Can. J. Math., Vol. XXXVII, No. 1, 1985, pp. 107-121

GEOMETRIC AND TOPOLOGICAL PROPERTIES OF CERTAIN w^* COMPACT CONVEX SETS WHICH ARISE FROM THE STUDY OF INVARIANT MEANS

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Introduction. Let E be a Banach space, A a subset of its dual E^*. $x_0 \in A$ is said to be a w^*G_δ point of A if there are $x_n \in E$ and scalars γ_n, $n = 1, 2, 3 \dots$ such that

$$\{x_0^*\} = \{x^* \in A; x^*(x_n) = \alpha_n \text{ for all } n\}.$$

Denote by $w^*G_\delta\{A\}$ the set of all w^*G_δ points of A.

If S is a semigroup of maps on E^* and $K \subset E^*$, denote by

$$F_K = \{x^* \in w^* \text{ cl } K; Sx^* = x^*\}$$

i.e., the set of points x^* in the w^*closure of K which are fixed points of S (i.e., $sx^* = x^*$ for each s in $S\}$. An operator will mean a bounded linear map on a Banach space and Co B will denote the convex hull of $B \subset E$.

We introduce hereby the following property for semigroups S:

Definition. We say that the semigroup *S has the w^*G_δ sequential property* if whenever a homomorphic image S' of S acts as a semigroup of w^* continuous operators on the dual E^* of any Banach space E, such that $S'K \subset K$ for some bounded convex $K \subset E^*$ then

$$w^*G_\delta\{F'_K\} \subset w^* \text{ seq cl } K.$$

Here

$$F'_K = \{x^* \in w^* \text{ cl } K; S'x^* = x^*\}$$

and w^* seq cl K (the w^*sequential closure of K) is as usual the set of $x^* \in E^*$ such that some sequence x_n^* in K satisfies

$$w^* \lim_{n\to\infty} x_n^* = x^*.$$

We have proved in part of Corollary 2.1 of **[4]** p. 29, in slightly different terminology, the following:

Received November 3, 1983. Part of the results obtained in this paper were contained in the lecture delivered by this author at the Conference on the Geometry of Normed Linear spaces in honor of Mahlon M. Day held in June 1983 at the University of Illinois. The results are dedicated to Mahlon M. Day with much gratitude and respect.

THEOREM 1. *Countable left-amenable semigroups have the w^*G_δ sequential property.*

The reason for making the above definition is the following proposition which implies that the w^*G_δ sequential property characterizes amenability in the class of countable groups. It is an application of a recent result of Losert and Rindler [**9**] and J. Rosenblatt [**12**].

PROPOSITION 2. *Let S be a nonamenable group. Then there exists a nonatomic probability space $(X, \mathscr{B}, p)$ on which S acts ergodically as a group of measure preserving maps (i.e., "nicely") such that if*

$$K = \mathrm{Co}\,\{0 \neq \psi \in L^\infty(X)^*,\ \psi \textit{ is multiplicative}\}$$

then

$$w^*G_\delta\{F_K\} \neq \emptyset \quad \textit{but} \quad w^*G_\delta\{F_K\} \cap w^* \text{ seq cl } K = \emptyset.$$

Hence S does not have the w^*G_δ sequential property.

The unifying thread of all the theorems in this paper is that they are applications of Theorem 1 and that they characterize amenability for countable groups.

We need some notation before we state them.

If A is a Banach algebra then

$$\Delta_A = \{0 \neq \phi \in A^*;\ \phi \text{ is multiplicative}\}.$$

We will assume that the Banach space E is canonically imbedded in E^{**}. $\sim$ denotes set theoretical difference and

$$E^{**} \sim E = \{\phi \in E^{**};\ \phi \notin E\}.$$

If $B \subset E^*$ then the set $B_0 \subset B$ is a w^*G_δ section of B if there are x_n in E and scalars α_n, $n = 1, 2, 3, \ldots$ such that

$$B_0 = \{b^* \in B;\ b^*(x_n) = \alpha_n \text{ for all } n\}.$$

If $a \in S$, $\delta_a \in l^1(S)$ denotes the point mass at a.

The following is Theorem 2.1 of [**12**]:

THEOREM. *Let S be a countable amenable semigroup of measure preserving maps acting ergodically on the nonatomic probability space $(X, \mathscr{B}, p)$. Then there exists an S invariant mean I on $L^\infty(X, \mathscr{B}, p)$ such that $I \neq p$. Necessarily*

$$I \in L^\infty(X)^* \sim L^1(X).$$

We note that any ϕ in $L^\infty(X)^*$ such that $\phi \geqq 0$ and $\phi(1) = 1$ is called a *mean*. (Analogously for means on function algebras.)

We apply our Theorem 1 and obtain the following improvement of Rosenblatt's theorem:

THEOREM 3. *Let $(X\mathscr{B}\mu)$ be a σ-finite nonatomic measure space and S a countable right amenable semigroup of operators*

$$s:L^{\infty}(X) = L^{\infty} \to L^{\infty}.$$

Let $M \subset \Delta_{L^{\infty}}$ be nonvoid such that $S^*M \subset M$ and let $K = \text{Co } M$. Then the set

$$F_K = \{\psi \in w^* \text{ cl } K; S^*\psi = \psi\}$$

and in fact any nonvoid w^*G_{δ} section F_K^0 of F_K satisfies

$$F_K^0 \cap \{L^{\infty}(X)^* \sim L^1(X)\} \neq \emptyset.$$

If S is any nonamenable group then S acts "nicely" on some nonatomic $(X, \mathscr{B}, p)$ yet $F_K \subset L^1(X)$ for $K = \text{Co } \Delta_{L^{\infty}}$.

Theorem 3 does not assert that the set F_K is big. It happens even for countable abelian semigroups S that S acts on some nonatomic $(X, \mathscr{B}, \mu)$ yet card $F_K = 1$ for some K as above. An easy such example is given after Theorem 3.

A convex subset K of E has the Radon Nikodym property (RNP) if for any finite measure space $(X, \mathscr{B}, \mu)$ any countably additive μ-continuous $m:\mathscr{B} \to E$ of bounded variation such that

$$\mu(A)^{-1}m(A) \in K \quad \text{whenever } \mu(A) \neq 0$$

is represented by a Bochner integrable function (see [**17**], p. 508).

If this can be done only with Pettis integrable functions then the set K is said to have the weak RNP (WRNP) (see for example [**14**]).

Our next results are concerned with the question of when does F_K, or some w^*G_{δ} section F_K^0 of F_K have the RNP (WRNP).

If S acts on $L^{\infty}(X, \mathscr{B}, \mu) = L^{\infty}$ and some finite set $M_0 \subset \Delta_{L^{\infty}}$ is S^* invariant and if $K_0 = \text{Co } M_0$ then clearly F_{K_0} is finite dimensional and afortiori has the RNP. We show that in a certain sense the converse is true provided S is a countable right amenable semigroup. We have:

THEOREM 4. *Let $(X\mathscr{B}\mu)$ be a σ-finite measure space, S a right amenable countable semigroup of bounded linear maps*

$$s:L^{\infty}(X) = L^{\infty} \to L^{\infty}.$$

Let $M \subset \Delta_{L^{\infty}}$ be such that $S^*M \subset M$ and let $K = \text{Co } M$.

If some nonvoid w^*G_{δ} section F_K^0 of F_K has the RNP *then there is some finite subset M_0 of M such that $S^*M_0 \subset M_0$.*

If S is any nonamenable group then there is some nonatomic $(X\mathscr{B}p)$ on which S acts "nicely" such that if $K = \text{Co } \Delta_{L^{\infty}}$ then F_K has the RNP *yet for no finite subset M_0 of M does $S^*M_0 \subset M_0$ hold.*

In the end we prove a proposition which reduces the WRNP case to the RNP case in case S is a semigroup of positive operators

$$s:L^{\infty}(X) \to L^{\infty}(X)$$

such that $s1 = 1$ for all s. This implies then the following.

COROLLARY 6. *Let $(X, \mathscr{B}, \mu)$ be σ-finite S a right amenable semigroup of positive operators*:

$$s:L^{\infty}(X) = L^{\infty} \to L^{\infty}$$

such that $s1 = 1$ for all s. Let $M \subset \Delta_{L^{\infty}}$ be such that $S^*M \subset M$ and $K = \mathrm{Co}\ M$. If F_K has the WRNP *then there is some finite $M_0 \subset M$ such that $S^*M_0 \subset M_0$.*

We have proved in [**4**] p. 25 much more than Theorem 1 above, and the proof in [**4**] is quite involved. We give hence, at the end of the paper, a direct simpler proof of Theorem 1.

1. The main results. We state for the purpose of applications a paraphrased version of our Corollary 2.1 of [**4**] p. 29:

THEOREM 1. *Countable left amenable semigroups have the w^*G_{δ}-sequential property.*

We postpone a direct proof to the end of the paper.

We apply in the next proposition an interesting result of Losert and Rindler [**9**] and J. Rosenblatt [**12**] and show that any nonamenable group S does not have the w^*G_{δ}-sequential property. Hence, the w^*G_{δ}-sequential property characterizes amenability in the class of countable groups.

PROPOSITION 2. *Let S be a nonamenable group. Then there exists a nonatomic probability space $(X, \mathscr{B}, p)$ on which S acts as a group of measure preserving maps such that if $l_sf(x) = f(sx)$, for s in S and f in $L^{\infty}(X) = L^{\infty}$,*

$$K = \mathrm{Co}(\Delta_{L^{\infty}}) \subset (L^{\infty})^*,$$

$$F_K = \{\psi \in w^* \text{ cl } K;\ l_S^*\psi = \psi\}$$

then

$$w^*G_{\delta}(F_K) \neq \emptyset$$

but

$$w^*G_{\delta}(F_K) \cap w^* \text{ seq cl } K = \emptyset.$$

In particular S does not have the w^*G_{δ}-sequential property.

Proof. There exists by [**9**] and [**12**] a nonatomic probability space $(X, \mathscr{B}, p)$ on which the nonamenable group S acts as a group of measure preserving maps $s:X \to X$ such that there exists a unique S-invariant mean on $L^\infty(X) = L^\infty$. Clearly

$$w^* \text{ cl } K = \{\psi \in L^\infty; \psi(1) = 1 = \|\psi\|\}$$

is the set of means on L^∞ and $F_K = \{p\}$ for this measure space. Trivially we have

$$w^* G_\delta(F_K) = \{p\} = F_K.$$

It is enough hence to show that p is not in w^* seq cl K. Assume that

$$p \in w^* \text{ seq cl } K$$

and let $\psi_n \in K = \text{Co } \Delta_{L^\infty}$ satisfy

$$w^* \lim \psi_n = p.$$

Then, by Grothendieck's theorem see **[2]** p. 156, Corollary 12 and the theorem on p. 179, we have that even w-lim $\psi_n = p$ i.e., $\psi_n \to p$ in $\sigma(L^{\infty *}, L^{\infty **})$. It follows that $p \in$ norm cl K since K is convex. Hence there is a sequence $\Phi_n \in K = \text{Co } \Delta_{L^\infty}$ such that

$$\|\Phi_n - p\|_{L^{\infty *}} \to 0.$$

Let $\delta_1, \ldots, \delta_n \in \Delta_{L^\infty}$ and $\alpha_1, \ldots, \alpha_n$ be scalars. Then since $L^\infty \approx C(\Delta_{L^\infty})$ one gets by a mild Tietze argument that

$$\left\|\sum_1^n \alpha_i \delta_i\right\|_{L^{\infty *}} = \sum_1^n |\alpha_i| = \left\|\sum_1^n \alpha_i \delta_i\right\|_{l^1(\Delta_{L^\infty})}.$$

By extending this isometry we get that

$$E = \left\{\sum_1^\infty \alpha_i \delta_i; \sum_1^\infty |\alpha_i| < \infty, \delta_i \in \Delta_{L^\infty}\right\}$$

is a closed subspace of $L^{\infty *}$ (and that Δ_{L^∞} is a canonical l^1 basis in $L^{\infty *}$). But moreover, the set

$$K_1 = \left\{\sum_1^\infty \alpha_i \delta_i; \alpha_i \geqq 0, \sum^\infty \alpha_i = 1, \delta_i \in \Delta_{L^\infty}\right\}$$

is weakly closed in $l^1(\Delta_{L^\infty})$ (just by pairing with the constant 1 and by one point support, functions in $l^\infty(\Delta_{L^\infty})$) and $\sigma(L^{\infty *}, L^{\infty **})$ restricted to E coincides with the weak topology of $l^1(\Delta_{L^\infty}) \approx E$. It follows hence that there are $\delta_i \in \Delta_{L^\infty}$, $\beta_i > 0$ with $\sum_1^\infty \beta_i = 1$ such that

$$p = \sum_1^\infty \beta_i \delta_i.$$

But then p has to contain atoms. In fact choose $N > 1$ such that

$$\sum_{N+1}^{\infty} \beta_j < \beta_1/2$$

and choose a set $C \subset X$ such that $\delta_1(1_C) = 1$ but $\delta_j(1_C) = 0$ if $2 \leqq i \leqq N$. Clearly $1_C \neq 0 \in L^\infty$. Let $D \subset C$ be in $\mathscr{B}$. If $\delta_1(1_D) = 1$ then $p(1_D) \geqq \beta_1$. If $\delta_1(1_D) = 0$ then

$$p(1_D) \leqq \beta_1/2.$$

Thus $\{p(1_D);\ D \subset C,\ D \in \mathscr{B}\}$ is not convex hence p restricted to $\{D \in \mathscr{B}, D \subset C\}$ contains atoms by Sack's theorem. This contradicts the fact that $(X, \mathscr{B}, p)$ is nonatomic.

Remark. This proposition answers in part our question on p. 29 of [**4**].

We will apply in what follows the w^*G_δ-sequential property to obtain an improvement of the following Theorem 2.1 of J. Rosenblatt [**12**]:

THEOREM 2.1. *Let S be a countable amenable semigroup of measure preserving transformations acting ergodically on the nonatomic probability space $(X, \mathscr{B}, p)$. Then p is not the unique S invariant mean on $L^\infty(X)$.*

Rosenblatt's proof of this theorem is not easy and he needs to prove a measure theoretical version of a theorem of Folner, using techniques developed by I. Namioka. This Theorem 2.1 and more, is in fact an immediate consequence of Corollary 2.1 and the proof of Theorem 4, p. 38 our memoir [**4**]. In fact, these imply that card $M_S \geqq 2^{\aleph_1}$ where M_S denotes the set of S invariant means on $L^\infty(X, \mathscr{B}, p)$ (see also p. 61 of [**4**]).

Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space S a semigroup of nonsingular measurable maps $s:X \to X$. Let $L^\infty(X\mathscr{B}\mu) = L^\infty$, $L^1(X\mathscr{B}\mu) = L^1$ and $l_s:L^\infty \to L^\infty$ be defined by

$$(l_sf)(x) = f(sx).$$

(l_s is the adjoint of the operator $l'_s:L^1 \to L^1$ defined by: if $g \in L^1$ and $\mu_g(A) = \int_A g d\mu$. Then

$$(l'_s\mu_g)(A) = \mu_g(s^{-1}A)$$

for $A \in \mathscr{B}$, $s \in S$.)

The operators $l_s:L^\infty \to L^\infty$ are multiplicative (i.e., $l_s(fg) = (l_sf)(l_sg)$). So are the operators of type $f \to f1_C$ where $C \in \mathscr{B}$. All such multiplicative operators $l:L^\infty \to L^\infty$ satisfy

$$l^*(\Delta_{L^\infty}) \subset \Delta_{L^\infty} \cup \{0\}.$$

Clearly, for all s, t in S, $l_sl_t = l_{ts}$ thus $l_s^*l_t^* = l_{st}^*$. Hence, if S if left amenable l_S is right amenable while l_S^* is again left amenable. We have now the following improvement of Rosenblatt's Theorem 2.1 of [**12**].

THEOREM 3. *Let* $(X, \mathscr{B}, \mu)$ *be a* σ*-finite nonatomic measure space and* S *a right amenable countable semigroup of bounded linear maps*

$$s:L^{\infty}(X) = L^{\infty} \rightarrow L^{\infty}.$$

Let $M \subset \Delta_{L^{\infty}}$ *be non void, such that* $S^*M \subset M$ *and* $K = \mathrm{Co}\, M$. *Then the set*

$$F_K = \{\psi \in w^* \text{ cl } K;\ S^*\psi = \psi\},$$

and in fact any nonvoid w^*G_{δ} *section* F_K^0 *of* F_K *satisfies*

$$F_K^0 \cap \{L^{\infty}(X)^* \sim L^1(X)\} \neq \emptyset.$$

Remarks. (a) The property described in Theorem 3 characterizes amenable groups in the class of countable groups: if S is a nonamenable group let $(X, \mathscr{B}, p)$ be the Losert-Rindler, Rosenblatt nonatomic probability space described in Proposition 2. Then the group l_S acts on $L^{\infty}(X)$. Let $K = \mathrm{Co}\, \Delta_{L^{\infty}}$. Then

$$\{p\} = F_K \subset L^1(X) \quad \text{and} \quad F_K \cap \{L^{\infty^*} \sim L^1\} = \emptyset.$$

(b) Theorem 3 does not assert that F_K is a "big set" if S is left amenable. It may in fact happen that card $F_K = 1$ even if S is abelian and arises from nonsingular point maps on $(X\mathscr{B}\mu)$ (see example at the end of this proof).

Proof. Clearly $F_K \neq \emptyset$ by the Markov-Kakutani-Day fixed point theorem. Let $f_n \in L^{\infty}$ and α_n be scalars $n = 1, 2, \ldots$ and assume that

$$F_K^0 \subset L^1(X) = L^1 \quad \text{where } f_n, \alpha_n \text{ determine } F_K^0.$$

We show then that $(X\mathscr{B}\mu)$ contain atoms, which cannot be. If $F_K^0 \subset L^1$ then F_K^0 is a weakly (i.e., $\sigma(L^1, L^{\infty})$) compact subset of L^1 and as such has to contain an exposed point, by a theorem of Amir and Lindenstrauss ([**8**] Theorem 6.4, p. 267). Hence there is some $f_0 \in L^{\infty}(X)$ and a scalar α_0 and some $\psi_0 \in F_K^0$ such that

$$\{\psi_0\} = \{\psi \in F_K^0;\ \psi f_0 = \alpha_0\} = \{\psi \in F_K;\ \psi f_n = \alpha_n \text{ for all } n \geqq 0\}.$$

Hence $\psi_0 \in w^*G_{\delta}(F_K)$. But the countable semigroup S is right amenable, hence S^* (which operates on L^{∞^*}) is left amenable. By our Theorem 1, S^* has the w^*G_{δ}-sequential property, thus

$$\psi_0 \in w^* \text{ seq cl } K.$$

Hence there is a sequence $\psi_n \in K = \mathrm{Co}\, M$ such that

$$w^* \lim \psi_n = \psi_0.$$

But the argument used in the proof of Proposition 2 shows that there are then $\beta_j > 0$ with $\sum \beta_j = 1$ and $\delta_j \in M$ such that

$$\psi_0 = \sum_1^\infty \beta_j \delta_j$$

and ψ_0 has to contain atoms. But $\psi_0 \in L^1(X)$ and $(X, \mathscr{B}, \mu)$ is nonatomic, hence ψ_0 does not contain atoms, which finishes the proof.

Remarks. (a) We only need in the above proof that the measure space $(X\mathscr{B}\mu)$ is nonatomic and localisable, i.e., (see [**16**] Theorem 5.1 or [**7**]) that it is a direct sum of pairwise disjoint finite nonatomic measure spaces $\sum \oplus (X_\alpha \mathscr{B}_\alpha \mu_\alpha)$.

(b) We show by an example that the set F_K may contain a unique element and certainly may have the RNP, in marked contrast with the results in [**6**] where analogous sets which arise from sets $K \subset L^1(X)$ with

$$F_K = \{\psi \in w^* \text{ cl } K;\ l_S^{**}\psi = \psi\}$$

do not have even the WRNP.

Example. Let R be the real line with Lebesgue measure λ. Let S be the rationals with the multiplication

$$r \cdot t = \max \{r, t\}.$$

Let S operate on R by $r \cdot x = \max \{r, x\}$. Then S is an extremely amenable abelian semigroup (see [**5**]) and as such has the fixed point property on compacta, by T. Mitchell's theorem (see for example [**5**]). Let

$$l_r;\ L^\infty(R) \to L^\infty(R)$$

be given by

$$(l_r f)x = f(r \cdot x) \quad \text{for } r \in S.$$

If $\psi \in \Delta_{L^\infty(R)}$ then there is some $\psi_0 \in w^* \text{ cl } \{l_S^*\psi\} \subset \Delta_{L^\infty(R)}$ such that

$$l_S^*\psi_0 = \psi_0.$$

If $M = \{\psi_0\}$ then

$$K = \text{Co } M = \{\psi_0\}.$$

Hence

$$F_K = \{\Phi \in w^* \text{ cl } K;\ l_S^*\Phi = \Phi\} = \{\psi_0\} \subset L^\infty(R)^* \sim L^1(R).$$

Our Theorem 3 hence cannot insure that the set F_K is necessarily "big" but just that

$$F_K \subset \{L^{\infty^*} \sim L^1\} \neq \emptyset.$$

This last fact can be put to special use in conjunction with Theorem 2.6 of our paper [**6**].

https://doi.org/10.4153/CJM-1985-009-9 Published online by Cambridge University Press

This example brings us to the following situation. Let $(X\mathscr{B}\mu)$ be a σ-finite measure space and assume that S is a semigroup of maps $s:L^{\infty}(X) \to L^{\infty}(X)$ which leaves invariant some finite set $M_0 \subset \Delta_{L^{\infty}(X)}$, i.e., $s^*M_0 = M_0$ for each s in S. If $K = \text{Co } M_0$ then

$$F_K = \{\psi \in w^* \text{ cl } K;\ S^*\psi = \psi\}$$

is finite dimensional and afortiori has the RNP. We prove in the next theorem that whenever a countable right amenable semigroup S is such that $S^*M \subset M$ for some $M \subset \Delta_{L^\infty}$ and some w^*G_δ-section of

$$F_K = \{\psi \in w^* \text{ cl } K;\ S^*\psi = \psi\}$$

has the RNP (where $K = \text{Co } M$) then there exists some finite set $M_0 \subset M$ such that $S^*M_0 \subset M_0$. Furthermore, we show that this "RNP-finite invariant property" characterizes amenability in the class of countable groups.

THEOREM 4. *Let $(X, \mathscr{B}, \mu)$ be a σ-finite measure space, S a right amenable countable semigroup of bounded linear maps $s:L^{\infty}(X) \to L^{\infty}(X)$. Let $M \subset \Delta_{L^{\infty}(X)}$ be such that $S^*M \subset M$ and let $K =$* Co *M. If some nonvoid w^*G_δ-section F_K^0 of*

$$F_K = \{\psi \in w^* \text{ cl } K,\ S^*\psi = \psi\}$$

has the RNP *then there is some finite subset M_0 of M such that $s^*M_0 = M_0$ for each $s \in S$.*

Remarks. (a) If S is any nonamenable group let $(X, \mathscr{B}, p)$ be the Losert-Rindler Rosenblatt nonatomic probability space on which S acts and for which p is the unique S-invariant mean on $L^{\infty}(X)$ (see Proposition 2). Let $M = \Delta_{L^{\infty}(X)}$, $K = \text{Co } M$. Then $F_K = \{p\}$ has the RNP but no finite subset M_0 of M satisfies $l_S^*M_0 \subset M_0$. Since such M_0 would imply that

$$p = k^{-1}\sum_{1}^{k} \delta_i \quad \text{for some } \{\delta_1, \ldots, \delta_k\} \subset M_0$$

i.e., that $\{p(A);\ A \in \mathscr{B}\}$ is finite. This contradicts the fact that p is nonatomic.

(b) The assumptions of Theorem 4 imply that if for each $\delta \in M$, the orbit $\{S^*\delta\}$ is infinite, then no nonvoid w^*G_δ-section F_K^0 of F_K can have the RNP.

(c) Norm separable w^* compact convex sets have the RNP (see **[17]** Proposition 1.10).

Proof. Assume that

$$F_K^0 = \{\psi \in F_K;\ \psi f_n = \alpha_n \text{ for } n \geqq 1\}$$

has the RNP. Then F_K^0 has a w^*G_δ point (see Lemma 9, p. 13 of [**11**]). For completeness here is a short proof: If F_K^0 has the RNP then Corollary 1.17 of [**17**], p. 512 implies that F_K^0 is affinely homeomorphic in the w^* topologies to a w^* compact convex subset K_1 of a dual Banach space Z^* which has the RNP. Hence Z is an Asplund space ([**17**], Theorem 2.8) and hence K_1 has w^* strongly exposed points ([**2**], p. 213). If $k_0 \in K_1$ is such, then k_0 is a w^*G_δ point of K_1. Hence its inverse image ψ_0 in F_K^0 (by the above w^*-w^* homeomorphism) is a w^*G_δ point of F_K^0 (this being a topological concept).

It follows that by adding a sequence $f'_n \in L^\infty(X)$ and scalars a'_n we can assume that

$$\{\psi_0\} = \{\psi \in F_K^0;\ \psi(f'_n) = \alpha'_n \text{ for } n \geqq 1\}.$$

However S^* is left amenable, hence has the w^*G_δ-sequential property. Hence there is a sequence $\psi_n \in \text{Co } M$ such that

$$w^* \lim \psi_n = \psi_0.$$

It follows (as in the proof of Proposition 2) that there are $c_n > 0$ with $\sum c_n = 1$ and $\delta_n \in M$ such that

$$\psi_0 = \sum_1^\infty c_n\delta_n \quad \text{and} \quad \delta_n \neq \delta_m \quad \text{if } n \neq m.$$

That same argument shows that the norm closed subspace $L(M)$ generated by M is isomorphic to $l^1(M)$. By the Hahn-Banach theorem, for any $f \in l^\infty(M)$ there is some $\Phi \in L^\infty(X)^{**}$ such that

$$\Phi\left(\sum_1^\infty \alpha_i\delta_i\right) = \sum_1^\infty \alpha_i f(\delta_i) \text{ whenever } \sum|\alpha_i| < \infty \text{ and } \delta_i \in M.$$

Assume that we rearrange the above c_i (which appear in ψ_0) such that

$$c_1 = c_2 = \ldots = c_N > c_i \quad \text{if } i \geqq N + 1.$$

Since $s^*\psi_0 = \psi_0$ we have

$$\sum_1^\infty c_n\delta_n = \sum_1^\infty c_n s^*(\delta_n) \quad \text{and} \quad s^*\delta_i \in M.$$

We claim that the set $M_0 = \{\delta_1, \ldots, \delta_N\} \subset M$ satisfies

$$s^*M_0 = M_0 \text{ for all } s \text{ in } S.$$

In fact assume that $i, j \leqq N$ are such that

$$s^*\delta_i = s^*\delta_j = \delta_k.$$

Let then $\Phi \in L^\infty(X)^{**}$ satisfy

$$\langle \Phi, \delta_k \rangle = 1 \text{ and } \Phi(\delta) = 0 \quad \text{if } \delta \neq \delta_k, \delta \in M.$$

Then

$$\langle \Phi, s^*\psi_0 \rangle \geqq c_i + c_j.$$

However $\langle \Phi, \psi_0 \rangle = c_k$. Thus $c_k \geqq c_i + c_j$. This implies that $i = j$ and s^* is one to one on M_0 and since $c_k \geqq c_i$ it follows that $k \leqq N$ i.e.,

$$s^*\psi_i \in M_0 \quad \text{for each } i \leqq N.$$

Since M_0 is finite $s^*M_0 = M_0$ for all s^* in S^*, which finishes the proof.

PROPOSITION 5. *Let S be a set of positive linear operators $s:L^\infty(X) \to L^\infty(X)$ with $s1 = 1$ where $(X\mathscr{B}\mu)$ is σ-finite (or localisable). Let $M \subset \Delta_{L^\infty} = \Delta$ be such that $S^*M \subset M$. Let $K =$ Co M. If the set*

$$F_K = \{\psi \in w^* \text{ cl } K; S^*\psi = \psi\}$$

has the WRNP *then it has the* RNP.

Proof. Identify $L^\infty(X) \approx C(\Delta) = C$ where $\Delta = \Delta_{L^\infty}$. Let

$$M_1 = w^* \text{ cl } M.$$

If $\psi \in C^*$, let μ_ψ be the Borel measure on Δ corresponding to ψ by the Riesz representation theorem.

We show at first that

$$L = \{\psi \in C^*(\Delta); S^*\psi = \psi, \mu_{|\psi|}(M_1) = |\psi|(1)\}$$

is a w^*-closed subspace of C^* which is closed under the lattice operations. Here

$$|\psi|(f) = \sup\{|\psi(g)|; 0 \leqq |g| \leqq f\} \quad \text{if } 0 \leqq f \in C$$

(see [**15**], Corollary 1, p. 72). Clearly $\mu_{|\psi|}(M_1) = |\psi|(1)$ is equivalent to $\mu_{|\psi|}(U) = 0$ where $U = \Delta \sim M_1$ is open.

Let

$$B_U = \{g \in C; \text{supp } g \subset U, |g| \leqq 1\}$$

where supp $g = \text{cl}\{x; g(x) \neq 0\}$. Then

$$\mu_{|\psi|}(U) = \sup\{|\psi|(f); 0 \leqq f \leqq 1, \text{supp} f \subset U\},$$

see [**13**]. It follows that

$$L = \{\psi \in C^*; S^*\psi = \psi, \psi(g) = 0 \text{ if } g \in B_U\}$$

and L is clearly a w^*-closed linear subspace of C^*. Fix now some ψ in L. Then

$$0 = |\psi|(|g|) \geqq ||\psi|(g)| \quad \text{for all } g \in B_U.$$

https://doi.org/10.4153/CJM-1985-009-9 Published online by Cambridge University Press

Hence

$$\psi^+(g) = \frac{1}{2}(|\psi| + \psi)(g) = 0 \quad \text{and}$$

$$\psi^-(g) = \frac{1}{2}(|\psi| - \psi)(g) = 0 \text{ for all } g \in B_U$$

(and $\psi^+ = \psi \vee 0$, $\psi^- = (-\psi) \vee 0$, [**15**], p. 72).

We still have to show that $s^*\psi^+ = \psi^+$, $s^*\psi^- = \psi^-$, and here we use an idea of Namioka: $\psi^+ \geqq \psi$ hence

$$s^*\psi^+ \geqq s^*\psi = \psi.$$

Since $s^*\psi^+ \geqq 0$ we have

$$s^*\psi^+ \geqq \psi \vee 0 = \psi^+$$

thus $s^*\psi^+ - \psi^+ \geqq 0$. But then

$$\|s^*\psi^+ - \psi^+\| = (s^*\psi^+ - \psi^+)(1) = \psi^+(1) - \psi^+(1) = 0.$$

We have shown that if ψ is in L so are ψ^+ and ψ^-.

Assume now that F_K has the WRNP. The canonical identification $L^\infty(X) \approx C(\Delta) = C$ identifies (see [**4**], Lemma 4, p. 31)

$$F_K \approx L_1^+ = \{0 \leqq \psi \in L; \psi(1) = 1\}$$
$$= \{0 \leqq \psi \in C^*; \psi(1) = 1, \mu_\psi(U) = 0, S^*\psi = \psi\}.$$

Thus L_1^+, the positive face of the unit sphere of L, has the WRNP and hence so does $L_1^+ - L_1^+$ by [**14**], Theorem 1(i). It follows that $\{\psi \in L; \|\psi\| \leqq 1\}$ has the WRNP i.e., by definition L has the WRNP. However, L is a Banach lattice and for such the WRNP implies the RNP by Proposition 8 of [**3**]. Thus L_1^+ has the RNP and so does F_K. Theorem 4 and Proposition 5 yield now

COROLLARY 6. *Let $(X\mathscr{B}\mu)$ be σ-finite S a right amenable countable semigroup of operators*

$$s:L^\infty(X) \to L^\infty(X) = L^\infty,$$

such that $s \geqq 0$ $s1 = 1$. Let $M \subset \Delta_{L^\infty}$ be such that $S^*M \subset M$ and let $K = \mathrm{Co}\, M$. If

$$F_K = \{\psi \in w^* \text{ cl } K; S^*\psi = \psi\}$$

has the WRNP *then there is some finite set $M_0 \subset M$ such that $s^*M_0 = M_0$ for all s in S.*

We will give now a self-contained

Proof of Theorem 1. Let E be a Banach space. Let S be a countable left amenable semigroup of linear maps $s:E^* \to E$ which are w^*-w^* continuous and let $K \subset E^*$ be bounded convex and such that $sK \subset K$ for each s in S. Let

$$F_K = \{y \in w^* \text{ cl } K;\ Sy = y\}$$

and let

$$y_0 \in w^*G_\delta(F_K).$$

Let $x_n \in E$, α_n be scalars such that

$$\{y_0\} = \{y \in w^* \text{ cl } K : Sy = y,\ y(x_n) = \alpha_n \quad \text{for all } n \geqq 1\}.$$

If $\Phi \in l^1(S)$ has finite support i.e.,

$$\Phi = \sum_1^n \alpha_i \delta_{s_i}$$

with $s_i \in S$, $\delta_{s_i} \neq \delta_{s_j}$ if $i \neq j$, define $T_\Phi : E^* \to E^*$ by

$$T_\Phi y = \sum_1^n \alpha_i s_i y.$$

If

$$M = \sup \{\|y\|;\ y \in K\}$$

then for each y in K we have

$$\|T_\Phi y\| \leqq \sum |\alpha_i| \|S_i y\| \leqq M \sum |\alpha_i| = M\,\|\Phi\|.$$

If

$$\Phi = \sum_1^n \alpha_i \delta_{s_i}$$

and $s \in S$ define

$$l_s \Phi = \sum_1^n \alpha_i \delta_{ss_i}.$$

Clearly $\|l_s\Phi\| \leqq \|\Phi\|$ for all Φ.

S is left amenable, hence there is, by Day's convergence to left invariance theorem [**1**], a sequence $\Phi_n \in \text{Co}\,\{\delta_s;\ s \in S\} \subset l^1(S)$ such that

$$\|l_s\Phi_n - \Phi_n\| \to 0 \quad \text{for each } s \text{ in } S$$

(the countability of S is used). It then follows that

$$\|(sT_{\Phi_n} - T_{\Phi_n})y\| = \|T_{l_s\Phi_n - \Phi_n}(y)\| \leqq \|l_s\Phi_n - \Phi_n\|\, M$$

for each y in w^* cl K. Thus

$$\|(sT_{\Phi_n} - T_{\Phi_n})y\| \to 0 \quad \text{if } n \to \infty$$

uniformly in $y \in w^*$ cl K.

Let v_β be a net in K such that $v_\alpha \to y_0$ in w^*. Then for fixed n and i we have

$$(T_{\Phi_n}v_\alpha)(x_i) \to (T_{\Phi_n}y_0)(x_i) = y_0(x_i) = \alpha_i.$$

Choose hence β_n such that

$$|(T_{\Phi_k}v_{\beta_n})(x_i) - \alpha_i| < \frac{1}{n} \quad \text{if } i, k \leqq n.$$

Then

$$\|sT_{\Phi_n}v_{\beta_n} - T_{\Phi_n}v_{\beta_n}\| = \|T_{l_s\Phi_n - \Phi_n}(v_{\beta_n})\|$$
$$\leqq \|l_s\Phi_n - \Phi_n\|\, M \to 0 \quad \text{if } n \to \infty.$$

Also

$$|(T_{\Phi_n}v_{\beta_n})(x_i) - \alpha_i| \to 0 \quad \text{if } n \to \infty \text{ for each fixed } i.$$

Let $u_n = T_{\Phi_n}v_{\beta_n}$. Then u_n belongs to K. We claim that $u_n \to y_0$ in w^*. If not there is some x_0 in E, $\delta > 0$ and a subsequence u_{n_k} such that

$$|(u_{n_k} - y_0)(x_0)| \geqq \delta > 0 \quad \text{for each } k.$$

And there is a subnet

$$w_\beta = u_{n_{k_\beta}} \to y_1$$

in w^* for some y_1 in E^*. Thus

$$|(y_1 - y_0)(x_0)| \geqq \delta \quad \text{and} \quad y_1 \neq y_0.$$

However $sw_\beta \to sy$ in w^* since each s is w^* continuous. And

$$\|sw_\beta - w_\beta\| \to 0 \quad \text{for each } s \text{ in } S.$$

Hence for each x in E

$$(sy_1 - y_1)(x) = \lim(sw_\beta - w_\beta)(x) = 0.$$

Thus $Sy_1 = y_1$. However $u_n(x_i) \to \alpha_i$ if $n \to \infty$ for each fixed i. Hence this will hold also for the subnet w_β i.e.,

$$w_\beta(x_i) \to \alpha_i \quad \text{for fixed } i.$$

But $w_\beta(x_i) \to y_1(x_i)$ for fixed i. It follows that

$$y_1(x_i) = \alpha_i \quad \text{for each } i \text{ and}$$

$$Sy_1 = y_1 \in w^* \text{ cl } K.$$

Thus

$$y_1 \in \{y \in w^* \text{ cl } K;\ Sy = y, y(x_i) = \alpha_i, i \geqq 1\} = \{y_0\}.$$

This contradicts the fact that $y_1 \neq y_0$. It follows that

$$y_0 = w^* \lim u_n \quad \text{and} \quad y_0 \in w^* \text{ seq cl } K.$$

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