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Geometric aspects on Humbert-Edge curves of type 5, Kummer surfaces and hyperelliptic curves of genus 2

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Abstract
In this work, we study the Humbert-Edge curves of type 5, defined as a complete intersection of four diagonal quadrics in $\mathbb{P}^5$. We characterize them using Kummer surfaces, and using the geometry of these surfaces, we construct some vanishing thetanulls on such curves. In addition, we describe an argument to give an isomorphism between the moduli space of Humbert-Edge curves of type 5 and the moduli space of hyperelliptic curves of genus 2, and we show how this argument can be generalized to state an isomorphism between the moduli space of hyperelliptic curves of genus $g = \frac{n-1}{2}$ and the moduli space of Humbert-Edge curves of type $n \geq 5$ where n is an odd number.

1. Introduction

W.L. Edge began the study of *Humbert's curves* in [5]; such curves are defined as canonical curves in $\mathbb{P}^4$ that are the complete intersection of three diagonal quadrics. A natural generalization of Humbert's curve was later introduced by Edge in [7]: irreducible, non-degenerated, and non-singular curves on $\mathbb{P}^n$ that are the complete intersection of $n-1$ diagonal quadrics. One important feature of Humbert-Edge curves of type n noted by Edge in [7] is that each one admits a *normal form*. Indeed, we can assume that the $n-1$ quadrics in $\mathbb{P}^n$ are given by

$$Q_i = \sum_{j=0}^{n} a_j^i x_j^2, \ \ i = 0, \ldots, n-1$$

where $a_j \in \mathbb{C}$ for all $j \in \{0, \ldots, n\}$ and $a_j \neq a_k$ if $j \neq k$, and here, a_j^i denotes the ith power of the a_j. We say that a curve satisfying this conditions is a *Humbert-Edge curve of type n*. Note that in the case of a Humbert's curve X, i.e. when $n = 4$, this form of the equations implies directly that X is contained in a degree four Del Pezzo surface.

The Humbert-Edge curves of type n for $n > 4$ has been studied in just a few works. Carocca, González-Aguilera, Hidalgo, and Rodríguez studied in [3] the Humbert-Edge curves from the point of view of uniformization and Klenian groups. Using a suitable form of the quadrics, Hidalgo presented in [11] and [12] a family of Humbert-Edge curves of type 5 whose fields of moduli are contained in $\mathbb{R}$ but none of their fields of definition are contained in $\mathbb{R}$. Frías-Medina and Zamora presented in [9] a characterization of Humbert-Edge curves using certain abelian groups of order 2^n and presented specializations admitting larger automorphism subgroups. Carvacho, Hidalgo, and Quispe determined in [4] the decomposition of the Jacobian of generalized Fermat curves and as a consequence for Humbert-Edge curves. Auffarth, Lucchini Arteche, and Rojas described in [1] the decomposition of the Jacobian of a Humbert-Edge

curve more precisely given the exact number of factors in the decomposition and their corresponding dimensions.

In the case $n = 5$, the normal form for these curves implies that they are contained in a special $K3$ surface, a *Kummer surface*. In this work, we study the Humbert-Edge curves of type 5 and determine some properties using the geometry of Kummer surfaces.

Recall that an *algebraic (complex) K3 surface* is a complete non-singular projective (compact connected complex) surface S such that $\omega_S \cong \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. Classically, a *(singular) Kummer surface* is a surface in $\mathbb{P}^3$ of degree 4 with 16 nodes and no other singularities. An important fact about Kummer surfaces is that are determined by the set of their nodes. One can take the resolution of singularities of a Kummer surface, and the obtained non-singular surface is a $K3$ surface. For our purposes, these non-singular models will be called Kummer surfaces.

The Kummer surfaces that we will take for our study are those coming from a hyperelliptic curve of genus 2. In such case, an important feature shared by Humbert-Edge curves of type 5 and Kummer surfaces is that they admit the automorphism subgroup generated by the natural involutions of $\mathbb{P}^5$ that change the ith coordinate by its negative:

$$\sigma_i : \quad \mathbb{P}^5 \qquad \rightarrow \qquad \mathbb{P}^5$$
$$(x_0 : \cdots : x_i \cdots : x_5) \mapsto (x_0 : \cdots : -x_i : \cdots : x_5)$$

These automorphisms along with Knutsen's result [14] on the existence of a $K3$ surface of degree $2n$ in $\mathbb{P}^{n+1}$ containing a smooth curve of genus g and degree d will enable us to characterize the Humbert-Edge curves of type 5 using the geometry of a Kummer surface.

This work is organized as follows. In Section 2, we review the construction of a Kummer surface from a two-dimensional torus and the conditions that ensure when a Kummer surface is projective. Later, we focus on the case of Kummer surfaces obtained from hyperelliptic curves of genus 2. Section 3 is split into three parts. First, in Section 3.1, we review the basic properties of Humbert-Edge curves of type 5 and characterize them using the geometry of the Kummer surface. Later, in Section 3.2, we present the construction of some odd theta characteristic on a Humbert-Edge curve of type 5 using the automorphisms σ_i's and some vanishing thetanulls using the Rosenhain tetrahedra associated with the Kummer surface. Finally, in Section 3.3 we use the embedding given in [3] to construct an isomorphism between the moduli space $\mathcal{HE}_5$ of Humbert-Edge curves of type 5 and the moduli space $\mathcal{H}_2$ of hyperelliptic curves of genus 2, and as a consequence, we obtain that $\mathcal{H}_2$ is a three-dimensional closed subvariety of $\mathcal{M}_{17}$. Moreover, we generalized this argument to show that there is an isomorphism between the moduli space $\mathcal{HE}_n$ of Humbert-Edge curves of type n, where $n \geq 5$ is an odd number, and the moduli space $\mathcal{H}_g$ of hyperelliptic curves of genus $g = \frac{n-1}{2}$.

2. Kummer surfaces

2.1. Construction from a two-dimensional torus

In this paper, the ground field is the complex numbers. In this section, we recall the construction of the Kummer surface associated with a two-dimensional torus.

Let T be a two-dimensional torus. Consider the involution $\iota : T \rightarrow T$ which sends $a \mapsto -a$ and takes the quotient surface $T/\langle \iota \rangle$. The surface $T/\langle \iota \rangle$ is known as the *singular Kummer surface* of T. It is well-known that this surface has 16 ordinary singularities, and by resolving them, we obtain a $K3$ surface called the *Kummer surface* of T and denoted by $\mathrm{Km}(T)$ (see e.g., [10, Theorem 3.4]). This procedure is called the *Kummer process*. Note that by construction, $\mathrm{Km}(T)$ has 16 disjoint smooth rational curves; indeed, they correspond to the singular points of the quotient surface. Nikulin proved in [15] the converse:

Theorem 2.1. *If a K3 surface S contains 16 disjoint smooth rational curves, then there exists a unique complex torus, up to isomorphism, such that S and the rational curves are obtained by the Kummer process. In particular, S is a Kummer surface.*

Note that the above construction holds true for any two-dimensional torus, not necessarily a projective one. In particular, with this process it is possible to construct $K3$ surfaces that are not projective. However, there is an equivalence between the projectivity of the torus and the associated $K3$ surface (see [**2**, Theorem 4.5.4]):

Theorem 2.2. *Let T be a two-dimensional torus. T is an abelian surface if and only if* $\mathrm{Km}(T)$ *is projective.*

Now, if A is a principally polarized abelian surface, then A is one of the following (see [**2**, Corollary 11.8.2]):

  (a) The Jacobian of a smooth hyperelliptic curve of genus 2 or
  (b) The canonical polarized product of two elliptic curves.

As we will see next, Case (a) is the one of our interest.

2.2. Hyperelliptic curves of genus 2 and Kummer surfaces

We are interested in $K3$ surfaces that are a smooth complete intersection of type $(2, 2, 2)$ in $\mathbb{P}^5$, i.e. that are a complete intersection of three quadrics. Moreover, we restrict to the case in which the quadrics are diagonal. The interest of having diagonal quadrics defining the $K3$ surface is that they enable us to work with hyperelliptic curves of genus 2.

Indeed, let C be the hyperelliptic curve of genus 2 given by the affine equation

$$y^2 = f(x) = (x - a_0)(x - a_1) \cdots (x - a_5), \tag{1}$$

where $a_0, \ldots, a_5 \in \mathbb{C}$ and $a_i \neq a_j$ if $i \neq j$. We can consider the jacobian surface $J(C)$ associated with C, and applying the Kummer process, we obtain that the $K3$ surface $\mathrm{Km}(J(C))$ is isomorphic to the surface in $\mathbb{P}^5$ defined by the complete intersection of the 3 diagonal quadrics by [**16**, Theorem 2.5]:

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \ i = 0, 1, 2. \tag{2}$$

In order to obtain a hyperelliptic curve of genus 2 beginning with a smooth $K3$ surface S in $\mathbb{P}^5$ given by the complete intersection of three diagonal quadrics, we may assume an additional hypothesis. Edge studied in [**6**] the Kummer surfaces defined by (2). One of his results establishes that whenever a surface X given by the intersection of three linearly independent quadrics has a common self-polar simplex Σ in $\mathbb{P}^5$ and contains a line in general position, then the equations defining X can be written with the form (2). Observe that this fact is equivalent to requiring that X contains 16 disjoint lines; indeed, using the natural involutions of $\mathbb{P}^5$ one can obtain the other lines.

Then, let S be a smooth $K3$ surface in $\mathbb{P}^5$ given by the complete intersection of the quadrics

$$Q_i = \sum_{j=0}^{5} a_{ij} x_j^2, \ \ i = 0, 1, 2,$$

where $a_{ij} \in \mathbb{C}$ for $i = 0, 1, 2$ and $j = 0, \ldots, 5$ and assume that S contains 16 disjoint lines. As a consequence, we may assume that S is given by the quadrics in (2) for some $a_i \in \mathbb{C}$ where $a_i \neq a_j$ if $i \neq j$. By [**15**, Theorem 1] there exists a unique (up to isomorphism) two-dimensional torus that gives rise to the surface S. Taking the hyperelliptic curve C given by the equation $y^2 = f(x) = (x - a_0)(x - a_1) \cdots (x - a_5)$, we obtain that S is isomorphic to $\mathrm{Km}(J(C))$.

From now on, we say that a Kummer surface is a smooth surface in $\mathbb{P}^5$ given by the complete intersection of three diagonal quadrics as in (2).

For a Kummer surface S given by (2), it is possible to give the parametric form of the 32 lines contained in S. Indeed, in [**6**] Edge noted that the equation of a line ℓ contained in S is given in the following parametric form:

$$\left(\frac{t+a_0}{\sqrt{f'(a_0)}}, \frac{t+a_1}{\sqrt{f'(a_1)}}, \frac{t+a_2}{\sqrt{f'(a_2)}}, \frac{t+a_3}{\sqrt{f'(a_3)}}, \frac{t+a_4}{\sqrt{f'(a_4)}}, \frac{t+a_5}{\sqrt{f'(a_5)}} \right). \tag{3}$$

Recall that the natural automorphisms $\sigma_i : x_i \mapsto -x_i$ of $\mathbb{P}^5$ act on S. Denote by E the group $\langle \sigma_0, \ldots, \sigma_5 \rangle$. Applying each element of E to the line ℓ, we obtain the other 32 lines on S. The identity gives the line ℓ and the remaining elements give the other 31 lines:

- $\ell_i := \sigma_i(\ell)$ for all $i \in \{0, \ldots, 5\}$,
- $\ell_{ij} := \sigma_i \sigma_j(\ell)$ for different $i, j \in \{0, \ldots, 5\}$
- $\ell_{ijk} := \sigma_i \sigma_j \sigma_k(\ell)$ for different $i, j, k \in \{0, \ldots, 5\}$.

It is well-known that a singular Kummer surface K is birational to a Weddle surface W (see e.g., [17, Proposition 1]). A *Weddle surface* is a quartic surface in $\mathbb{P}^3$ with six nodes. The 32 lines on S have a geometric interpretation in both K and W as Edge pointed in [6]. Indeed, the projection π of S from ℓ is a Weddle surface W and it occurs:

- $\pi(\ell_i) = k_i$ is a node on W, for all $i = 0, \ldots, 5$,
- $\pi(\ell_{ij})$ is the line through k_i and k_j, for different indices $i, j \in \{0, \ldots, 5\}$,
- $\pi(\ell_{ijh})$ is the line in the intersection of the plane generated by k_i, k_j, k_h with the complementary plane, for different indices $i, j, h \in \{0, \ldots, 5\}$, and
- $\pi(\ell)$ is the cubic on W through the six nodes k_i's.

On the other hand, since S is the resolution of singularities of K it occurs:

- The 16 nodes of K correspond to the 16 lines ℓ_i and ℓ_{ijk}, and
- The conics of contact of K with its 16 tropes correspond to the 16 lines ℓ and ℓ_{ij}.

A *trope* is a plane which intersects the quartic along a conic. The nodes and the tropes of a singular Kummer surface provide an interesting configuration on it.

Definition 2.3. *Let Γ be a set of 16 planes and 16 points in $\mathbb{P}^3$.*

- *Γ is a $(16, 6)$-configuration if every plane contains exactly 6 of the 16 points and every point lies in exactly 6 of the 16 planes. The 16 planes are called* special planes.
- *A $(16, 6)$-configuration is* non-degenerate *if every two special planes share exactly two points of the configuration and every pair of points is contained in exactly two special planes.*
- *An* abstract $(16, 6)$-configuration *is a 16×16 matrix (a_{ij}) whose entries are ones or zeros, with exactly 6 ones in each row and in each column. The rows of the matrix are called* points *of the configuration, and the columns are called* planes *of the configuration. The ith point belongs to the jth plane if and only if $a_{ij} = 1$.*

Gonzalez-Dorrego classified in [10] the non-degenerate $(16, 6)$-configurations and used them to classify the singular Kummer surfaces. Given a singular Kummer surface, the nodes and the tropes establish a non-degenerate $(16, 6)$-configuration (see [10, Corollary 2.18]), and conversely, given a $(16, 6)$-configuration, there exists a singular Kummer surface whose associated $(16, 6)$-configuration is the given one (see [10, Theorem 2.20]).

Definition 2.4. *A* Rosenhain tetrahedron *in an abstract $(16, 6)$-configuration is a set of 4 points and 4 planes such that each plane contains exactly 3 points and each point belongs to exactly 3 planes. The 4 points are the* vertices *of the tetrahedron. An* edge *is a pair of vertices, and a* face *is a triple of vertices.*

Rosenhain tetrahedra always exist in a singular Kummer surface; in fact, there exist 80 of them ([10, Corollary 3.21]). Moreover, these tetrahedra are relevant because using them we can construct

divisors that are linearly equivalent and whose class induces the closed embedding to $\mathbb{P}^5$ (see [**10**, Proposition 3.22 and Remark 3.24]):

Proposition 2.5. *Given a Rosenhain tetrahedron on a singular Kummer surface K, let D be the divisor on the associated Kummer surface S given by the sum of proper transforms of the 4 conics in which the planes meet on K and the 4 exceptional divisors corresponding to the 4 nodes. Then, the linear equivalence class of D is independent of the choice of the Rosenhain tetrahedron. In addition, $D^2 = 8$, $\dim|D| = 5$, and the linear system $|D|$ induces a closed embedding of S in $\mathbb{P}^5$ as the complete intersection of three quadrics.*

These divisors will be used in the next section to construct vanishing thetanulls on Humbert-Edge curves of type 5.

3. Humbert-Edge curves of type 5

3.1. Properties and characterization

Here, we review the main properties of the Humbert-Edge curves of type 5 and present a characterization using the lines lying on a Kummer surface.

Definition 3.1. *An irreducible, non-degenerate, and non-singular curve $X_5 \subseteq \mathbb{P}^5$ is a Humbert-Edge curve of type 5 if it is the complete intersection of 4 diagonal quadrics $Q_0, \ldots, Q_3$:*

$$Q_i = \sum_{j=0}^{5} a_{ij} x_j^2, \quad i = 0, \ldots, 3.$$

The basic properties of a Humbert-Edge curve of type 5 are stated below.

Lemma 3.2. *Let $X_5 \subset \mathbb{P}^5$ be a Humbert-Edge curve of type 5. The following hold:*

1. *X_5 is a curve of degree 16.*
2. *The genus of X_5 is equal to $g(X_5) = 17$.*
3. *Every 4-minor of the matrix (a_{ij}) is non-degenerate.*
4. *X_5 is non-trigonal.*

The diagonal form of the equations defining a Humbert-Edge curve X_5 of type 5 implies that it admits the action of the group E generated by the six involutions $\sigma_i : x_i \mapsto -x_i$ acting with fixed points and whose product is the identity. Moreover, these involutions establish a relation between the Humbert-Edge curves of type 5 and the Humbert's curves in $\mathbb{P}^4$. For every $i = 0, \ldots, 5$, we can consider the covering $\pi_i : X_5 \to X_5/\langle\sigma_i\rangle$ induced by the involution σ_i. This is a two-to-one covering ramified at 16 points obtained as the intersection points of X_5 with the hyperplane $V(x_i)$. In addition, the quotient of $X_5/\langle\sigma_i\rangle$ is a Humbert's curve in $\mathbb{P}^4$. This double covering can be interpreted geometrically as the projection of X_5 with center e_i onto the hyperplane $V(x_i)$.

Next result shows that a Humbert-Edge curve of type 5 is always contained in a Kummer surface. It is a consequence of the fact noted by Edge in [**7**] that a Humbert-Edge curve of type n can be written in a normal form.

Proposition 3.3. *Let $X_5 \subset \mathbb{P}^5$ be a Humbert-Edge curve of type 5. There exists a Kummer surface in $\mathbb{P}^5$ which contains X_5.*

Proof. Assume that X_5 is given by the equations

$$Q_i = \sum_{j=0}^{5} a_{ij} x_j^2, \quad i = 0, \ldots, 3$$

where $a_{ij} \in \mathbb{C}$. For each $j = 0, \ldots, 5$, consider the coefficients a_{ij} as the entries of the point $p_j = \left(a_{0j} : a_{1j} : a_{2j} : a_{3j}\right)$ in the projective space $\mathbb{P}^3$. We have six points $p_0, \ldots, p_5$ in $\mathbb{P}^3$ that are in general position, so there exists a unique rational normal curve $C \subset \mathbb{P}^3$ through these points. Finally, take a change of coordinates of $\mathbb{P}^3$ such that C is in the standard parametric form, then we may assume that $p_j = \left(1 : a_j : a_j^2 : a_j^3\right)$ for all $j = 0, \ldots, 5$. Therefore, we obtain that X_5 is given by the equations

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \quad i = 0, \ldots, 3 \tag{4}$$

with $a_j \in \mathbb{C}, j = 0, \ldots, 5$ and $a_j \neq a_k$ if $j \neq k$. The quadrics Q_0, Q_1, Q_2 define the Kummer surface associated with the hyperelliptic curve $y^2 = \prod_{j=0}^{5} (x - a_j)$. Therefore, X_5 is contained in a Kummer surface. $\qquad\square$

Remark 3.4. *Note that given a Humbert-Edge curve X_5 of type 5 in normal form* (4), *by the above proposition it is always possible to find a hyperelliptic curve C such that the Kummer surface $\mathrm{Km}(J(C))$ contains X_5. Reciprocally, given a hyperelliptic curve C of genus 2, by the discussion in Section 2.2, in a natural way the associated Kummer surface $\mathrm{Km}(J(C))$ contains a Humbert-Edge curve of type 5 whose equations are in the normal form* (4). *Denote by $\mathcal{M}_g$ the moduli space of smooth and irreducible curves of genus g. Note that Proposition 3.3 lets us see that Humbert-Edge curves of type 5 depend on three parameters in $\mathcal{M}_{17}$; in fact in Section 3.3, we prove that the moduli space of Humbert-Edge curves of type 5 is isomorphic to the moduli space of hyperelliptic curves of genus 2.*

Next we present a characterization for Humbert-Edge curves of type 5 using the lines on Kummer surfaces.

Theorem 3.5. *Let $X \subset \mathbb{P}^5$ be an irreducible, non-degenerate, and non-singular curve of degree 16 and genus 17. The following statements are equivalent:*

   (i) *X is a Humbert-Edge curve of type 5.*
  (ii) *X admits six involutions $\sigma_0, \ldots, \sigma_5$ such that $\langle \sigma_0, \ldots, \sigma_5 \rangle \cong (\mathbb{Z}/2\mathbb{Z})^5$, $\sigma_0 \cdots \sigma_5 = 1$ and the quotient $X/\langle \sigma_i \rangle$ is a Humbert's curve for every $i = 0, \ldots, 5$.*
 (iii) *There exists a Kummer surface S which contains X and such that the intersection of X with the 16 lines ℓ and ℓ_{ij} is at most one point.*
 (iv) *There exists a Kummer surface S which contains X and such that the intersection of X with the 16 lines ℓ_i and ℓ_{ijk} is at most one point.*

Proof. (i)$\Leftrightarrow$(ii) We have this equivalence by [**9**, Theorem 3.4].

(i)$\Rightarrow$(iii) Assume that X is Humbert-Edge curve of type 5. Proposition 3.3 implies that X is contained in a Kummer surface S. So, we may assume the existence of different scalars $a_0, \ldots, a_5 \in \mathbb{C}$ such that X is given by the equations

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \quad i = 0, \ldots, 3$$

and S is given by the equations Q_0, Q_1, Q_2. Consider the line ℓ in parametric form as in (3). A direct computation shows that for every $t \in \mathbb{C}$,

$$Q_3 \left(\frac{t + a_0}{\sqrt{f'(a_0)}}, \frac{t + a_1}{\sqrt{f'(a_1)}}, \frac{t + a_2}{\sqrt{f'(a_2)}}, \frac{t + a_3}{\sqrt{f'(a_3)}}, \frac{t + a_4}{\sqrt{f'(a_4)}}, \frac{t + a_5}{\sqrt{f'(a_5)}} \right) = 1.$$

Therefore, X does not intersect the line ℓ and the diagonal form of the third equation implies that X also does not intersect the lines ℓ_{ij} for every $i, j \in \{0, \ldots, 5\}$ with $i \neq j$.

(iii)$\Rightarrow$(i) Assume that X is contained in a Kummer surface S defined by the equations

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \quad i = 0, 1, 2,$$

where $a_j \neq a_k$ if $j \neq k$, and such that X does not intersect the lines ℓ, ℓ_{jk} in two different points for all $j, k \in \{0, \ldots, 5\}$ with $j \neq k$. We denote $f(x) = \prod_{j=0}^{5} (x - a_j)$. Since S is a $K3$ surface of type $(2, 2, 2)$ in $\mathbb{P}^5$ containing X, our situation should be one of the cases determined by Knutsen in [**14**, Theorem 6.1 (3)]. In fact, we are in Case a) of the latter, in Knutsen's notation we have $n = 4$, $d = 16$, $g = 17$, and $g = d^2/16 + 1$. Moreover, such a result implies that X is the complete intersection of S and a hypersurface of degree $d/8$, i.e. X is the complete intersection of four quadrics. Denote the fourth quadric by

$$Q = \sum_{j=0}^{5} d_j x_j^2 + \sum_{0 \leq k < j \leq 5} d_{jk} x_j x_k.$$

Now, when we evaluate the quadric Q in the parametric form of ℓ we obtain a quadratic equation with parameter t with leading coefficient

$$\frac{1}{\prod_{j=0}^{5} f'(a_j)} \left(\sum_{j=0}^{5} f'(a_0) \cdots \widehat{f'(a_j)} \cdots f'(a_5) d_j + \sum_{0 \leq k < j \leq 5} f'(a_0) \cdots \sqrt{f'(a_k)} \sqrt{f'(a_j)} \cdots f'(a_5) d_{jk} \right).$$

The hypothesis that Q does not intersect the line ℓ in two different points implies that such coefficient vanishes (the coefficient of t could vanish but in such case the constant term must be different from zero). This also occurs for all of the 15 lines ℓ_{jk} by hypothesis, and then, we have 16 imposed conditions. The leading coefficient for the line ℓ_{jk} can be deduced from the above one, in fact, since the line ℓ_{jk} is obtained from ℓ by the application of $\sigma_j \sigma_k$, it is enough to add a negative sign to the coefficient of the terms d_{rs} whenever r or s are equal to j or k. Solving the linear system in the variables d_j's and d_{jk}'s, we obtain that all the d_{jk}'s are equal to zero, that $d_0, d_1, d_2, d_3,$ and d_4 are free parameters and

$$d_5 = -f'(a_5) \left(\frac{d_0}{f'(0)} + \frac{d_1}{f'(a_1)} + \frac{d_2}{f'(a_2)} + \frac{d_3}{f'(a_3)} + \frac{d_4}{f'(a_4)} \right). \tag{5}$$

Therefore, X is the complete intersection of four diagonal quadrics in $\mathbb{P}^5$ and we conclude that it is a Humbert-Edge curve of type 5.

(iii)$\Leftrightarrow$(iv) As above, assuming that X is contained in a Kummer surface S defined by the equations

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \quad i = 0, 1, 2,$$

where $a_j \neq a_k$ if $j \neq k$, by [**14**, Theorem 6.1 (3)] we ensure that X is the complete intersection of S and a hypersurface of degree 2. Under the hypothesis of (iii) or (iv), when we solve the system of equations in the parameter t as we previously did, we obtain that the coefficient of every mixed term vanishes and d_5 has the form of (5). The remaining conditions, (iv) or (iii), respectively, do not impose new conditions on the coefficients. $\qquad \square$

3.2. Theta characteristics

In this section, we use the coverings given by the subgroups generated by involutions σ_i's and the Rosenhain tetrahedra of singular Kummer surfaces to construct theta characteristics on a Humbert-Edge curve of type 5. We recall the definition of a theta characteristic and a vanishing thetanull.

Definition 3.6. *Let X be an algebraic curve. A line bundle L on X is a* theta characteristic *if $L^2 \sim K_X$. A theta characteristic L is* even *(respectively,* odd*) if $h^0(L)$ is even (respectively, odd). A* vanishing thetanull *is an even theta characteristic L such that $h^0(L) > 0$.*

Recall that given a Humbert-Edge curve X_5 of type 5, for every $i = 0, \ldots, 5$ the double covering $\pi_i : X_5 \rightarrow X_5/\langle \sigma_i \rangle$ is ramified at 16 points obtained as the intersection points of X_5 with the hyperplane $V(x_i)$. Denote by $R_i = p_{i1} + \cdots + p_{i16}$ the ramification divisor for every $i = 0, \ldots, 5$.

Proposition 3.7. *Let $X_5 \subset \mathbb{P}^5$ be a Humbert-Edge curve of type 5. X_5 admits 26 odd theta characteristics with 3 sections, 6 of them correspond to the line bundle associated with the ramification divisors, and the remaining 20 are induced by the coverings associated with the subgroups generated by three different involutions σ_i, σ_j and σ_k for $i, j, k \in \{0, \ldots, 5\}$.*

Proof. For distinct $i, j \in \{0, \ldots, 5\}$, consider the subgroup generated by the involutions σ_i and σ_j and take the induced covering of degree four $\pi_{ij} : X_5 \rightarrow X_5/\langle \sigma_i, \sigma_j \rangle$. This is a simply ramified covering with the 32 ramified points $p_{i1}, \ldots, p_{i16}, p_{j1}, \ldots, p_{j16}$. Since $X_5/\langle \sigma_i, \sigma_j \rangle = E_{ij}$ is an elliptic curve, it follows that

$$K_{X_5} \sim \pi_{ij}^*(K_{E_{ij}}) + R_i + R_j = R_i + R_j.$$

So, $K_{X_5} \sim R_i + R_j$ for all $i, j \in \{0, \ldots, 5\}$. Fix and index $i \in \{0, \ldots, 5\}$ and take $j, k \in \{0, \ldots, 5\} \setminus \{i\}$ with $j \neq k$. Using the fact that

$$R_i + R_k \sim K_{X_5} \sim R_k + R_j,$$

we have that $R_i \sim R_j$. Thus, $K_{X_5} \sim R_i + R_j \sim 2R_i$ and R_i is a theta characteristic.

Now, let $i, j, k \in \{0, \ldots, 5\}$ be different indices. The covering of degree eight $\pi_{ijk} : X_5 \rightarrow X_5/\langle \sigma_i, \sigma_j, \sigma_k \rangle$ is a simply ramified covering in the 48 points $p_{i1}, \ldots, p_{i16}, p_{j1}, \ldots, p_{j16}, p_{k1}, \ldots, p_{k16}$. Note that $X_5/\langle \sigma_i, \sigma_j, \sigma_k \rangle \cong \mathbb{P}^1$. Then,

$$K_{X_5} \sim \pi_{ijk}^*(K_{\mathbb{P}^1}) + R_i + R_j + R_k \sim \pi_{ijk}^*(K_{\mathbb{P}^1}) + R_i + K_{X_5}.$$

Therefore, we have that $\pi_{ijk}^*(-K_{\mathbb{P}^1}) \sim R_i$ and we conclude that $\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2))$ is a theta characteristic of X_5.

Next step is to compute $h^0(\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2)))$. To do so, we will use the fact that $h^0(\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2))) = h^0(\pi_{ijk_*}\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2)))$. The covering π_{ijk} is determined by a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ such that $\mathcal{L}^8 = \mathcal{O}_{\mathbb{P}^1}(\pi_{ijk_*}(R_i + R_j + R_k))$, and in addition, we have that $\pi_{ijk_*}\mathcal{O}_{X_5} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-7}$. By the projection formula:

$$
\begin{aligned}
\pi_{ijk_*}\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2)) &= \mathcal{O}_{\mathbb{P}^1}(2) \otimes \pi_{ijk_*}\mathcal{O}_{X_5} \\
&= \mathcal{O}_{\mathbb{P}^1}(2) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-7}) \\
&= (\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{O}_{\mathbb{P}^1}) \oplus (\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{L}^{-1}) \oplus \cdots \oplus (\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{L}^{-7}).
\end{aligned}
$$

From the equality $\mathcal{L}^8 = \mathcal{O}_{\mathbb{P}^1}(\pi_{ijk_*}(R_i + R_j + R_k))$, we get that the degree of $\mathcal{L}$ is equal to 6, and this implies that $\mathcal{O}_{\mathbb{P}^1}(2) \otimes \mathcal{L}^{-n}$ has no sections for every $n = 1, \ldots, 7$. Therefore, $\pi_{ijk_*}\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2)) = \mathcal{O}_{\mathbb{P}^1}(2)$, and it follows that $h^0(\pi_{ijk}^*(\mathcal{O}_{\mathbb{P}^1}(2))) = 3$. Finally, the line bundle $\mathcal{O}_{X_5}(R_i)$ has 3 sections since $R_i \sim \pi_{ijk}^*(-K_{\mathbb{P}^1})$. □

Using the geometry of the Kummer surface, we can determine some vanishing thetanulls on a Humbert-Edge curve of type 5.

Proposition 3.8. *Let $X_5 \subset \mathbb{P}^5$ be a Humbert-Edge curve of type 5. X_5 admits 80 vanishing thetanulls with 6 sections corresponding to Rosenhain tetrahedra of the associated singular Kummer surface K.*

Proof. By Proposition 3.3, X_5 is contained in a Kummer surface S. Denote by K the singular Kummer surface associated with S. For a Rosenhain tetrahedron on K, let D be the associated divisor in S (see Proposition 2.5). By [**14**, Proposition 3.1], we have that X_5 and D are dependent in Pic(S); in fact, X_5 appears as an element in the linear system $|2D|$. Using the fact that the canonical bundle of S is trivial and that $X_5 \in |2D|$, the adjunction formula implies that

$$K_{X_5} = (K_S + X_5)|_{X_5} = (2D)|_{X_5} = 2D|_{X_5}.$$

Thus, the divisor $D|_{X_5}$ is a theta characteristic. Only the calculation of $h^0(D|_{X_5})$ remains. Twisting the exact sequence of sheaves

$$0 \to \mathcal{O}_S(-X_5) \to \mathcal{O}_S \to \mathcal{O}_{X_5} \to 0$$

by $\mathcal{O}_S(D)$, we obtain the exact sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S(D) \to \mathcal{O}_{X_5}(D) \to 0,$$

and therefore, we obtain the exact sequence in cohomology

$$0 \to H^0(S, -D) \to H^0(S, D) \to H^0(X_5, D|_{X_5}) \to H^1(S, -D).$$

We have $H^0(S, -D) = 0$ and since D is very ample, by Mumford vanishing theorem we obtain that $H^1(S, -D) = 0$. Then, $H^0(S, D) = H^0(X_5, D|_{X_5})$ and from $\dim|D| = 5$ it follows that $h^0(X_5, D|_{X_5}) = 6$. We conclude the proof recalling that there are 80 Rosenhain tetrahedra associated with a singular Kummer surface (see Proposition 2.5). □

We conclude this subsection with the following remarks:

- The way to construct the vanishing thetanulls for a Humbert-Edge curve of type 5 using the geometry of a Kummer surface differs completely from the classical case: a Humbert's curve X admits exactly 10 vanishing thetanulls and can be constructed by taking the quotients of X by the subgroup generated by two involutions (see the proof of [**9**, Theorem 2.2]), the geometry of the Del Pezzo surface containing X is not involved in such process.
- The procedure used in Proposition 3.8 to construct the vanishing thetanulls holds true for every smooth curve on degree 16 and genus 17 on a Kummer surface S. Indeed, if Y is any smooth curve of degree 16 and genus 17 contained in S, then using again [**14**, Proposition 3.1] we have that Y and D are dependent on Pic(S) and the previous argument holds.

3.3. Moduli space of Humbert-Edge curves of type 5

As we mention in Remark 3.4, given a Humbert-Edge curve of type 5 it is always possible to associate a hyperelliptic curve of genus 2 and vice versa. Here, we discuss about this fact, and using the results of Carocca, Gónzalez-Aguilera, Hidalgo, and Rodríguez [**3**], we prove that the moduli space of Humbert-Edge curves of type 5 is isomorphic to the moduli space of hyperelliptic curves of genus 2.

Given a Humbert-Edge curve X_5 of type 5, using Edge's idea of considering the coefficients as points in $\mathbb{P}^3$ and the unique rational normal curve in $\mathbb{P}^3$ through them, one is able to write down the equations of X_5 in normal form (see the proof of Proposition 4):

$$Q_i = \sum_{j=0}^{5} a_j^i x_j^2, \quad i = 0, \ldots, 3, \tag{6}$$

where $a_j \in \mathbb{C}$ for $j = 0, \ldots, 5$ and $a_j \neq a_k$ if $j \neq k$. Note that since the rational normal curve that we are considering is constructed via the Veronese map $\nu : \mathbb{P}^1 \to \mathbb{P}^3$, one can fix three points in $\mathbb{P}^1$, and therefore, X_5 depends only on three different complex numbers. Thus, we can assume that we are fixing the points $0, 1$, and ∞, and then, we have three free different parameters λ_1, λ_2, and λ_3 defining the curve X_5. Since this idea can be carried out in the general case of Humbert-Edge curves of type n, with this fact in mind in [**3**, Section 4.1] the authors found an embedding for Humbert-Edge curves of type n in $\mathbb{P}^n$ in such way that the equations depend on $n - 2$ different parameters. In the particular case of the Humbert-Edge curve X_5 of type 5, the equations take the form

$$x_0^2 + x_1^2 + x_2^2 = 0$$
$$\lambda_1 x_0^2 + x_1^2 + x_3^2 = 0$$
$$\lambda_2 x_0^2 + x_1^2 + x_4^2 = 0$$
$$\lambda_3 x_0^2 + x_1^1 + x_5^2 = 0,$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0, 1\}$ are different complex numbers. To emphasize the dependence on the parameters $\lambda_1, \lambda_2, \lambda_3$ and considering that we are fixing $0, 1, \infty$, we denote this curve as $X_5(\lambda_1, \lambda_2, \lambda_3)$. Also, note that if we consider the degree 32 map given by

$$\pi_{(\lambda_1, \lambda_2, \lambda_3)} : X_5(\lambda_1, \lambda_2, \lambda_3) \to \qquad \mathbb{P}^1$$
$$(x_0 : \ldots : x_5) \mapsto -\left(\frac{x_1}{x_0}\right)^2,$$

then

$$\{0, 1, \infty, \lambda_1, \lambda_2, \lambda_3\} \tag{7}$$

is the branch locus of $\pi_{(\lambda_1, \lambda_2, \lambda_3)}$.

On the other hand, if C is a hyperelliptic curve of genus 2, then we can write the equation which defines C as

$$y^2 = (x - a_0)(x - a_1) \cdots (x - a_5), \tag{8}$$

where $a_j \in \mathbb{C}$ for $j = 0, \ldots, 5$ and $a_j \neq a_k$ if $j \neq k$. Since there always exists an automorphism of $\mathbb{P}^1$ which carries a tuple of different complex numbers (a_0, a_1, a_2) to $(0, 1, \infty)$, we may assume that C is given by the equation

$$y^2 = x(x - 1)(x - \lambda_1)(x - \lambda_2)(x - \lambda_3), \tag{9}$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0, 1\}$ are different. Similarly as before, we denote by $C(\lambda_1, \lambda_2, \lambda_3)$ the hyperelliptic curve of genus 2 with parameters $0, 1, \infty, \lambda_1, \lambda_2$ and λ_3. In addition, since $C(\lambda_1, \lambda_2, \lambda_3)$ is a hyperelliptic curve of genus 2, there exists a degree 2 map $\rho_{(\lambda_1, \lambda_2, \lambda_3)} : C(\lambda_1, \lambda_2, \lambda_3) \to \mathbb{P}^1$ whose branch locus is precisely given by (7).

In both cases of Humbert-Edge curves of type 5 and hyperelliptic curves of genus 2, given such a curve we have a map to $\mathbb{P}^1$ with a specific branch locus. In fact, the branch locus determines the curve modulo isomorphism. Indeed, Hidalgo, Reyes-Carocca, and Valdés determined in [**13**, Section 2.3] when two generalized Fermat curves are isomorphic, in particular, the following result can be deduced:

Proposition 3.9. *Let $X_5(\lambda_1, \lambda_2, \lambda_3)$ and $X_5(\mu_1, \mu_2, \mu_3)$ be Humbert-Edge curves of type 5. The following statements are equivalent:*

1. *$X_5(\lambda_1, \lambda_2, \lambda_3)$ is isomorphic to $X_5(\mu_1, \mu_2, \mu_3)$.*
2. *There exists a Möbius transformation M such that*

$$\{M(0), M(1), M(\infty), M(\lambda_1), M(\lambda_2), M(\lambda_3)\} = \{0, 1, \infty, \mu_1, \mu_2, \mu_3\}.$$
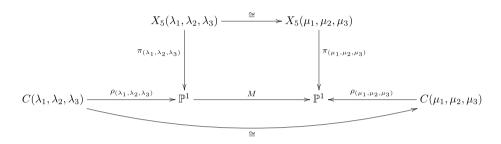
In the case of hyperelliptic curves of genus 2, we have an analogous result (see [**8**, Section III.7.3]):

Proposition 3.10. *Let $C(\lambda_1, \lambda_2, \lambda_3)$ and $C(\mu_1, \mu_2, \mu_3)$ be hyperelliptic curves of genus 2. The following statements are equivalent:*

1. *$C(\lambda_1, \lambda_2, \lambda_3)$ is isomorphic to $C(\mu_1, \mu_2, \mu_3)$.*
2. *There exists a Möbius transformation M such that*

$$\{M(0), M(1), M(\infty), M(\lambda_1), M(\lambda_2), M(\lambda_3)\} = \{0, 1, \infty, \mu_1, \mu_2, \mu_3\}.$$

The above results can be summarized in the following commuting diagram:

Now, we briefly discuss the construction of the moduli space of Humbert-Edge curves of type 5. To construct such moduli space, we follow Section 4.2 of [**3**]. Let HE_5 be the set of all Humbert-Edge curves of type 5. Since any Humbert-Edge curve of type 5 has genus 17, we have a map $r_5 : HE_5 \to \mathcal{M}_{17}$ defined by $r_5(X_5(\lambda_1, \lambda_2, \lambda_3)) = [X_5(\lambda_1, \lambda_2, \lambda_3)]$, where $\mathcal{M}_{17}$ is the moduli space of curves of genus 17. By Proposition 3.9, we can consider the isomorphism class of a Humbert-Edge curve of type 5, and this gives an equivalence relation on HE_5. Let $\mathcal{HE}_5$ be the set obtained from this equivalence relation. We have a projection map $p_5 : HE_5 \to \mathcal{HE}_5$, and we have a well-defined map $q_5 : \mathcal{HE}_5 \to \mathcal{M}_{17}$ so that $r_5 = q_5 \circ p_5$. We have that $\mathcal{HE}_5$ is a moduli space, and we call it *the moduli space of Humbert-Edge curves of type* 5, that is the set of Humbert-Edge curves of type 5 modulo isomorphism.

We denote by $\mathcal{H}_2$ the moduli space of hyperelliptic curves of genus 2. By Propositions 3.10 and 3.9, we have a well-defined map

$$f_5 : \mathcal{H}_2 \to \mathcal{HE}_5$$
$$[C(\lambda_1, \lambda_2, \lambda_3)] \mapsto [X_5(\lambda_1, \lambda_2, \lambda_3)]$$

By construction, it is immediate that f_5 is a surjective map. The following proposition deals with the injectivity.

Proposition 3.11. *The above map $f_5 : \mathcal{H}_2 \to \mathcal{HE}_5$ is injective. Therefore, f_5 is an isomorphism of moduli spaces.*

Proof. Let $C(\lambda_1, \lambda_2, \lambda_3)$ and $C(\mu_1, \mu_2, \mu_3)$ be hyperelliptic curves of genus 2. Assume that $f_5([C(\lambda_1, \lambda_2, \lambda_3)]) = f_5([C(\mu_1, \mu_2, \mu_3)])$; that is $[X_5(\lambda_1, \lambda_2, \lambda_3)] = [X_5(\mu_1, \mu_2, \mu_3)]$. Then, there exists an isomorphism between $X_5(\lambda_1, \lambda_2, \lambda_3)$ and $X_5(\mu_1, \mu_2, \mu_3)$. By Proposition 3.9, there exists a Möbius transformation M such that

$$\{M(0), M(1), M(\infty), M(\lambda_1), M(\lambda_2), M(\lambda_3)\} = \{0, 1, \infty, \mu_1, \mu_2, \mu_3\}.$$

Thus, by Proposition 3.10 we conclude that the hyperelliptic curves $C(\lambda_1, \lambda_2, \lambda_3)$ and $C(\mu_1, \mu_2, \mu_3)$ are isomorphic. $\qquad\square$

On the other hand, in [**3**, Proposition 4.3] the authors proved that the map $q_5 : \mathcal{HE}_5 \to \mathcal{M}_{17}$ is injective. Thus, using the isomorphism f_5 we obtain the following:

Corollary 3.12. *The moduli space $\mathcal{H}_2$ of hyperelliptic curves of genus 2 is a three-dimensional closed algebraic variety in $\mathcal{M}_{17}$ via the composition $q_5 \circ f_5 : \mathcal{H}_2 \hookrightarrow \mathcal{M}_{17}$.*

We conclude this paper noting that in the general case, the moduli space of Humbert-Edge curves of type n, where $n \geq 5$ is an odd number, is isomorphic to the moduli space of hyperelliptic curves of genus $\frac{n-1}{2}$. In general, a Humbert-Edge curve $X_n(\lambda_1, \ldots, \lambda_{n-2})$ of type n in $\mathbb{P}^n$ can be written as

$$
\begin{aligned}
x_0^2 + x_1^2 + x_2^2 &= 0 \\
\lambda_1 x_0^2 + x_1^2 + x_3^2 &= 0 \\
\lambda_2 x_0^2 + x_1^2 + x_4^2 &= 0 \\
&\vdots \\
\lambda_{n-2} x_0^2 + x_1^1 + x_n^2 &= 0,
\end{aligned}
$$

where $\lambda_1, \ldots, \lambda_{n-2} \in \mathbb{C} \setminus \{0, 1\}$ are different (see [**3**, Section 4.1]).

Since we have an analogous of Proposition 3.9 for the general case (see [**13**, Section 2.3]), then the argument to construct the moduli space of Humbert-Edge curves of type 5 in fact holds true for the general case of Humbert-Edge curves of type n (see Section 4.2 of [**3**]). Therefore, we can consider *the moduli space $\mathcal{HE}_n$ of Humbert-Edge curves of type n*, and we have a well-defined map $q_n : \mathcal{HE}_n \to \mathcal{M}_{g_n}$ where $\mathcal{M}_{g_n}$ is the moduli space of curves of genus $g_n = 2^{n-2}(n-3) + 1$.

On the other hand, a hyperelliptic curve $C(\lambda_1, \ldots, \lambda_{2g-1})$ of genus g is given by the equation

$$
y^2 = x(x - 1)(x - \lambda_1) \cdots (x - \lambda_{2g-1}),
$$

where $\lambda_1, \ldots, \lambda_{2g-1} \in \mathbb{C} \setminus \{0, 1\}$ are different. Since also Proposition 3.10 can be generalized in this general context (see [**8**, Section III.7.3]), under the assumptions that $n \geq 5$ is odd and $n - 2 = 2g - 1 = d$, we can define a surjective map

$$
\begin{aligned}
f_d : \mathcal{H}_g &\to \mathcal{HE}_n \\
[C(\lambda_1, \ldots, \lambda_d)] &\mapsto [X_n(\lambda_1, \ldots, \lambda_d)],
\end{aligned}
$$

where $\mathcal{H}_g$ denotes the moduli space of hyperelliptic curves of genus $g = \frac{n-1}{2}$. Finally, applying the argument of Proposition 3.11 and using the fact that the natural map $q_n : \mathcal{HE}_n \to \mathcal{M}_{g_n}$ is injective (see [**3**, Proposition 4.3]) we conclude the following:

Proposition 3.13. *If $n \geq 5$ is an odd number and $n - 2 = 2g - 1 = d$, then the map $f_d : \mathcal{H}_g \to \mathcal{HE}_n$ is an isomorphism of moduli spaces. In particular, we have that $\mathcal{H}_g$ is an $(n - 2)$-dimensional closed variety in $\mathcal{M}_{g_n}$ via the composition $q_n \circ f_d : \mathcal{H}_g \hookrightarrow \mathcal{M}_{g_n}$, where $g_n = 2^{n-2}(n-3) + 1$.*

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