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SOME INEQUALITIES FOR THE NUMERICAL RADIUS FOR HILBERT SPACE OPERATORS

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Abstract

We introduce some new refinements of numerical radius inequalities for Hilbert space invertible operators. More precisely, we prove that if $T \in \mathcal{B}(\mathcal{H})$ is an invertible operator, then $\|T\| \leq \sqrt{2}\omega(T)$.

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1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and let $\mathcal{B}^{-1}(\mathcal{H})$ denote the set of all invertible operators in $\mathcal{B}(\mathcal{H})$. For $T \in \mathcal{B}(\mathcal{H})$, let

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$$

and

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\},$$

respectively, denote the numerical radius and operator norm of T. It is well known that $\omega(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$ and that, for all $T \in \mathcal{B}(\mathcal{H})$,

$$\omega(T) \leq \|T\| \leq 2\omega(T). \tag{1.1}$$

In [1], Berger proved that for any $T \in \mathcal{B}(\mathcal{H})$ and natural number n,

$$\omega(T^n) \leq \omega^n(T).$$

Also, Holbrook in [6] showed that, for any $A, B \in \mathcal{B}(\mathcal{H})$,

$$\omega(AB) \leq 4\omega(A)\omega(B). \tag{1.2}$$

In the case $AB = BA$,

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

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If A and B are operators in $\mathcal{B}(\mathcal{H})$, we write the direct sum $A \oplus B$ for the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathcal{H} \oplus \mathcal{H}$. Thus,

$$\|A \oplus B\| = \max\{\|A\|, \|B\|\} \tag{1.3}$$

and

$$\omega(A \oplus B) = \max\{\omega(A), \omega(B)\}. \tag{1.4}$$

The following result from [5] may be stated as well: if T is normal, then

$$\|T^n\| = \|T\|^n \quad (n \in \mathbb{N})$$

and

$$\omega(T) = \|T\|. \tag{1.5}$$

In [3], Dragomir has shown that if $T \in \mathcal{B}(\mathcal{H})$, $s \in \mathbb{C} - \{0\}$, $r \in \mathbb{R}$ are such that $\|T - sI\| \le r$, then

$$\sqrt{1 - \frac{r^2}{|s|^2}}\,\|T\| \le \omega(T) \quad (\text{for } r < |s|). \tag{1.6}$$

In Section 2, we establish a considerable improvement of inequalities (1.1) and (1.2). Also, for $T \in \mathcal{B}(\mathcal{H})$, we find an upper bound for $\omega^2(T) - \omega(T^2)$ and consider some further inequalities for invertible operators.

2. Main results

In order to derive our main results, we need the following lemma.

Lemma 2.1. *Let $\mathcal{H}$ be a Hilbert space. If $a, b \in \mathcal{H}$ and $t \in \mathbb{R}$, then*

$$\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \le \|a\|^2\|b - ta\|^2. \tag{2.1}$$

Proof. Since $|\mathrm{Re}\langle a, b\rangle| \le |\langle a, b\rangle|$, the discriminant of the quadratic polynomial

$$q(t) = \|a\|^4 t^2 - 2\mathrm{Re}\langle a, b\rangle\|a\|^2 t + |\langle a, b\rangle|^2$$

is not positive. This implies that $q(t) \ge 0$ for all $t \in \mathbb{R}$. Hence,

$$\|a\|^2\|b\|^2 - |\langle a, b\rangle|^2 \le \|a\|^4 t^2 - 2\mathrm{Re}\langle a, b\rangle\|a\|^2 t + \|a\|^2\|b\|^2 = \|a\|^2\|b - ta\|^2. \qquad \square$$

Now we are in a position to give a new proof for the inequality (1.6).

Theorem 2.2. *If $T \in \mathcal{B}(\mathcal{H})$, $\beta \in \mathbb{C} - \{0\}$ and $r \in \mathbb{R}$ are such that $\|T - \beta I\| \le r$, then*

$$\sqrt{1 - \frac{r^2}{|\beta|^2}}\,\|T\| \le \omega(T) \quad (\textit{for } r < |\beta|). \tag{2.2}$$

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PROOF. Suppose that $x \in \mathcal{H}$ with $\|x\| = 1$. Choose $a = Tx, b = \beta x$ in (2.1) to give

$$\|Tx\|^2\|\beta x\|^2 - |\langle Tx, \beta x\rangle|^2 \le \|Tx\|^2\|tTx - \beta x\|^2,$$

whence

$$\|Tx\|^2 - |\langle Tx, x\rangle|^2 \le \|Tx\|^2 \frac{\|tTx - \beta x\|^2}{|\beta|^2}.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ gives

$$\|T\|^2 - \omega^2(T) \le \|T\|^2 \frac{\|tT - \beta I\|^2}{|\beta|^2}.$$

By hypothesis, $\|T - \beta I\| \le r$, so taking $t = 1$ gives

$$\Big(1 - \frac{r^2}{|\beta|^2}\Big)\|T\|^2 \le \omega^2(T). \qquad \square$$

We need the following lemma to give some applications of the inequality (2.2).

LEMMA 2.3 [2]. *If $a, b, e \in \mathcal{H}$ and $\|e\| = 1$, then*

$$2|\langle a, e\rangle\langle e, b\rangle| \le \|a\|\,\|b\| + |\langle a, b\rangle|. \tag{2.3}$$

THEOREM 2.4. *If $T \in \mathcal{B}(\mathcal{H}), \beta \in \mathbb{C} - \{0\}$ and $r \in \mathbb{R}$ are such that $\|T - \beta I\| \le r$, then*

$$\Big(2 - \frac{|\beta|^2}{|\beta|^2 - r^2}\Big)\omega^2(T) \le \omega(T^2) \quad (\text{for } r < |\beta|).$$

PROOF. Putting $a = Tx, b = T^*x$ and $e = x$, $\|x\| = 1$ in Lemma 2.3 gives

$$2|\langle Tx, x\rangle|^2 \le |\langle T^2x, x\rangle| + \|T^*x\|\,\|Tx\|.$$

Taking the supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ gives

$$2\omega^2(T) \le \omega(T^2) + \|T\|^2 \le \omega(T^2) + \frac{|\beta|^2}{|\beta|^2 - r^2}\omega^2(T)$$

by (2.2). Hence,

$$\Big(2 - \frac{|\beta|^2}{|\beta|^2 - r^2}\Big)\omega^2(T) \le \omega(T^2). \qquad \square$$

We use the following lemma due to Dragomir and Sandor [4] to improve the second inequality (1.1). See also [7] for more information.

LEMMA 2.5 [4]. *If $a, b \in \mathcal{H}$ and $p \ge 2$, then*

$$\|a\|^p + \|b\|^p \le \tfrac{1}{2}(\|a + b\|^p + \|a - b\|^p). \tag{2.4}$$

THEOREM 2.6. *If $T \in \mathcal{B}^{-1}(\mathcal{H})$, $s = \inf_{\|x\|=1} \|Tx\|/\|T\|$ and $p \ge 2$, then*

$$\|T\|^p \le \frac{\|T + T^*\|^p + \|T - T^*\|^p}{2(1 + s^p)} \le \frac{2^p\omega^p(T)}{1 + s^p}.$$

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PROOF. For the first inequality, put $a = Tx$ and $b = T^*x$, where $x \in \mathcal{H}$ and $\|x\| = 1$, in (2.4). Then

$$\|Tx\|^p + \|T^*x\|^p \le \tfrac{1}{2}(\|Tx + T^*x\|^p + \|Tx - T^*x\|^p).$$

Now, by the definition of s,

$$s^p\|T\|^p + \|T^*x\|^p \le \tfrac{1}{2}(\|Tx + T^*x\|^p + \|Tx - T^*x\|^p).$$

Taking the supremum over x,

$$(1 + s^p)\|T\|^p \le \tfrac{1}{2}(\|T + T^*\|^p + \|T - T^*\|^p).$$

For the second inequality, since $(T^* + T)$ and $(T - T^*)$ are normal, (1.5) yields

$$\begin{aligned}\|T + T^*\|^p + \|T - T^*\|^p &= \omega^p(T + T^*) + \omega^p(T - T^*)\\ &\le (\omega(T) + \omega(T^*))^p + (\omega(T) + \omega(T^*))^p = 2^{p+1}\omega^p(T).\end{aligned}$$

Therefore,

$$\|T\|^p \le \frac{\|T + T^*\|^p + \|T - T^*\|^p}{2(1 + s^p)} \le \frac{2^p\omega^p(T)}{1 + s^p}.$$

□

REMARK 2.7. If $T \in \mathcal{B}^{-1}(\mathcal{H})$, $p \ge 2$ and $s = \inf_{\|x\|=1} \|T^*x\|/\|T\|$, employing an argument similar to that used in the proof of Theorem 2.6,

$$\|T\|^p \le \frac{\|T + T^*\|^p + \|T - T^*\|^p}{2(1 + s^p)} \le \frac{2^p\omega^p(T)}{1 + s^p}.$$

REMARK 2.8. If $T \in \mathcal{B}^{-1}(\mathcal{H})$, $p = 2$ and $s = \inf_{\|x\|=1} \|T^*x\|/\|T\|$, the parallelogram law gives

$$\|T\|^2 \le \frac{\|T + T^*\|^2 + \|T - T^*\|^2}{2(1 + s^2)} \le \frac{4\omega^2(T)}{1 + s^2}.$$

COROLLARY 2.9. *For $A, B \in \mathcal{B}^{-1}(\mathcal{H})$, define*

$$\alpha_1 = \inf_{\|x\|=1} \frac{\|Ax\|}{\|A\|}, \quad \alpha_2 = \inf_{\|x\|=1} \frac{\|A^*x\|}{\|A\|}, \quad \beta_1 = \inf_{\|x\|=1} \frac{\|Bx\|}{\|B\|}, \quad \beta_2 = \inf_{\|x\|=1} \frac{\|B^*x\|}{\|B\|}.$$

If $\alpha = \max\{\alpha_1, \alpha_2\}$ and $\beta = \max\{\beta_1, \beta_2\}$, then

$$\|A\| \le \frac{2\omega(A)}{\sqrt[p]{1 + \alpha^p}} \tag{2.5}$$

and

$$\omega(AB) \le \frac{4\omega(A)\omega(B)}{\sqrt[p]{(1 + \alpha^p)(1 + \beta^p)}}. \tag{2.6}$$

PROOF. The inequality (2.5) follows from Theorem 2.6 and Remark 2.7. Similarly,

$$\|B\| \le \frac{2\omega(B)}{\sqrt[p]{1 + \beta^p}}. \tag{2.7}$$

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For the inequality (2.6), observe, using (1.1) in the first inequality and (2.5) and (2.7) in the third, that

$$\begin{aligned}\omega(AB) &\le \|AB\| \\ &\le \|A\|\,\|B\| \le \frac{4\omega(A)\omega(B)}{\sqrt[p]{(1+\alpha^p)(1+\beta^p)}}.\end{aligned}$$ □

The inequalities (2.5) and (2.6) strengthen (1.1) and (1.2), respectively.

Theorem 2.10. *If* $T \in \mathcal{B}^{-1}(\mathcal{H}), \beta \in \mathbb{C} - \{0\}$ *and* $s = \inf_{\|x\|=1} \|Tx\|/\|T\|$, *then*

$$\omega^2(T) - \omega(T^2) \le \inf_{\beta} \frac{\|\beta T \pm T^*\|^2}{1+s^2}.$$

Proof. Put $a = \beta Tx$ and $b = T^*x$, where $x \in \mathcal{H}, \|x\| = 1$, in (2.1). We deduce that

$$\|\beta Tx\|^2\|T^*x\|^2 - |\langle \beta Tx, T^*x\rangle|^2 \le \|\beta Tx\|^2\|\beta Tx - T^*x\|^2.$$

Taking the supremum over $x \in \mathcal{H}, \|x\| = 1$ gives

$$\sup_{\|x\|=1} (\|Tx\|\,\|T^*x\|)^2 \le \omega^2(T^2) + \|T\|^2\|t\beta T - T^*\|^2. \tag{2.8}$$

On the other hand, by (2.3) with $a = Tx, b = T^*x, e = x$,

$$2|\langle Tx, x\rangle|^2 - |\langle T^2x, x\rangle| \le \|Tx\|\,\|T^*x\|$$

and taking the supremum over $x \in \mathcal{H}, \|x\| = 1$ gives

$$2\omega^2(T) - \omega(T^2) \le \sup_{\|x\|=1} (\|Tx\|\,\|T^*x\|).$$

Hence, by (2.8) for $t = 1$,

$$(2\omega^2(T) - \omega(T^2))^2 \le \omega^2(T^2) + \|T\|^2\|\beta T - T^*\|^2 \tag{2.9}$$

and, applying (2.9) and Theorem 2.6,

$$4\omega^4(T) - 4\omega^2(T)\omega(T^2) \le \|T\|^2\|\beta T - T^*\|^2 \le \frac{4\omega^2(T)}{1+s^2}\|\beta T - T^*\|^2.$$

Consequently,

$$\omega^2(T) - \omega(T^2) \le \frac{\|\beta T - T^*\|^2}{1+s^2}$$

and, finally,

$$\omega^2(T) - \omega(T^2) \le \inf_{\beta} \frac{\|\beta T - T^*\|^2}{1+s^2}.$$

Replacing T by iT gives the related inequality

$$\omega^2(T) - \omega(T^2) \le \inf_{\beta} \frac{\|\beta T + T^*\|^2}{1+s^2}.$$ □

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Corollary 2.11. *If $T \in \mathcal{B}^{-1}(\mathcal{H})$, $\alpha, \beta \in \mathbb{C} - \{0\}$, $r \in \mathbb{R}$ are such that $\|T - \alpha I\| \le r$, then*

$$\omega^2(T) - \omega(T^2) \le \frac{|\alpha|^2}{4(|\alpha|^2 - r^2)} \inf_{\beta} \|\beta T \pm T^*\|^2.$$

Proof. By Theorem 2.10,

$$\omega^2(T) - \omega(T^2) \le \frac{\|T\|^2}{4\omega^2(T)} \|\beta T \pm T^*\|^2.$$

From the hypothesis $\|T - \alpha I\| \le r$ and Theorem 2.2,

$$\omega^2(T) - \omega(T^2) \le \frac{|\alpha|^2}{4(|\alpha|^2 - r^2)} \inf_{\beta} \|\beta T \pm T^*\|^2. \qquad \square$$

From Theorem 2.6, we have an interesting result for invertible operators.

Theorem 2.12. *Let $T \in \mathcal{B}^{-1}(\mathcal{H})$. Then $\|T\|^2 \le 2\omega^2(T)$ or $\|T^{-1}\|^2 \le 2\omega^2(T^{-1})$.*

Proof. For any $x \in \mathcal{H}$, $\|x\| = 1$,

$$\frac{1}{\|T^{-1}\|} \le \|Tx\|.$$

Since $(\|T^{-1}\| \, \|T\|)^{-1} \le s = \inf_{\|x\|=1} \|Tx\|/\|T\|$,

$$\|T\|^2 \le \frac{\|T + T^*\|^2 + \|T - T^*\|^2}{2(\|T^{-1}\| \, \|T\|)^{-2} + 1)},$$

by Remark 2.8, and so

$$\frac{1}{\|T^{-1}\|^2} + \|T\|^2 \le \frac{1}{2}(\|T + T^*\|^2 + \|T - T^*\|^2) \le 4\omega^2(T).$$

If $\|T^{-1}\| \le \|T\|$, then

$$\frac{1}{\|T\|^2} + \|T\|^2 \le 4\omega^2(T).$$

Replacing T by $T/\|T\|$ in the last inequality gives

$$\|T\|^2 \le 2\omega^2(T). \tag{2.10}$$

If, on the other hand, $\|T\| \le \|T^{-1}\|$, by replacing T by T^{-1} in (2.10), we deduce the desired result. $\square$

Corollary 2.13. *If $T \in \mathcal{B}^{-1}(\mathcal{H})$, then*

$$\max\{\|T\|, \|T^{-1}\|\} \le \sqrt{2} \max\{\omega(T), \omega(T^{-1})\}.$$

Proof. Let $A = \begin{bmatrix} T & 0 \\ 0 & T^{-1} \end{bmatrix}$, so that $A^{-1} = \begin{bmatrix} T^{-1} & 0 \\ 0 & T \end{bmatrix}$. By Theorem 2.12,

$$\|A\| \le \sqrt{2}\omega(A) \quad \text{or} \quad \|A^{-1}\| \le \sqrt{2}\omega(A^{-1}).$$

Since $\|A\| = \|A^{-1}\|$ and $\omega(A) = \omega(A^{-1})$, the result follows from (1.3) and (1.4). $\square$

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The following theorem is a considerable improvement of the second inequality in (1.1) for Hilbert space invertible operators.

THEOREM 2.14. *If* $T \in \mathcal{B}^{-1}(\mathcal{H})$, *then*

$$\|T\| \leq \sqrt{2}\omega(T).$$

PROOF. First we show that if $\|T^{-1}\| \leq \frac{1}{4}$, then $\|T\| \leq \sqrt{2}\omega(T)$. By (1.2),

$$\omega(TT^{-1}) \leq 4\omega(T)\omega(T^{-1})$$

and so

$$\tfrac{1}{4} \leq \omega(T)\omega(T^{-1}).$$

Since $\omega(T^{-1}) \leq \frac{1}{4}$, it follows that $\omega(T^{-1}) \leq \omega(T)$ and, by Corollary 2.13,

$$\|T\| \leq \sqrt{2}\omega(T).$$

Now take $T \in \mathcal{B}^{-1}(\mathcal{H})$ and put $A = 4\|T^{-1}\|T$. Since $\|A^{-1}\| = \frac{1}{4}$,

$$\|A\| \leq \sqrt{2}\omega(A),$$

which leads to

$$\big\|4\|T^{-1}\|T\big\| \leq \sqrt{2}\omega(4\|T^{-1}\|T)$$

and the result follows from the fact that $\omega(\cdot)$ is a norm. □

COROLLARY 2.15. *If* $A, B \in \mathcal{B}^{-1}(\mathcal{H})$, *then*

$$\omega(AB) \leq 2\omega(A)\omega(B).$$

PROOF. By Theorem 2.14,

$$\|A\| \leq \sqrt{2}\omega(A)$$

and also

$$\|B\| \leq \sqrt{2}\omega(B).$$

Therefore,

$$\omega(AB) \leq \|A\|\,\|B\| \leq 2\omega(A)\omega(B),$$

which is exactly the desired result. □

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