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Interpolation of Morrey Spaces on Metric Measure Spaces

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Abstract. In this article, via the classical complex interpolation method and some interpolation methods traced to Gagliardo, the authors obtain an interpolation theorem for Morrey spaces on quasi-metric measure spaces, which generalizes some known results on $\mathbb{R}^n$.

1 Introduction

In 1938, due to the applications in elliptic partial differential equations, Morrey [21] introduced a class of function spaces, nowadays named after him. In recent years, there is an increasing interest in applications of Morrey spaces in various areas of analysis, such as partial differential equations, potential theory and harmonic analysis; we refer, for example, to [1–4, 18, 20, 22, 23, 33] and their references.

Let $(\mathscr{X}, d, \mu)$ be a *quasi-metric measure space*, which means that $\mathscr{X}$ is a non-empty set, d a *quasi-metric* (that is, for all $x, y, z \in \mathscr{X}$, it holds that $d(x, y) \in [0, \infty)$, $d(x, y) = d(y, x)$, and $d(x, y) \le K[d(x, z) + d(z, y)]$, where $K \in [1, \infty)$ is a constant independent of x, y, z) and μ a *non-negative measure*. Let $0 < p \le u \le \infty$. Recall that the *Morrey space* $\mathcal{M}^u_p(\mathscr{X})$ is defined to be the space of all locally p-integrable functions f on $\mathscr{X}$ such that

$$\|f\|_{\mathcal{M}^u_p(\mathscr{X})} := \sup_{B\subset \mathscr{X}} [\mu(B)]^{1/u-1/p} \left[\int_B |f(x)|^p \, d\mu(x)\right]^{1/p} < \infty, \tag{1.1}$$

where the supremum is taken over all balls in $\mathscr{X}$.

Obviously, $\mathcal{M}^p_p(\mathscr{X}) = L^p(\mathscr{X})$. As a natural generalization of Lebesgue spaces, the interpolation properties of Morrey spaces became an interesting question. The first result on this problem is due to Stampacchia [31] and, independently, Campanato and Murthy [8]: they obtained an interpolation property for linear operators from Lebesgue spaces to Morrey spaces on $\mathbb{R}^n$ and showed that, if a linear operator T is bounded from $L^{q_i}(\mathbb{R}^n)$ to Morrey spaces $\mathcal{M}^{u_i}_{p_i}(\mathbb{R}^n)$ with operator norm M_i, $i \in \{0, 1\}$, then T is also bounded from $L^q(\mathbb{R}^n)$ to $\mathcal{M}^u_p(\mathbb{R}^n)$ when

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$1/q = (1-\theta)/q_0 + \theta/q_1$, $1/u = (1-\theta)/u_0 + \theta/u_1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$ for some $\theta \in (0,1)$ with the operator norm not more than a positive constant multiple of $M_0^{1-\theta} M_1^{\theta}$. In 1969, Peetre [25] found that the previous conclusion still holds true when $\big(L^{q_0}(\mathbb{R}^n), L^{q_1}(\mathbb{R}^n)\big)$ and $L^q(\mathbb{R}^n)$ are replaced, respectively, by a certain abstract pair (A_0, A_1) and an interpolation space A constructed from (A_0, A_1).

However, the converse result is in general not true. In 1995, Ruiz and Vega [27] proved that, when $n \geq 2$, $u \in (0,n)$, $\theta \in (0,1)$ and $1 \leq p_2 < p_3 < \frac{n-1}{u} < p_1 < \infty$, for any given $C \in (0,\infty)$, there exists a positive continuous linear operator $T\colon \mathcal{M}^u_{p_i}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$, $i \in \{1,2,3\}$, with the operator norm satisfying $\|T\|_{\mathcal{M}^u_{p_i}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)} \leq K_i$, $i \in \{1,2\}$, but $\|T\|_{\mathcal{M}^u_{p_3}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)} > C K_1^{1-\theta} K_2^{\theta}$ for $\frac{1}{p_3} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$. This implies the lack of convexity of operators on Morrey spaces. For the case dimension $n = 1$, Blasco, Ruiz and Vega [5] in 1999 proved that, for a particular u, if $1 < p_0 < p_1 < u < \infty$, then there exist $q_0, q_1 \in (1,\infty)$ and a positive continuous linear operator T which is bounded from $\mathcal{M}^u_{p_i}(\mathbb{R})$ to $L^{q_i}(\mathbb{R})$, $i \in \{0,1\}$, but not bounded from $\mathcal{M}^u_p(\mathbb{R})$ to $L^q(\mathbb{R})$, where $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. These counterexamples show that the Morrey spaces have no interpolation property in general.

Nevertheless, under some restriction, the Morrey spaces also have some interpolation properties. Let $1 < p_0 \leq u_0 < \infty$, $1 < p_1 \leq u_1 < \infty$, $\theta \in (0,1)$, $1/u = (1-\theta)/u_0 + \theta/u_1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Recently, in [32], it was proved that, if

$$p_0 u_1 = p_1 u_0, \tag{1.2}$$

then

$$\begin{aligned}[\mathring{\mathcal{M}}^{u_0}_{p_0}(\mathbb{R}^n), \mathring{\mathcal{M}}^{u_1}_{p_1}(\mathbb{R}^n)]_\theta &= [\mathring{\mathcal{M}}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\theta \\ &= [\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathring{\mathcal{M}}^{u_1}_{p_1}(\mathbb{R}^n)]_\theta = \mathring{\mathcal{M}}^{u}_{p}(\mathbb{R}^n),\end{aligned} \tag{1.3}$$

where the space $\mathring{\mathcal{M}}^u_p(\mathbb{R}^n)$ denotes the closure of the Schwartz functions in $\mathcal{M}^u_p(\mathbb{R}^n)$. Also recently, Lemarié–Rieusset [18, Theorem 3(ii)] showed that for p_0, p_1, u_0, u_1, θ, p and u as above,

$$[\mathcal{M}^{u_0}_{p_0}(\mathbb{R}^n), \mathcal{M}^{u_1}_{p_1}(\mathbb{R}^n)]_\theta = \mathcal{M}^u_p(\mathbb{R}^n) \quad \text{if and only if (1.2) holds,}$$

which gives the sufficient and necessary condition ensuring the interpolation property of Morrey spaces on $\mathbb{R}^n$; see [18, Theorem 3].

The main purpose of the present article is to establish the interpolation properties of Morrey spaces on quasi-metric measure spaces $\mathscr{X}$. The used interpolation methods include the $\langle\,\cdot\,\rangle_\theta$-method, the $\pm$ method traced to Gagliardo, and the classical complex interpolation method.

We begin with some basic notation. Let X_0, X_1 be a *couple of quasi-Banach spaces*, which are continuously embedding into a large Hausdorff topological vector space Y.

The *space* $X_0 + X_1$ is defined by

$$X_0 + X_1 := \big\{ h \in Y : \text{there exists } h_i \in X_i,\ i \in \{0,1\}, \text{ such that } h = h_0 + h_1 \big\}, \tag{1.4}$$

and its *norm* is given by

$$\|h\|_{X_0+X_1} := \inf\{\|h_0\|_{X_0} + \|h_1\|_{X_1} : h = h_0 + h_1, h_0 \in X_0 \text{ and } h_1 \in X_1\}. \tag{1.5}$$

Next we recall the definition of complex interpolation (see, for example, [7, 16]). Let X be a quasi-Banach space, $U := \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$, and let $\overline{U}$ be its closure. Here and in what follows, for any $z \in \mathbb{C}$, $\operatorname{Re} z$ denotes its *real part*. A map $f : U \to X$ is said to be *analytic* if, for any given $z_0 \in U$, there exist $\eta \in (0,\infty)$ and $\{h_n\}_{n=0}^{\infty} \subset X$ such that $f(z) = \sum_{n=0}^{\infty} h_n(z - z_0)^n$ for all $z \in U$ is uniformly convergent for $|z - z_0| < \eta$. A quasi-Banach space X is called *analytically convex* if there exists a positive constant C such that, for any analytic function $f : U \to X$ which is continuous on the closed strip $\overline{U}$,

$$\max_{z\in U} \|f(z)\|_X \le C \max_{\operatorname{Re} z\in\{0,1\}} \|f(z)\|_X.$$

Suppose that $X_0 + X_1$ is analytically convex. The *set* $\mathcal{F} := \mathcal{F}(X_0, X_1)$ is defined to be the set of all functions $f : U \to X_0 + X_1$ such that

(i) f is analytic and *bounded* in $X_0 + X_1$, which means that $f(U) := \{f(z) : z \in U\}$ is a bounded subset of $X_0 + X_1$;
(ii) f is extended continuously to the closure $\overline{U}$ of the strip U such that the traces $t \mapsto f(j + it)$ are bounded continuous functions into X_j, $j \in \{0, 1\}$, $t \in \mathbb{R}$.

We endow $\mathcal{F}$ with the *quasi-norm*

$$\|f\|_{\mathcal{F}} := \max\Big\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{X_1}\Big\}.$$

Let X_0, X_1 be two quasi-Banach spaces such that $X_0 + X_1$ is analytically convex. Then the *complex interpolation space* $[X_0, X_1]_\theta$ with $\theta \in (0, 1)$ is defined by

$$[X_0, X_1]_\theta := \{g \in X_0 + X_1 : \text{there exists } f \in \mathcal{F} \text{ such that } f(\theta) = g\},$$

and its *norm* given by $\|g\|_{[X_0,X_1]_\theta} := \inf_{f\in\mathcal{F}}\{\|f\|_{\mathcal{F}} : f(\theta) = g\}$.

Now we turn to some interpolation methods traced to Gagliardo (see, for example, [12]). A quasi-Banach space X is called an *intermediate space* with respect to $X_0 + X_1$ if and only if $X_0 \cap X_1 \subset X \subset X_0 + X_1$ with continuous embeddings. If X is an *intermediate space* with respect to $X_0 + X_1$, let X° be the *closure of* $X_0 \cap X_1$ *in* X. The *Gagliardo closure* of X with respect to $X_0 + X_1$, denoted by $X^\sim$, is defined as follows: $a \in X^\sim$ if and only if there exists a sequence $\{a_i\}_{i\in\mathbb{N}} \subset X$ such that $a_i \to a$ as $i \to \infty$ in $X_0 + X_1$ and $\|a_i\|_X \le \lambda$ for some $\lambda < \infty$ and all $i \in \mathbb{N}$. Moreover, let $\|a\|_{X^\sim} := \inf\{\lambda\}$.

Definition 1.1 Let (X_0, X_1) be a pair of quasi-Banach spaces and let $\theta \in (0, 1)$.

(i) (*The* $\langle\cdot\rangle_\theta$*-method*) We say $a \in \langle X_0, X_1\rangle_\theta$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any bounded sequence

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$\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$, $\sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i$ converges in X_j, $j \in \{0,1\}$. Moreover, for $j \in \{0,1\}$,

$$\Big\| \sum_{i\in\mathbb{Z}} \varepsilon_i 2^{i(j-\theta)} a_i \Big\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for some nonnegative constant C, independent of $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Let $\|a\|_{\langle X_0,X_1\rangle_\theta} := \inf\{C\}$.

(ii) (*The* $\pm$ *method*) We say $a \in \langle X_0, X_1, \theta\rangle$ if there exists a sequence $\{a_i\}_{i\in\mathbb{Z}} \subset X_0 \cap X_1$ such that $a = \sum_{i\in\mathbb{Z}} a_i$ in $X_0 + X_1$ and, for any finite subset $F \subset \mathbb{Z}$ and bounded sequence $\{\varepsilon_i\}_{i\in\mathbb{Z}} \subset \mathbb{C}$, and $j \in \{0,1\}$,

$$\Big\| \sum_{i\in F} \varepsilon_i 2^{i(j-\theta)} a_i \Big\|_{X_j} \le C \sup_{i\in\mathbb{Z}} |\varepsilon_i|$$

for some constant C independent of F, $\{\varepsilon_i\}_{i\in\mathbb{Z}}$ and $\{a_i\}_{i\in\mathbb{Z}}$. Let $\|a\|_{\langle X_0,X_1,\theta\rangle} := \inf\{C\}$.

We remark that the $\langle\cdot\rangle_\theta$-method is a special case of the $\langle\cdot\rangle_\phi$-method introduced in [24, 26], and the $\pm$-method is originally from [13, 14, 24]. Obviously, $\langle X_0, X_1\rangle_\theta \subset \langle X_0, X_1, \theta\rangle$.

Now we formulate the main result of the present article as follows.

Theorem 1.2 *Let $\theta \in (0,1)$, $0 < p_0 \le u_0 \le \infty$, $0 < p_1 \le u_1 \le \infty$ and $0 < p \le u \le \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$. If (1.2) holds true, then*

$$\langle \mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X})\rangle_\theta = (\mathcal{M}^{u}_{p}(\mathscr{X}))^\circ \tag{1.6}$$

and

$$\langle \mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X}), \theta\rangle = \mathcal{M}^{u}_{p}(\mathscr{X}). \tag{1.7}$$

Moreover, when $p_0, p_1 \in [1,\infty]$, it further holds that

$$(\mathcal{M}^{u}_{p}(\mathscr{X}))^\circ = [\mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X})]_\theta.$$

We remark that Theorem 1.2 generalizes the corresponding interpolation result of Morrey spaces in [34] in the case $\mathscr{X} = \mathbb{R}^n$ to any quasi-metric measure space $\mathscr{X}$. The proof of this theorem is given in Section 2. Actually, we prove a more general result in Theorem 2.3 which covers Theorem 1.2. Different from the approach used in [34], wherein the interpolation of Morrey spaces was obtained via establishing the interpolation result for the corresponding sequence spaces with respect to Morrey spaces, the main idea for proving Theorem 1.2 here is to calculate the Calderón product of Morrey spaces themselves and then use a general result of Nilsson [24] on the relation between the Calderón product and the Gagliardo interpolation.

We also point out that, due to the counterexample constructed by Lemarié–Rieusset in [18, Section 6], the condition (1.2) in Theorem 1.2 is also necessary when $\mathscr{X} = \mathbb{R}^n$, $1 < p_0 \le u_0 < \infty$ and $1 < p_1 \le u_1 < \infty$.

As an immediate consequence of (1.7) in Theorem 1.2 and [14, Proposition 6.1], we have the following result. We omit the details.

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Corollary 1.3 *Let θ, p_0, p_1, p, u_0, u_1 and u be as in Theorem 1.2 such that (1.2) holds true, and (A_0, A_1) a couple of quasi-Banach spaces.*

(i) *If a linear operator T is bounded from $\mathcal{M}^{u_j}_{p_j}(\mathscr{X})$ to A_j with operator norms M_j, $j \in \{0,1\}$, then T is also bounded from $\mathcal{M}^u_p(\mathscr{X})$ to $\langle A_0, A_1, \theta\rangle$, and the operator norm is not more than a positive constant multiple of $M_0^{1-\theta} M_1^\theta$.*

(ii) *If a linear operator T is bounded from A_j to $\mathcal{M}^{u_j}_{p_j}(\mathscr{X})$ with operator norms M_j, $j \in \{0,1\}$, then T is also bounded from $\langle A_0, A_1, \theta\rangle$ to $\mathcal{M}^u_p(\mathscr{X})$, and the operator norm is not more than a positive constant multiple of $M_0^{1-\theta} M_1^\theta$.*

Remark 1.4 We remark that, by (1.6), the conclusions in Corollary 1.3 still hold true if we replace $\mathcal{M}^u_p(\mathscr{X})$ and $\langle A_0, A_1, \theta\rangle$, respectively, by $(\mathcal{M}^u_p(\mathscr{X}))^\circ$ and $\langle A_0, A_1\rangle_\theta$ (see [26]).

We also observe that the proof of the complex interpolation of Morrey spaces on $\mathbb{R}^n$ in [18, Theorem 3] does not depend on any geometrical properties of Euclidean spaces. Hence, by a proof similar to that used for [18, Theorem 3], we also know that, if (1.2) holds true, then

$$[\mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X})]_\theta = \mathcal{M}^u_p(\mathscr{X}),$$

where $1 < p_0 \le u_0 < \infty$, $1 < p_1 \le u_1 < \infty$, $\theta \in (0,1)$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$. This observation, together with Theorem 1.2, induces the following result. We omit the details.

Corollary 1.5 *Let $\theta \in (0,1)$, $1 < p_0 \le u_0 \le \infty$, $1 < p_1 \le u_1 \le \infty$, and $1 < p \le u \le \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$. If (1.2) holds true, then*

$$\begin{aligned}\langle \mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X}), \theta\rangle &= [\mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X})]_\theta \\ &= \langle \mathcal{M}^{u_0}_{p_0}(\mathscr{X}), \mathcal{M}^{u_1}_{p_1}(\mathscr{X})\rangle_\theta = \mathcal{M}^u_p(\mathscr{X}).\end{aligned}$$

Again, Corollary 1.5 generalizes Lemarié–Rieusset [18, Theorem 3(ii)] on $\mathbb{R}^n$ to any quasi-metric measure space $\mathscr{X}$. Also, by [18, Theorem 3(ii)], we know that (1.2) is also necessary for the conclusions of Corollary 1.5. When $\mathscr{X} := \mathbb{R}^n$, $\mathcal{M}^u_p(\mathscr{X})$ is replaced by $\mathring{\mathcal{M}}^u_p(\mathbb{R}^n)$ and at least one of $\{\mathcal{M}^{u_j}_{p_j}(\mathscr{X})\}_{j=0}^1$ by the corresponding $\{\mathring{\mathcal{M}}^{u_j}_{p_j}(\mathscr{X})\}_{j=0}^1$, some complex interpolation theorems similar to Corollary 1.5 were obtained in [32] (see also (1.3)).

Remark 1.6 Let p_0, u_0, p_1, u_1, p, u be as in Corollary 1.5 such that (1.2) holds.

(i) By Corollary 1.5, we know that the conclusions in Corollary 1.3 are also true if we replace $\langle A_0, A_1, \theta\rangle$ by $\langle A_0, A_1\rangle_\theta$ or $[A_0, A_1]_\theta$ (see, for example, [26] and [16, Theorem 8.1]).

(ii) When $\mathscr{X} := \mathbb{R}^n$, as an immediate consequence of (1.3), the conclusions in Corollary 1.3 are also true if we replace $\mathcal{M}^u_p(\mathscr{X})$ by $\mathring{\mathcal{M}}^u_p(\mathbb{R}^n)$, $\langle A_0, A_1, \theta\rangle$ by $[A_0, A_1]_\theta$, and at least one of $\{\mathcal{M}^{u_j}_{p_j}(\mathscr{X})\}_{j=0}^1$ by the corresponding $\{\mathring{\mathcal{M}}^{u_j}_{p_j}(\mathscr{X})\}_{j=0}^1$ (see, for example, [16, Theorem 8.1]).

(iii) Observe that if $\mathscr{X} := \mathbb{R}^n$ and if $\mathcal{M}_{p_j}^{u_j}(\mathscr{X})$, A_j, and $\mathcal{M}_p^u(\mathscr{X})$ are replaced by $\mathring{\mathcal{M}}_{p_j}^{u_j}(\mathbb{R}^n)$, some Lebesgue space, and $\mathring{\mathcal{M}}_p^u(\mathbb{R}^n)$, respectively, then Corollary 1.3(i) is just [4, Theorem 18] of Adams and Xiao, in which (1.2) is also needed.

We point out that since all results of this article hold true for quasi-metric measure spaces, they have wide generality, and, in particular, they hold true for both spaces of homogeneous type in the sense of Coifman and Weiss [9,10] and non-homogeneous spaces in the sense of Hytönen [15].

Finally, we make some conventions on notation. Throughout the paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $A \lesssim B$ means $A \le CB$, where C is a positive constant independent of A and B.

2 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. To this end, we need to consider the Calderón product of Morrey spaces.

We start by recalling the notion of the Calderón product; see, for example, [7,16]. A quasi-Banach space X of complex-valued measurable functions is called a *quasi-Banach lattice* if, for any $f \in X$ and a function g satisfying $|g| \le |f|$, we have $g \in X$ and $\|g\|_X \le \|f\|_X$. Given two quasi-Banach lattices X_0 and X_1, and $\theta \in (0,1)$, their *Calderón product* $X_0^{1-\theta}X_1^\theta$ is defined by

(2.1)
$$\begin{aligned} X_0^{1-\theta}X_1^\theta := \{&f \text{ is a complex-valued measurable function}:\\ &\text{there exist } f^0 \in X_0 \text{ and } f^1 \in X_1 \text{ such that } |f| \le |f^0|^{1-\theta}|f^1|^\theta\}, \end{aligned}$$

and its *norm* is given by $\|f\|_{X_0^{1-\theta}X_1^\theta} := \inf\{\|f^0\|_{X_0}^{1-\theta}\|f^1\|_{X_1}^\theta\}$, where the infimum is taken over all $f^i \in X_i$, $i \in \{0,1\}$, such that $|f| \le |f^0|^{1-\theta}|f^1|^\theta$.

It was proved in [7] that if X_0 and X_1 are two quasi-Banach lattices, then $X_0^{1-\theta}X_1^\theta$ is complete and it is also easy to see that Morrey spaces are quasi-Banach lattices. Moreover, we have the following conclusion.

Proposition 2.1 *Let $\theta \in (0,1)$, $0 < p_0 \le u_0 \le \infty$, $0 < p_1 \le u_1 \le \infty$ and $0 < p \le u \le \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$. If (1.2) holds true, then*

$$[\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^\theta = \mathcal{M}_p^u(\mathscr{X}).$$

Proof Let $f \in [\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^\theta$. Then by (2.1) we know that there exist $f_0 \in \mathcal{M}_{p_0}^{u_0}(\mathscr{X})$ and $f_1 \in \mathcal{M}_{p_1}^{u_1}(\mathscr{X})$ such that, for almost every $x \in \mathscr{X}$, $|f(x)| \le |f_0(x)|^{1-\theta}|f_1(x)|^\theta$ and

$$\|f_0\|_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})}^{1-\theta}\|f_1\|_{\mathcal{M}_{p_1}^{u_1}(\mathscr{X})}^\theta \lesssim \|f\|_{[\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^\theta}. \tag{2.2}$$

Thus, for any ball $B \subset \mathscr{X}$, by the Hölder inequality, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$, we conclude that

$$\begin{aligned}
&\frac{1}{[\mu(B)]^{1/p-1/u}}\left[\int_B |f(x)|^p\,d\mu(x)\right]^{1/p}\\
&\quad\le \frac{1}{[\mu(B)]^{1/p-1/u}}\left[\int_B |f_0(x)|^{(1-\theta)p}|f_1(x)|^{\theta p}\,d\mu(x)\right]^{1/p}\\
&\quad\le \frac{1}{[\mu(B)]^{1/p-1/u}}\left[\int_B |f_0(x)|^{p_0}\,d\mu(x)\right]^{\frac{1-\theta}{p_0}}\left[\int_B |f_1(x)|^{p_1}\,d\mu(x)\right]^{\frac{\theta}{p_1}}\\
&\quad= \left\{\frac{1}{[\mu(B)]^{1/p_0-1/u_0}}\left[\int_B |f_0(x)|^{p_0}\,d\mu(x)\right]^{\frac{1}{p_0}}\right\}^{1-\theta}\\
&\qquad\times\left\{\frac{1}{[\mu(B)]^{1/p_1-1/u_1}}\left[\int_B |f_1(x)|^{p_1}\,d\mu(x)\right]^{\frac{1}{p_1}}\right\}^{\theta}.
\end{aligned}$$

Thus, from this and the definition of Morrey spaces (see (1.1)), together with (2.2), we deduce that

$$\|f\|_{\mathcal{M}_p^u(\mathscr{X})} \le \|f_0\|^{1-\theta}_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})}\|f_1\|^{\theta}_{\mathcal{M}_{p_1}^{u_1}(\mathscr{X})} \lesssim \|f\|_{[\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^{\theta}},$$

which implies that $f \in \mathcal{M}_p^u(\mathscr{X})$ and hence

$$[\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^{\theta} \subset \mathcal{M}_p^u(\mathscr{X}). \tag{2.3}$$

Conversely, let $f \in \mathcal{M}_p^u(\mathscr{X})$ and define, for all $x \in \mathscr{X}$,

$$\widetilde{f_0}(x) := |f(x)|^{p/p_0} \quad\text{and}\quad \widetilde{f_1}(x) := |f(x)|^{p/p_1}.$$

Notice that for any ball $B \subset \mathscr{X}$, by the definition of $\widetilde{f_0}$ and (1.2), together with the definition of $\mathcal{M}_p^u(\mathscr{X})$ (see (1.1)), we see that

$$\begin{aligned}
&\frac{1}{[\mu(B)]^{1/p_0-1/u_0}}\left[\int_B |\widetilde{f_0}(x)|^{p_0}\,d\mu(x)\right]^{1/p_0}\\
&\quad= \frac{1}{[\mu(B)]^{1/p_0-1/u_0}}\left\{\left[\int_B |f(x)|^p\,d\mu(x)\right]^{1/p}\right\}^{p/p_0}\\
&\quad= \left\{\frac{1}{[\mu(B)]^{1/p-1/u}}\left[\int_B |f(x)|^p\,d\mu(x)\right]^{1/p}\right\}^{p/p_0}\\
&\quad\le \|f\|^{p/p_0}_{\mathcal{M}_p^u(\mathscr{X})}.
\end{aligned}$$

This implies that $\widetilde{f_0} \in \mathcal{M}_{p_0}^{u_0}(\mathscr{X})$ and

$$\|\widetilde{f_0}\|_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})} \le \|f\|^{p/p_0}_{\mathcal{M}_p^u(\mathscr{X})}. \tag{2.4}$$

Similarly, by the definition of $\widetilde{f}_1$, (1.1) and (1.2), we conclude that $\widetilde{f}_1 \in \mathcal{M}^{u_1}_{p_1}(\mathscr{X})$ and

$$\|\widetilde{f}_1\|_{\mathcal{M}^{u_1}_{p_1}(\mathscr{X})} \le \|f\|^{p/p_1}_{\mathcal{M}^u_p(\mathscr{X})}. \tag{2.5}$$

Moreover, by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, together with the definitions of $\widetilde{f}_0$ and $\widetilde{f}_1$, we see that, for all $x \in \mathscr{X}$,

$$|f(x)| = |f(x)|^{p(1-\theta)/p_0}\,|f(x)|^{p\theta/p_1} = |\widetilde{f}_0(x)|^{1-\theta}\,|\widetilde{f}_1(x)|^{\theta}.$$

From this, the definition of $\|\cdot\|_{[\mathcal{M}^{u_0}_{p_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}^{u_1}_{p_1}(\mathscr{X})]^{\theta}}$ (see (2.1)), (2.4), and (2.5), together with $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$, it further follows that

$$\|f\|_{[\mathcal{M}^{u_0}_{p_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}^{u_1}_{p_1}(\mathscr{X})]^{\theta}} \le \|\widetilde{f}_0\|^{1-\theta}_{\mathcal{M}^{u_0}_{p_0}(\mathscr{X})}\,\|\widetilde{f}_1\|^{\theta}_{\mathcal{M}^{u_1}_{p_1}(\mathscr{X})} \le \|f\|_{\mathcal{M}^u_p(\mathscr{X})} < \infty,$$

which implies that $f \in [\mathcal{M}^{u_0}_{p_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}^{u_1}_{p_1}(\mathscr{X})]^{\theta}$ and hence

$$\mathcal{M}^u_p(\mathscr{X}) \subset [\mathcal{M}^{u_0}_{p_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}^{u_1}_{p_1}(\mathscr{X})]^{\theta}. \tag{2.6}$$

Thus, combining (2.3) and (2.6), we see that

$$[\mathcal{M}^{u_0}_{p_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}^{u_1}_{p_1}(\mathscr{X})]^{\theta} = \mathcal{M}^u_p(\mathscr{X}),$$

which completes the proof of Proposition 2.1. ∎

Proposition 2.1 in the case when $u_0 = p_0$ and $u_1 = p_1$ coincides with the Calderón product property for Lebesgue spaces on metric measure spaces. We refer, for example, to [17, Formula 1.6.1] and [19, p. 179, Exercise 3], for the Calderón product of Lebesgue spaces on $\mathbb{R}^n$. Recently, Sickel, Skrzypczak, and Vybíral in [30, Lemma 8] obtained the Calderón product between two weighted Lebesgue spaces, with $p_0, p_1 \in (0,\infty]$, on $\mathbb{R}^n$. In case $p_0, p_1 \in [1,\infty]$, this result can also be found in [6, Exercise 4.3.8]. We also refer, for example, to [11, 16, 32] for the extension to the Calderón product of sequence spaces with respect to function spaces.

Let X be a quasi-Banach lattice and $q \in [1,\infty]$. The *q-convexification* of X, denoted by $X^{(q)}$, is defined as follows: $x \in X^{(q)}$ if and only if $|x|^q \in X$, and let $\|x\|_{X^{(q)}} := \||x|^q\|^{1/q}_X$ (see, for example, [24]). A quasi-Banach lattice X is said to *be of type* $\mathfrak{C}$ if there exists an equivalent quasi-norm on X such that, for some $q \in [1,\infty]$, $X^{(q)}$ is a Banach lattice in this norm (see, for example, [24]).

The following result on the Calderón product and the interpolation for quasi-Banach lattices being of type $\mathfrak{C}$ is a special case of the general result [24, Theorem 2.1] obtained by Nilsson.

Theorem 2.2 *Let X_0 and X_1 be two quasi-Banach lattices of type $\mathfrak{C}$. Then*

$$\langle X_0, X_1\rangle_\theta = (X_0^{1-\theta}X_1^{\theta})^{\circ}$$

and

$$X_0^{1-\theta}X_1^{\theta} \subset \langle X_0, X_1, \theta\rangle \subset (X_0^{1-\theta}X_1^{\theta})^{\sim}.$$

Notice that for all $\delta \in \big(0, \min(1,p)\big]$, the $1/\delta$-convexification $\big(\mathcal{M}_p^u(\mathscr{X})\big)^{(1/\delta)}$ of the Morrey space is a Banach space, namely, the Morrey space is of type $\mathfrak{E}$. Then, applying Proposition 2.1 and Theorem 2.2, we have the following conclusion, which covers Theorem 1.2 as a special case. The approach used to prove Theorem 2.3 is inspired by the proof of [11, Theorem 8.5].

Theorem 2.3 *Let $\theta \in (0,1)$, $0 < p_0 \le u_0 \le \infty$, $0 < p_1 \le u_1 \le \infty$ and $0 < p \le u \le \infty$ such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $\frac{1}{u} = \frac{1-\theta}{u_0} + \frac{\theta}{u_1}$. If (1.2) holds true, then*

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathscr{X}), \mathcal{M}_{p_1}^{u_1}(\mathscr{X})\rangle_\theta = \big([\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^{\theta}\big)^\circ = (\mathcal{M}_p^u(\mathscr{X}))^\circ \tag{2.7}$$

and

$$\langle \mathcal{M}_{p_0}^{u_0}(\mathscr{X}), \mathcal{M}_{p_1}^{u_1}(\mathscr{X}), \theta\rangle = [\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^{\theta} = \mathcal{M}_p^u(\mathscr{X}). \tag{2.8}$$

In particular, when $p_0, p_1 \in [1,\infty]$, it further holds that

$$(\mathcal{M}_p^u(\mathscr{X}))^\circ = [\mathcal{M}_{p_0}^{u_0}(\mathscr{X}), \mathcal{M}_{p_1}^{u_1}(\mathscr{X})]_\theta. \tag{2.9}$$

Proof It is easy to see that (2.7) is an immediate consequence of Proposition 2.1 and Theorem 2.2, by observing that the Morrey space is of type $\mathfrak{E}$.

Moreover, from Proposition 2.1, Theorem 2.2 and the fact that the Morrey space is of type $\mathfrak{E}$ again, it also follows that

$$\mathcal{M}_p^u(\mathscr{X}) = [\mathcal{M}_{p_0}^{u_0}(\mathscr{X})]^{1-\theta}[\mathcal{M}_{p_1}^{u_1}(\mathscr{X})]^{\theta} \subset \langle \mathcal{M}_{p_0}^{u_0}(\mathscr{X}), \mathcal{M}_{p_1}^{u_1}(\mathscr{X}), \theta\rangle \subset [\mathcal{M}_p^u(\mathscr{X})]^\sim.$$

Thus, to show (2.8), we only need to prove $[\mathcal{M}_p^u(\mathscr{X})]^\sim \subset \mathcal{M}_p^u(\mathscr{X})$.

Let $f \in [\mathcal{M}_p^u(\mathscr{X})]^\sim$. By the definition of $[\mathcal{M}_p^u(\mathscr{X})]^\sim$, there exists a sequence $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{M}_p^u(\mathscr{X})$ such that

$$\lim_{i\to\infty} \|f_i - f\|_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})+\mathcal{M}_{p_1}^{u_1}(\mathscr{X})} = 0$$

and

$$\|f_i\|_{\mathcal{M}_p^u(\mathscr{X})} \lesssim \|f\|_{[\mathcal{M}_p^u(\mathscr{X})]^\sim} \tag{2.10}$$

for all $i \in \mathbb{N}$. Then, for all $i \in \mathbb{N}$, by $f_i - f \in \mathcal{M}_{p_0}^{u_0}(\mathscr{X}) + \mathcal{M}_{p_1}^{u_1}(\mathscr{X})$ (see (1.4) and (1.5)), we know that there exist $f_i^0 \in \mathcal{M}_{p_0}^{u_0}(\mathscr{X})$ and $f_i^1 \in \mathcal{M}_{p_1}^{u_1}(\mathscr{X})$ such that, for almost every $x \in \mathscr{X}$, $f_i(x) - f(x) = f_i^0(x) + f_i^1(x)$ and

$$\|f_i^0\|_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})} + \|f_i^1\|_{\mathcal{M}_{p_1}^{u_1}(\mathscr{X})} \lesssim \|f_i - f\|_{\mathcal{M}_{p_0}^{u_0}(\mathscr{X})+\mathcal{M}_{p_1}^{u_1}(\mathscr{X})} \to 0, \tag{2.11}$$

as $i \to \infty$.

Fix $x_0 \in \mathscr{X}$, and let $B_m := B(x_0, 2^m) \cap \mathscr{X}$ for all $m \in \mathbb{N}$. Let $j \in \{0,1\}$. Since $\|f_i^j\|_{\mathcal{M}_{p_j}^{u_j}(\mathscr{X})} \to 0$ as $i \to \infty$ (by (2.11)) and $\mathcal{M}_{p_j}^{u_j}(\mathscr{X}) \subset L_{\mathrm{loc}}^{p_j}(\mathscr{X})$ (see (1.1)), it follows that $\{f_i^j\}_{i\in\mathbb{N}}$ converges in measure on B_1 and hence, by the Riesz theorem,

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there exist subsequences, denoted again by $\{f_i^j\}_{i\in\mathbb{N}}$, such that $f_i^j \to 0$, as $i \to \infty$, for almost every $x \in B_1$. Repeating this argument on B_2, we find subsequences, denoted again by $\{f_i^j\}_{i\in\mathbb{N}}$, that converge to 0 almost everywhere on B_2. By this procedure, we conclude that for any ball $B \subset \mathscr{X}$, there exist subsequences $\{f_{i_k}^j\}_{k\in\mathbb{N}}$ of $\{f_i^j\}_{i\in\mathbb{N}}$ such that $f_{i_k}^j \to 0$, as $i_k \to \infty$, almost everywhere on B. Therefore, applying the Fatou lemma, by the definition of the Morrey space (see (1.1)) and (2.10), we see that

$$\begin{aligned}
[\mu(B)]^{1/u-1/p}\left[\int_B |f(x)|^p\,dx\right]^{1/p} &= [\mu(B)]^{1/u-1/p}\left[\int_B \lim_{k\to\infty}|f_{i_k}(x)|^p\,dx\right]^{1/p}\\
&\le \varliminf_{k\to\infty}\,[\mu(B)]^{1/u-1/p}\left[\int_B |f_{i_k}(x)|^p\,dx\right]^{1/p}\\
&\le \varliminf_{k\to\infty}\,\|f_{i_k}\|_{\mathcal{M}_p^u(\mathscr{X})} \lesssim \|f\|_{[\mathcal{M}_p^u(\mathscr{X})]^\sim}.
\end{aligned}$$

Thus, by this, the arbitrariness of the ball B, and (1.1), we know that $f \in \mathcal{M}_p^u(\mathscr{X})$ and $[\mathcal{M}_p^u(\mathscr{X})]^\sim \subset \mathcal{M}_p^u(\mathscr{X})$, which completes the proof of (2.8).

Finally, the equality (2.9) follows from (2.7), the fact that the Morrey space is of type $\mathfrak{C}$, and a general result by Shestakov [28,29] which says that if X_0, X_1 are Banach lattices, then $(X_0^{1-\theta}X_1^{\theta})^\circ = [X_0, X_1]_\theta$. This finishes the proof of Theorem 2.3. ■

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