gamma)} |\Gamma^{S''}|.$$

However, for any set $S'' \in \mathscr{R}_k''(\tilde{U}, \Gamma)$, we have $|\Gamma^{S''}| = 1$, since

$$\mathscr{R}_k''(\tilde{U}, \Gamma) = \mathscr{P}^k(U) \cap \mathscr{P}_W''(\Gamma) = \{R | R = W(\gamma), \gamma \in \Gamma_{\tilde{G}}\}$$

and, by assumption, Γ is a set of prime graphs. Hence, the following equation holds:

$$(17) \qquad \sum_{k=0}^{K(\tilde{U}, \Gamma)} (-1)^k \sum_{S'' \in \mathscr{R}_k''(\tilde{U}, \Gamma)} |\Gamma^{S''}| = |\Gamma_{\tilde{G}}{}^{\mathrm{e}}| - |\Gamma_{\tilde{G}}{}^{\mathrm{o}}|.$$

Substituting (16) into (15) with (17) taken into account, we obtain (2) so that also Corollary 1 is proved.

Appendix 2. The function H, appearing in (4) and (7), has been defined as follows:

$$H(i, j) = \sum_{k=0}^{j} (k + i)! \binom{j}{k}.$$

Some interesting properties of this function make its computation easier. All statements given here are proved in [**12**]. Let us define

$$C(j) = \sum_{k=0}^{j} D_{k,j} = \sum_{k=0}^{j} k! \binom{j}{k}.$$

For such an integer function $C(j)$, the following recurrence relationship holds:

$$C(j) = 1 + jC(j - 1); \qquad C(0) = 1.$$

Now, for $H(i, j)$, we have:

$$\begin{aligned} H(i, 0) &= i!, \\ H(0, j) &= C(j), \\ H(i, j) &= H(i+1, j-1) + H(i, j-1). \end{aligned}$$

Note that the last equation may also be written in the following way:

$$H(i, j) = H(i-1, j+1) - H(i-1, j).$$

Using these formulas we can quickly calculate Table I.

Appendix 3. Let S be a set of links contained in $\mathscr{R}_k'(\bar{U}, \Psi)$; then, from definitions, $G_S = (X, S)$ is always a subroute of some open routes between the two fixed vertices (extreme vertices) of G_n. Specifically, $|\Psi^S|$ denotes the number of open routes of Ψ passing through all the links of S. Now, T_S denotes the set of the free vertices of G_S, i.e. the set of those vertices of G_S which are neither a fixed extreme of G_n nor adjacent to any link of S. There is no difficulty in showing that

$$|T_S| = n - k - p - 2,$$

where $n = |X|$, $k = |S|$, and p is the number of free components appearing in G_S. The proof is tedious and will not be presented here. On the other hand, a similar proof may be found in [**5**].

First, oriented graphs, namely flow diagrams, are considered. All open routes of Ψ^S, passing through the same subset of T_S consisting of j free vertices of G_S, differ from one another only for the order in which the free vertices and the free components of G_S are joined (permutations of $j + p$ elements). Since j free vertices may be chosen in any way among $n - k - p - 2$ (combinations of class j of $n - k - p - 2$ elements), it follows that

$$(18) \quad |\Psi^S| = \sum_{s=0}^{n-k-p-2} (s+p)! \binom{n-k-p-2}{s} = H(p, n-k-p-2).$$

The same line of reasoning may be followed for networks, but a further argument must be supplied. In fact, every free component of G_S can now be covered both in one and its opposite direction. Therefore, when j free vertices have been selected and the order of these vertices and of the p free components of G_S has been specified, 2^p possible distinct chains of Ψ^S remain which must be counted. Thus, for networks, (18) becomes:

$$(19) \quad |\Psi^S| = \sum_{s=0}^{n-k-p-2} 2^p (s+p)! \binom{n-k-p-2}{s} = 2^p H(p, n-k-p-2).$$

Now, this and (18) are equivalent to (4).

Substituting (4) into (3) and taking into account (7), we see that (8) can be directly derived; in fact, only a suitable classification (with respect to p) of the elements of $\mathscr{R}_k'(\bar{U}, \Psi)$ has been performed.

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Proof of Theorem 3 *and Corollary* 3. Let us consider, now, the closed routes (Theorem 3). In what follows, our main concern will be with flow diagrams, since the corresponding formulas for networks can easily be drawn by following a line of reasoning which is quite similar to the very simple one we used above, dealing with open routes. Since closed routes do not have any extreme vertices, the number of free vertices of G_S is given by

$$|T_S| = n - k - q \quad \text{for all } S \in \mathscr{R}_k'(\bar{U}, \Phi),$$

where q is the number of components appearing in G_S. First of all, the case $k = 0$ will be considered. In order to construct the formula for $F_0(n) = |\Phi|$ given in Table II, we note that all circuits contained in Φ have a length $s \geqq 2$ and that all circuits of length s differ from one another only in the order in which its s vertices are arranged in a closed sequence (permutations of $s - 1$ elements). Since s vertices may be chosen freely among n in $\binom{n}{s}$ distinct ways, it follows that:

$$F_0(n) = \sum_{s=2}^{n} (s-1)!\binom{n}{s} = \sum_{s=0}^{n} (s-1)!\binom{n}{s} + 1 - n = H(-1, n) + 1 - n.$$

Now, the case $k = 1$ will be dealt with. Note that all the circuits of Φ^S, $|S| = 1$, meeting with a specified set of s free vertices of G_S, differ from one another only in the order in which the s free vertices are arranged with respect to the only component ($q = 1$) of G_S. Since s free vertices may be chosen among $n - 2$ in $\binom{n-2}{s}$ distinct ways, we have:

$$F_1(n) = \sum_{s=0}^{n-2} s!\binom{n-2}{s} = H(0, n-2).$$

When the general case is considered ($S \in \mathscr{R}_k'(\bar{U}, \Phi)$), it must be pointed out that all circuits of Φ^S meeting with a specified set of s free vertices of G_S differ from one another only in the order in which the free vertices and the components of G_S are arranged in a closed sequence (permutations of $s + q - 1$ elements). Since s free vertices may be chosen among $n - k - q$ in $\binom{n-k-q}{s}$ distinct ways, it follows that:

$$|\Phi^S| = \sum_{s=0}^{n-k-q} (s+q-1)!\binom{n-k-q}{s} = H(q-1, n-k-q).$$

This completes the proof of Theorem 3, for flow diagrams. A different proof of the corresponding formulas which are valid for networks may be found in [**5**].

Finally, Corollary 3 can be obtained from Theorem 3 simply by substituting (9) in (10) and classifying, with respect to q, the elements of $\mathscr{R}_k'(\bar{U}, \Phi)$.

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