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Cover Product and Betti Polynomial of Graphs

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Abstract. The cover product of disjoint graphs G and H with fixed vertex covers $C(G)$ and $C(H)$, is the graph $G \circledast H$ with vertex set $V(G) \cup V(H)$ and edge set

$$E(G) \cup E(H) \cup \{\{i,j\} : i \in C(G), j \in C(H)\}.$$

We describe the graded Betti numbers of $G \circledast H$ in terms of those of G and H. As applications we obtain: (i) For any positive integer k there exists a connected bipartite graph G such that $\operatorname{reg} R/I(G) = \mu_S(G) + k$, where, $I(G)$ denotes the edge ideal of G, $\operatorname{reg} R/I(G)$ is the Castelnuovo–Mumford regularity of $R/I(G)$ and $\mu_S(G)$ is the induced or strong matching number of G; (ii) The graded Betti numbers of the complement of a tree depends only upon its number of vertices; (iii) The h-vector of $R/I(G \circledast H)$ is described in terms of the h-vectors of $R/I(G)$ and $R/I(H)$. Furthermore, in a different direction, we give a recursive formula for the graded Betti numbers of chordal bipartite graphs.

1 Introduction

Let $R := \Bbbk[x_1, \dots, x_n]$ be the polynomial ring over a field $\Bbbk$ with $\deg x_i = 1$. The *edge ideal* for a (simple) graph G, with vertex set $\{1, \dots, n\}$, is the monomial ideal $I(G) = \langle x_i x_j : \{i,j\} \text{ is an edge of } G \rangle$. In general, each monomial ideal $I \subseteq R$ has associated a minimal graded free resolution

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{pj}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1j}} \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

where $R(-j)$ denotes the R-module obtained by shifting the degrees of R by j, and the nonnegative integers $\beta_{ij}(R/I) := \beta_{ij}$ are called the *graded Betti numbers* of R/I. A basic problem in commutative algebra is to describe these numbers as well as some homological invariants associated with them. But, even for edge ideals, which are quadratic and square-free, these problems are wide open. An interpretation of these invariants in terms of combinatorial information encoded in the graph have been the focus of much research over the last number of years; see, *e.g.*, [8, 17–20, 24, 32, 34], or [14, 23] for surveys on these developments. Two such invariants are the *projective dimension* and the (Castelnuovo–Mumford) *regularity*:

$$\operatorname{pdim} R/I := \max\{i : \beta_{ij} \neq 0\} \quad \text{and} \quad \operatorname{reg} R/I := \max\{j - i : \beta_{ij} \neq 0\}.$$

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Projective dimension and regularity tell us the length and the width of the minimal resolution of R/I. So, these two invariants provide an estimate of the complexity of computing a minimal resolution.

For edge ideals, one can mention some representative results about these invariants:

(i) Let $\alpha(G)$ denote the cardinality of a largest minimal vertex cover of G (definitions are given below). Then $\operatorname{pdim} R/I(G) \geq \alpha(G)$, with equality if G is sequentially Cohen–Macaulay (say, chordal graphs [10], or more generally graphs with no chordless cycles of length other than 3 or 5 [33]).

(ii) For any graph G, it holds that $\operatorname{reg} R/I(G) \geq \mu_S(G)$, where $\mu_S(G)$ denotes the induced or strong matching number of G [19, Lemma 2.2]. If $\beta_{ij}(R/I(G)) \neq 0$ for a graph G, then $i+1 \leq j \leq 2i$ [14, Thm. 3.2.3], so $\operatorname{pdim} R/I(G) \geq \operatorname{reg} R/I(G) \geq \mu_S(G)$. If a bipartite graph G is unmixed or sequentially Cohen–Macaulay, then $\operatorname{reg} R/I(G) = \mu_S(G)$; see [20] and [30], respectively. The same is true for chordal bipartite graphs, or more generally for weakly chordal graphs [34, Prop. 20].

(iii) Let G be a chordal graph and let $\overline{G}$ denote its complement. If $i \geq 1$ and $j \neq i+1$, then $\beta_{ij}(R/I(\overline{G})) = 0$; otherwise, $\beta_{ij}(R/I(\overline{G})) = \sum(c(H)-1)$, where H runs over all the induced subgraphs of G with j vertices, and $c(H)$ denotes the number of connected components of H [8, Thm. 3.2] (Theorem 5.6).

To our knowledge, not much is known even for bipartite graphs. After computing many examples, we were led to suspect that for any connected bipartite graph G, $\mu_S(G) \leq \operatorname{reg} R/I(G) \leq \mu_S(G)+1$, but this turned out to be false. In fact, with this problem in mind we were guided to our main result, Theorem 3.5, and, as one of its applications we will prove that for any positive integer k there exists a connected bipartite graph such that $\operatorname{reg} R/I(G) = \mu_S(G)+k$. As another application, we will show that the graded Betti numbers of the complement of a tree depend only upon its number of vertices.

Define the *Betti polynomial* of R/I as follows:

$$B(R/I; x, y) = \sum_{i=0}^{p} \sum_{j} \beta_{ij} x^i y^{j-i}.$$

Note that the x-degree and the y-degree correspond to the projective dimension and regularity of R/I. Betti polynomials were introduced in [8, Def. 6.1]; we interchange the role of the variables x and y used there. Here we study the Betti polynomial in the case that I is the edge ideal of a graph. To keep our notation simple we will write $B(G; x, y)$ instead of $B(R/I(G); x, y)$, and similarly for $\beta_{ij}(G)$, $\operatorname{pdim}(G)$ and $\operatorname{reg}(G)$. We remark that if G has no edges, then $B(G; x, y) := 1$. Betti polynomials provide a compact way of encoding the graded Betti numbers. For instance, in [17, Thm. 5.2.4] it is shown that

$$\beta_{ij}(K_{m,n}) = \begin{cases} \sum_{r+s=i+1;\ r,s\geq 1} \binom{m}{r}\binom{n}{s} & \text{if } j = i+1, \\ 0 & \text{if } j \neq i+1, \end{cases}$$

where $K_{m,n}$ denotes the complete bipartite graph. From this result (Example 3.3), it follows that

$$B(K_{m,n};x,y) = 1 + x^{-1}y\big[(1+x)^m - 1\big]\big[(1+x)^n - 1\big]. \tag{1.1}$$

As we will see in our main result, Theorem 3.5, finding Betti polynomials, rather than describing the Betti numbers explicitly, will provide us with a tool to describe relationships among graded Betti numbers of different graphs or their induced subgraphs.

A set of vertices of a graph G is said to be *independent* if no two vertices are joined by an edge. A *vertex cover* of G is a set of vertices $C \subseteq V(G)$ such that $e \cap C \neq \varnothing$ for any edge e of G, or equivalently if $V(G)\backslash C$ is an independent set of G. We define the *cover product* of two disjoint graphs G and H, with fixed vertex covers $C(G)$ and $C(H)$, respectively, as the graph $G \circledast H$ with vertex set $V(G) \cup V(H)$ and edge set

$$E(G) \cup E(H) \cup \{\{i,j\} : i \in C(G), j \in C(H)\}.$$

Our aim is to describe the Betti polynomial of $G \circledast H$ in terms of those of G and H. To simplify notation we do not specify in $G \circledast H$ the dependence of the vertex covers $C(G)$ and $C(H)$. The cover product of two graphs is a natural generalization of the join of two graphs, but to our knowledge, it has not been studied yet.

Our main result is the following theorem.

***Theorem* 3.5** *Let G and H be graphs with vertex covers $C(G)$ and $C(H)$, respectively. Set $m = |C(G)|$ and $n = |C(H)|$. Then*

$$B(G \circledast H;x,y) = (1+x)^n \widetilde{B}(G;x,y) + (1+x)^m \widetilde{B}(H;x,y) + B(K_{m,n};x,y),$$

where $\widetilde{B}(G;x,y) = B(G;x,y) - 1$ and $K_{m,n}$ denotes the complete bipartite graph on $m+n$ vertices.

Since the Betti polynomial of the complete bipartite graph depends only upon the numbers of vertices m and n, we remark that Theorem 3.5 describes the Betti polynomial of $G \circledast H$ in terms of those of G and H and just the cardinalities of the vertex covers $C(G)$ and $C(H)$. A combinatorial understanding of this result would be helpful in the study of graded Betti numbers.

Several applications are given:

- Recall that the *induced* or *strong matching number* of G, $\mu_S(G)$ (following the notation in [11]), is the largest k such that the disjoint union of k edges is an induced subgraph of G. In the direction of studying $\operatorname{reg}(G)$ for bipartite graphs, as suggested in [30, Question 3.5], we obtain *Corollary* 3.7: *For any positive integer k there exists a connected bipartite graph G such that $\mu_S(G) = 2k$ and $\operatorname{reg}(G) = \mu_S(G) + k$.*
- The h-vector of $R/I(G \circledast H)$ can be described in terms of the h-vectors of $R/I(G)$ and $R/I(H)$; see Theorem 4.1. We remark that our main result and Theorem 4.1 generalize the results [24, Cor. 3.4, Cor. 4.11] and [32, Lemma 5.4], where the vertex covers consist of all the vertices of G and H, respectively.
- A graph is said to be *chordal* if it contains no induced cycle with four or more vertices. For a chordal graph G, a description of $B(\overline{G};x,y)$, where $\overline{G}$ denotes the complement of G, is well known; see Theorem 5.6 below. However, from that description it is not evident that the Betti polynomial of the complement of a tree only

depends upon its number of vertices. We observe *Corollary* 5.3: *If* G *is a tree on* $n+1$ *vertices, then* $B(\overline{G}; x, y) = 1 + x^{-1}y[\binom{n}{2}x^2 + \cdots + (n-1)\binom{n}{n}x^n]$.

Finally, in a different direction, we show (see Proposition 6.7) that there is a recursive formula for the graded Betti numbers of chordal bipartite graphs. A consequence of this result is that these numbers do not depend on the characteristic of the field.

A collection of interesting open problems related to Betti numbers can be found in [25]. In what follows, we refer to [2] and [3] for unexplained terminology on graph theory and algebraic combinatorics, respectively.

2 Some Examples

Since we will describe the Betti polynomial of $G \circledast H$ in terms of those of G and H, let us recall the Betti polynomials of some other graphs. Denote by K_n the complete graph on n vertices and by C_n the cycle on n vertices.

Example 2.1 ([17, Thm. 5.1.1])

$$\begin{aligned} B(K_n; x, y) &= 1 + xy\big[1 + 2(1+x) + \cdots + (n-1)(1+x)^{n-2}\big] \\ &= 1 + x^{-1}y\big[\tbinom{n}{2}x^2 + \cdots + (n-1)\tbinom{n}{n}x^n\big]. \end{aligned}$$

Example 2.2 ([5, 6]) For $n \geq 4$,

$$B(\overline{C_n}; x, y) = B(K_{n-1}; x, y) + xy\big[x^{n-2} - (1+x)^{n-2} + x^{n-3}y\big].$$

Example 2.3 It is well known that Betti numbers depends upon the characteristic of the field. Let G be the graph on 11 vertices given in [19], up to relabeling:

$$\begin{aligned} G = \big\{ &\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,8\}, \{8,9\}, \{9,10\}, \{10,11\}, \\ &\{1,11\}, \{1,6\}, \{1,7\}, \{2,5\}, \{2,9\}, \{2,11\}, \{3,5\}, \{3,7\}, \{3,8\}, \{4,6\}, \\ &\{4,10\}, \{4,11\}, \{5,9\}, \{6,9\}, \{8,10\}\big\}. \end{aligned}$$

In characteristic 0, using CoCoA [1],

$$\begin{aligned} B_0(G; x, y) = 1 &+ 25xy + 80x^2y + 25x^2y^2 + 95x^3y + 152x^3y^2 + 40x^4y \\ &+ 356x^4y^2 + 6x^5y + 400x^5y^2 + 245x^6y^2 + 80x^7y^2 + 11x^8y^2, \end{aligned}$$

while in characteristic 2, $B_2(G; x, y) = B_0(G; x, y) + x^8y^3 + x^9y^2$.

Note 2.4 In [26], the subgraph polynomial of a graph was introduced: $S(G; x, y) = \sum_{i,j} b_{ij}x^iy^j$, where b_{ij} is the number of subgraphs of G with i edges and j vertices; we interchange the role of the variables x and y used there. One may notice that this polynomial is similar to the Betti polynomial, but associated with the Taylor resolution of $R/I(G)$ [22, Section 6.1]. Since the Hilbert series $\mathrm{Hilb}(R/I(G), t)$ can be computed from any free resolution of $R/I(G)$ [3, Lemma 4.1.13], it follows that if G is a graph on n vertices, then $(1-t)^n\,\mathrm{Hilb}(R/I(G), t) = S(G; -1, t)$, which is the main result in [9] and [26].

Let I and J be monomial ideals of a polynomial ring S over a field, with generating sets over disjoint sets of variables. If $F_\bullet$ and $G_\bullet$ are minimal graded free resolutions of S/I and S/J, respectively, then $F_\bullet \otimes G_\bullet$ is a minimal graded free resolution of $S/(I+J)$ [18, Lemma 2.1]. So, for the disjoint union $G \sqcup H$ of the graphs G and H, we have the following lemma.

Lemma 2.5 $B(G \sqcup H; x, y) = B(G; x, y) B(H; x, y)$.

The following well-known results are immediate from Lemma 2.5 ([17, Prop. 2.2.8], [34, Lemma 8], [8, Lemma 6.2], [8, Lemma 6.3], respectively)

Corollary 2.6

(i) $\operatorname{pdim}(G \sqcup H) = \operatorname{pdim}(G) + \operatorname{pdim}(H)$;
(ii) $\operatorname{reg}(G \sqcup H) = \operatorname{reg}(G) + \operatorname{reg}(H)$;
(iii) *if v is an isolated vertex, then* $B(G; x, y) = B(G\backslash\{v\}; x, y)$;
(iv) *if $e = uv$ is an isolated edge, then* $B(G; x, y) = (1 + xy) B(G\backslash\{u, v\}; x, y)$.

3 Cover Product

For a vertex v of G, define $N(v) = \{w \in V(G) : vw \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$. For a subset of vertices W of G, denote by $G\backslash W$ the subgraph of G obtained by deleting the vertices in W. Abusing notation, we write $G\backslash v$ instead of $G\backslash\{v\}$. A useful result is the following lemma.

Lemma 3.1 ([8, Lemma 6.4]) *Let G be a graph with a vertex v and a set of vertices $U = \{u_1, \dots, u_k\}$, all different from v. If $N(v) \subseteq N(u)$ for all $u \in U$, then*

$$B(G; x, y) = B(G\backslash v; x, y) + (1 + x)^k \big[B(G\backslash U; x, y) - B(G\backslash(U \cup \{v\}); x, y) \big].$$

Example 3.2 Let u be a vertex of the graph G and let G^u be the graph obtained from G by duplicating the vertex u, *i.e.*,

$$V(G^u) = V(G) \cup \{v'\} \quad \text{and} \quad E(G^u) = E(G) \cup \{u'v' : u' \in N(u)\},$$

where v' is a new vertex. (For more about duplicating a vertex, see [21, 27]). By Lemma 3.1, with $U = \{u\}$ and $v := v'$, it follows that

$$B(G^u; x, y) = (2 + x) B(G; x, y) - (1 + x) B(G\backslash u; x, y).$$

Example 3.3 $B(K_{m,n}; x, y) = 1 + x^{-1} y [1 + (1 + x)^{m+n} - (1 + x)^m - (1 + x)^n]$. In fact,

$$\begin{aligned} B(K_{m,n}; x, y) &= B(K_{m,n-1}; x, y) + (1 + x)^{n-1} \big[B(K_{m,1}; x, y) - 1 \big] \\ &= 1 + x^{-1} y \big[1 + (1 + x)^{m+n-1} - (1 + x)^m - (1 + x)^{n-1} \big] \\ &\quad + (1 + x)^{n-1} \big[1 + x^{-1} y [1 + (1 + x)^{m+1} - (1 + x)^m - (1 + x)] \big] \\ &\quad - (1 + x)^{n-1} \\ &= 1 + x^{-1} y \big[1 + (1 + x)^{m+n} - (1 + x)^m - (1 + x)^n \big]. \end{aligned}$$

Now write $\widetilde{B}(G; x, y) := B(G; x, y) - 1$.

Lemma 3.4 ([24, Cor. 3.4], [32, Lemma 5.4]) *Let G and H be two graphs with m and n vertices, respectively. Then, with the vertex covers $C(G) = V(G)$ and $C(H) = V(H)$,*

$$B(G \circledast H; x, y) = (1+x)^n \widetilde{B}(G; x, y) + (1+x)^m \widetilde{B}(H; x, y) + B(K_{m,n}; x, y).$$

The following theorem is our main result, and is a generalization of Lemma 3.4.

Theorem 3.5 *Let G and H be graphs with vertex covers $C(G)$ and $C(H)$, respectively. Set $m = |C(G)|$ and $n = |C(H)|$. Then*

$$B(G \circledast H; x, y) = (1+x)^n \widetilde{B}(G; x, y) + (1+x)^m \widetilde{B}(H; x, y) + B(K_{m,n}; x, y). \tag{3.1}$$

Proof We proceed by induction on $|V(G)|+|V(H)|$. Suppose $|V(G)| = |V(H)| = 1$. If one of $C(G)$ or $C(H)$ is the empty set, $K_{m,n}$ has no edges, and so $B(K_{m,n}; x, y) = 1$. (Recall that in the introduction, we defined the Betti polynomial of a graph with no edges to be identically 1.) In this case both sides of equation (3.1) are equal to 1. In the case where $C(G) = V(G)$ and $C(H) = V(H)$, both sides of equation (3.1) are equal to $B(K_{1,1}; x, y)$. So, the initial step is verified.

Now let G and H be arbitrary graphs. If $C(G) = V(G)$ and $C(H) = V(H)$, the result is Lemma 3.4. Thus, we may assume that H has a vertex $v \notin C(H)$. Since $C(H)$ is a vertex cover of H, in the graph $G \circledast H$ it holds that $N(v) \subseteq C(H)$, and hence $N(v) \subseteq N(u)$ for all $u \in U := C(G)$. So, by Lemma 3.1,

$$\begin{aligned} B(G \circledast H; x, y) &= B\big((G \circledast H)\backslash v; x, y\big) \\ &\quad + (1+x)^m \big[B((G \circledast H)\backslash U; x, y) - B((G \circledast H)\backslash (U \cup \{v\}); x, y)\big]. \end{aligned}$$

By induction, taking the vertex covers $C(G)$ for G and $C(H)$ for $H\backslash v$,

$$B(G \circledast (H\backslash v); x, y) = (1+x)^n \widetilde{B}(G; x, y) + (1+x)^m \widetilde{B}(H\backslash v; x, y) + B(K_{m,n}; x, y).$$

Then, noticing that $B((G \circledast H)\backslash v; x, y) = B(G \circledast (H\backslash v); x, y)$,

$$\begin{aligned} B(G \circledast H; x, y) &= (1+x)^n \widetilde{B}(G; x, y) + (1+x)^m \widetilde{B}(H\backslash v; x, y) + B(K_{m,n}; x, y) \\ &\quad + (1+x)^m [\widetilde{B}(H; x, y) - \widetilde{B}(H\backslash v; x, y)] \\ &= (1+x)^n \widetilde{B}(G; x, y) + (1+x)^m \widetilde{B}(H; x, y) + B(K_{m,n}; x, y). \end{aligned}$$ ∎

Corollary 3.6 *Let G and H be graphs with vertex covers $C(G)$ and $C(H)$, respectively. Set $m = |C(G)|$ and $n = |C(H)|$. Then*

(i) $\operatorname{pdim}(G \circledast H) = \max\{n + \operatorname{pdim}(G), m + \operatorname{pdim}(H), m + n - 1\}$,
(ii) $\operatorname{reg}(G \circledast H) = \max\{\operatorname{reg}(G), \operatorname{reg}(H)\}$.

Corollary 3.7 *For any positive integer k there exists a connected bipartite graph G such that $\mu_S(G) = 2k$ and $\operatorname{reg}(G) = \mu_S(G) + k$, where $\mu_S(G)$ is the induced matching number of G.*

Proof It is well known [34, Prop. 9] that for the cycle on n vertices, C_n, $n \geq 3$, it holds that $\operatorname{reg}(C_n) = \lfloor (n+1)/3 \rfloor$. In particular, $\operatorname{reg}(C_8) = 3 = \mu_S(C_8) + 1$. Let H be the disjoint union of k copies of C_8. Take $G = H \circledast v$, where v is a new vertex, and choose $C(H)$ to be a vertex cover of H with no adjacent vertices. Then $\mu_S(G) = 2k$ and $\operatorname{reg}(G) = 3k = \mu_S(G) + k$. ∎

The complete multipartite graph is the graph $K_{n_1,\dots,n_k}$ $(k \geq 2)$ in which vertices are adjacent if and only if they belong to different partite sets; *i.e.*, if $V_1, \dots, V_k$ are disjoint sets of vertices, with $n_i = |V_i|$, $i = 1, \dots, k$, then $K_{n_1,\dots,n_k}$ is the graph with vertices $\cup V_i$ and edges $\{\{i, j\} : i \in V_p, j \in V_q, p \neq q\}$. Set $n = \sum n_i$.

Corollary 3.8 ([17, Thm. 5.3.8])

$$B(K_{n_1,\dots,n_k}; x, y) = 1 + x^{-1}y\Big[1 + (k-1)(1+x)^n - \sum_{i=1}^{k}(1+x)^{n-n_i}\Big].$$

Proof Proceeding by induction on k, if $k = 2$, the right-hand side of the equation is equal to

$$1 + x^{-1}y\big[1 + (1+x)^{n_1+n_2} - (1+x)^{n_1} - (1+x)^{n_2}\big],$$

which coincides with equation (1.1). So we assume that $k > 2$.

Let us consider V_{n_k} as a graph with no edges; in particular, $\widetilde{B}(V_{n_k}; x, y) = 0$. By Theorem 3.5,

$$\begin{aligned}
&B(K_{n_1,\dots,n_k}; x, y)\\
&\quad = (1+x)^{n_k}\widetilde{B}(K_{n_1,\dots,n_{k-1}}; x, y) + (1+x)^{n-n_k}\widetilde{B}(V_{n_k}; x, y) + B(K_{n-n_k,n_k}; x, y)\\
&\quad = (1+x)^{n_k}\Big[x^{-1}y\Big[1 + (k-2)(1+x)^{n-n_k} - \sum_{i=1}^{k-1}(1+x)^{(n-n_k)-n_i}\Big]\Big]\\
&\qquad + 1 + x^{-1}y\big[1 + (1+x)^n - (1+x)^{n-n_k} - (1+x)^{n_k}\big]\\
&\quad = 1 + x^{-1}y\Big[1 + (k-1)(1+x)^n - \sum_{i=1}^{k}(1+x)^{n-n_i}\Big].
\end{aligned}$$

∎

Let $G_1, \dots, G_k$ be graphs over disjoint vertex sets. Let $C(G_i)$ be a fixed vertex cover of G_i, $i = 1, \dots, k$. Define the *multipartite cover product* of the graphs $G_1, \dots, G_k$ as the graph $\circledast_{i=1}^{k} G_i$, with vertices $\bigcup V(G_i)$ and edges

$$\bigcup_{i=1}^{k} E(G_i) \cup \big\{\{r, s\} : r \in C(G_p), s \in C(G_q), p \neq q\big\}.$$

Set $n_i = |V(G_i)|$, $n = \sum n_i$, $c_i = |C(G_i)|$ and $c = \sum c_i$. A repeated application of Theorem 3.5 yields the following theorem.

Theorem 3.9

$$B(\circledast_{i=1}^{k} G_i; x, y) = \sum_{i=1}^{k}(1+x)^{c-c_i}\widetilde{B}(G_i; x, y) + B(K_{c_1,\dots,c_k}; x, y).$$

Sketch of Proof Proceed by induction on k.

$$
\begin{aligned}
B(\circledast_{i=1}^{k} G_i; x, y)
&= (1+x)^{c_k}\widetilde{B}(\circledast_{i=1}^{k-1} G_i; x, y) + (1+x)^{c-c_k}\widetilde{B}(G_k; x, y) + B(K_{c-c_k, c_k}; x, y)\\
&= (1+x)^{c_k}\Big[\sum_{i=1}^{k-1}(1+x)^{c-c_k-c_i}\widetilde{B}(G_i; x, y) + B(K_{c_1,\dots,c_{k-1}}; x, y) - 1\Big]\\
&\quad + (1+x)^{c-c_k}\widetilde{B}(G_k; x, y) + B(K_{c-c_k, c_k}; x, y)\\
&= \sum_{i=1}^{k-1}(1+x)^{c-c_i}\widetilde{B}(G_i; x, y) + (1+x)^{c_k}\widetilde{B}(K_{c_1,\dots,c_{k-1}}; x, y)\\
&\quad + (1+x)^{c-c_k}\widetilde{B}(G_k; x, y) + B(K_{c-c_k, c_k}; x, y)\\
&= \sum_{i=1}^{k}(1+x)^{c-c_i}\widetilde{B}(G_i; x, y) + (1+x)^{c_k}\widetilde{B}(K_{c_1,\dots,c_{k-1}}; x, y) + B(K_{c-c_k, c_k}; x, y)\\
&= \sum_{i=1}^{k}(1+x)^{c-c_i}\widetilde{B}(G_i; x, y) + B(K_{c_1,\dots,c_k}; x, y). \qquad \blacksquare
\end{aligned}
$$

4 h-vectors

Let Δ be a simplicial complex of dimension $r-1$, so r is the largest cardinality of a face. Its *f-polynomial* is $f(\Delta, t) := t^r + f_1 t^{r-1} + \cdots + f_r$, where f_i is the number of faces of cardinality i, and its *h-polynomial* is $h(\Delta, t) := f(\Delta, t-1)$. If $\operatorname{Hilb}(R/I_\Delta, t)$ denotes the Hilbert series of the Stanley–Reisner ring R/I_Δ and

$$h(\Delta, t) = t^r + h_1 t^{r-1} + \cdots + h_s t^{r-s},$$

with $h_s \neq 0$, then

$$(1-t)^r \operatorname{Hilb}(R/I_\Delta, t) = 1 + h_1 t + \cdots + h_s t^s. \tag{4.1}$$

The polynomial in the right-hand side of equation (4.1) is known as the h-vector of R/I_Δ. Let

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{pj}} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1j}} \longrightarrow R \longrightarrow R/I_\Delta \longrightarrow 0$$

be a minimal graded free resolution of R/I_Δ, where $R = k[x_1, \dots, x_n]$, the x_i's are indeterminates, and n is the number of vertices of Δ. It can be verified [3, Cor. 4.1.14] that

$$\sum_{i=0}^{p}\sum_j \beta_{ij}(-1)^i t^j = (1-t)^{n-r}(1 + h_1 t + \cdots + h_s t^s). \tag{4.2}$$

The collection of independent sets of a graph G are the faces of a simplicial complex, known as the *independence complex* of the graph, and which, for simplicity, we denote by ΔG. The Stanley–Reisner ring associated with ΔG is the ring $R/I_{\Delta G}$, where $I_{\Delta G}$ is the ideal generated by the non-faces of ΔG. Since the edges of G are the minimal non-independent sets of the graph, it holds that $I_{\Delta G} = I(G)$; *i.e.*, $R/I(G)$ is precisely the Stanley–Reisner ring associated with the independence complex ΔG.

Let $\vec{h}(G, t) := 1 + h_1 t + \cdots + h_s t^s$ be the h-vector of $R/I(G) = R/I_{\Delta G}$. Let $G_1, \dots, G_k$ be graphs over disjoint sets of vertices. Set $n_i = |V(G_i)|$ and $n = \sum n_i$. Let $C(G_i)$ be

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a fixed vertex cover of G_i, $i = 1, \dots, k$. Set $c_i = |C(G_i)|$ and $c = \sum c_i$. Furthermore, $r_i = \dim \Delta G_i + 1$ and $r = \dim \Delta(\circledast_{i=1}^k G_i) + 1$.

Theorem 4.1

$$(1-t)^{n-r}\vec{h}(\circledast_{i=1}^k G_i, t) = \sum_{i=1}^k (1-t)^{n_i - r_i + c - c_i}\vec{h}(G_i, t) - (k-1)(1-t)^c.$$

Proof

$$\begin{aligned}
(1-t)^{n-r}\vec{h}(\circledast G_i, t) &= B(\circledast G_i; -t, t), \text{ (equation (4.2))}\\
&= \sum_{i=1}^k (1-t)^{c-c_i}\widetilde{B}(G_i; -t, t) + B(K_{c_1,\dots,c_r}; -t, t), \text{ (Theorem 3.9)}\\
&= \sum_{i=1}^k (1-t)^{n_i - r_i + c - c_i}\vec{h}(G_i, t) - \sum (1-t)^{c-c_i} + B(K_{c_1,\dots,c_k}; -t, t)\\
&= \sum_{i=1}^k (1-t)^{n_i - r_i + c - c_i}\vec{h}(G_i, t) - (k-1)(1-t)^c, \text{ (Corollary 3.8).}
\end{aligned}$$

■

5 Complements of Trees

Define $T_1(x) = T_2(x) = 0$, and, for $n \geq 3$,

$$T_n(x) = (xy)^{-1}\big[B(K_{n-1}; x, y) - 1\big] = 1 + 2(1+x) + \cdots + (n-2)(1+x)^{n-3}.$$

Lemma 5.1 *For $1 \leq k \leq n$ it holds that*

$$\begin{aligned}
B(K_n; x, y) = (1+x)\widetilde{B}(K_{n-1}; x, y) &+ B(K_{1,n-k}; x, y)\\
&+ xy(1+x)^{n-k}\big[T_{k+1}(x) - (1+x)T_k(x)\big].
\end{aligned}$$

We recall some more definitions. A *clique* of a graph is a set of pairwise adjacent vertices. A vertex v is said to be *simplicial* if $N(v)$ is a clique. It is well known that chordal graphs always have a simplicial vertex [7]. Actually, a graph is chordal if and only if one can repeatedly find a simplicial vertex and delete it from the graph until no vertex is left [12].

Corollary 5.2 *Let G be a connected chordal graph such that any two maximal cliques intersect in at most two vertices. If G has n vertices and n_k maximal cliques of cardinality k, then*

$$B(\overline{G}; x, y) = B(K_{n-1}; x, y) - xy\sum_{k\geq 3} n_k(1+x)^{n-k}T_k(x).$$

Proof Proceed by induction on n. If $n = 1$ or 2, both sides of the equation are equal to 1. Assume $n \geq 3$. Let v be a simplicial vertex of G and set $i = |N(v)|$. We have $E(\overline{G}) = E(\overline{G\backslash v}) \cup \{uv : u \in V(G)\backslash N(v)\}$. Let $G' = \overline{G\backslash v}$ and H' be the graph consisting only of the vertex v. Fix $C(G') = V(G)\backslash N[v]$ and $C(H') = \{v\}$. Note that $\overline{G} = G' \circledast H'$ and $G\backslash v$ is connected.

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Let m_k be the number of maximal cliques of cardinality k in $G\backslash v$. Then $m_i = n_i + 1$, $m_{i+1} = n_{i+1} - 1$, and $m_k = n_k$ for $k \neq i, i+1$.

$$\begin{aligned}
B(\overline{G}; x, y) &= B(G' \circledast H'; x, y)\\
&= (1+x)\widetilde{B}(\overline{G\backslash v}; x, y) + B(K_{1,n-i-1}; x, y), \quad \text{(Theorem 3.5)}\\
&= (1+x)\big[\widetilde{B}(K_{n-2}; x, y) - xy \sum_{k\geq 3} m_k (1+x)^{n-1-k} T_k(x)\big] + B(K_{1,n-i-1}; x, y)\\
&= (1+x)\widetilde{B}(K_{n-2}; x, y) - xy \sum_{k\geq 3} m_k (1+x)^{n-k} T_k(x) + B(K_{1,n-i-1}; x, y)\\
&= (1+x)\widetilde{B}(K_{n-2}; x, y) - xy \sum_{k\geq 3} n_k (1+x)^{n-k} T_k(x) + B(K_{1,n-i-1}; x, y)\\
&\quad + xy(1+x)^{n-(i+1)} T_{i+1}(x) - xy(1+x)^{n-i} T_i(x)\\
&= B(K_{n-1}; x, y) - xy \sum_{k\geq 3} n_k (1+x)^{n-k} T_k(x), \quad \text{(Lemma 5.1)}. \qquad \blacksquare
\end{aligned}$$

Corollary 5.3 *If G is a tree with $n+1$ vertices, then*

$$B(\overline{G}; x, y) = B(K_n, x, y) = 1 + x^{-1} y \Big[\binom{n}{2} x^2 + \cdots + (n-1)\binom{n}{n} x^n \Big].$$

Proof Apply Corollary 5.2, with $n_k = 0$ for all $k \geq 3$. $\blacksquare$

Corollary 5.4 *Let G be a connected chordal graph with n vertices. There are integers $m_3, m_4, \dots, m_\ell$, such that*

$$B(\overline{G}; x, y) = B(K_{n-1}; x, y) - xy \sum_{k\geq 3} m_k (1+x)^{n-k} T_k(x).$$

Proof Proceed by induction on n, cases $n = 1, 2$ being clear. We illustrate one more case. For $n = 3$, if G is a path on three vertices, take $0 = m_3 = m_4 = \cdots$. If G is a triangle, take $m_3 = 1$, $0 = m_4 = m_5 = \cdots$. Let v be a simplicial vertex of G and set $j = |N(v)|$. We proceed similarly as in the proof of Corollary 5.2.

$$\begin{aligned}
B(\overline{G}; x, y) &= (1+x)\widetilde{B}(\overline{G\backslash v}; x, y) + B(K_{1,n-j-1}; x, y), \quad \text{(Theorem 3.5)}\\
&= (1+x)\widetilde{B}(K_{n-2}; x, y) - xy \sum_{k\geq 3} m'_k (1+x)^{n-k} T_k(x) + B(K_{1,n-j-1}; x, y)\\
&= (1+x)\widetilde{B}(K_{n-2}; x, y) - xy \sum_{k\geq 3} m'_k (1+x)^{n-k} T_k(x) + B(K_{1,n-j-1}; x, y)\\
&\quad \pm xy(1+x)^{n-(j+1)} T_{j+1}(x) \pm xy(1+x)^{n-j} T_j(x), \quad \text{(Adding and subtracting)}\\
&= B(K_{n-1}; x, y) - xy \sum_{k\geq 3} m_k (1+x)^{n-k} T_k(x), \quad \text{(Lemma 5.1)}. \qquad \blacksquare
\end{aligned}$$

Question 5.5 Can the m_k's of Corollary 5.4 be described in terms of the combinatorics of the graph?

An explicit description of $B(\overline{G}; x, y)$ is given in [8].

Theorem 5.6 ([8, Thm. 3.2]) *Let G be a chordal graph. Then*

$$B(\overline{G}; x, y) = 1 + x^{-1} y \sum_H (c(H) - 1) x^{|H|},$$

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where H runs over all the induced subgraphs of G, and $c(H)$ denotes the number of connected components of H.

6 Recursive Formulae

Following [13], an edge $e = uv$ is called a *splitting edge* if $N(u) \subseteq N[v]$ or $N(v) \subseteq N[u]$. To simplify notation, for an edge $e = uv$ we define $N[e] = N(u) \cup N(v)$, and write $G\backslash e$ to mean the graph obtained from G by deleting the edge e, but not its vertices. Another useful result is the following theorem.

Theorem 6.1 ([13, Thm. 3.7]) *Let e be a splitting edge of G and $k = |N[e]| - 2$. Then*

$$B(G;x,y) = B(G\backslash e;x,y) + xy(1+x)^k B(G\backslash N[e];x,y).$$

Example 6.2 Return to Example 3.2. Let $G^{\underline{u}}$ be the graph obtained from G^u by adding the edge $e = uv'$. This edge e is a splitting edge of $G^{\underline{u}}$, so, by Theorem 6.1,

$$B(G^{\underline{u}};x,y) = (2+x)B(G;x,y)-(1+x)B(G\backslash u;x,y)+xy(1+x)^{|N(u)|}B(G\backslash N[u];x,y).$$

Recall that a graph is chordal if and only if one can repeatedly find a simplicial vertex and delete it from the graph until no vertex is left [12]. Since any edge incident to a simplicial vertex is a splitting edge, Theorem 6.1 gives a recursive way to compute the Betti polynomial of a chordal graph.

Corollary 6.3 *Let v be a simplicial vertex of G. Let $e_1, \dots, e_r$ be all the edges containing v and $k_i = |N[e_i]| - 2$. Then*

$$B(G;x,y) = B(G\backslash v;x,y) + xy\sum_{i=1}^{r}(1+x)^{k_i} B(G\backslash N[e_i];x,y).$$

Proof Apply Theorem 6.1 to the graphs G, $G\backslash e_1$, $(G\backslash e_1)\backslash e_2, \dots$. ∎

Corollary 6.3 may be rewritten as follows.

Corollary 6.4 *Let v be a simplicial vertex of G. Let $v_1, \dots, v_r$ be all the vertices adjacent to v and $k_i = |N[v_i]| - 2$. Then*

$$B(G;x,y) = B(G\backslash v;x,y) + xy\sum_{i=1}^{r}(1+x)^{k_i} B\big(G\backslash N[v_i];x,y\big).$$

Example 6.5 Let P_n denote the chordless *path graph* on n vertices. It follows from Corollary 6.4 that, for $n \geq 3$,

$$B(P_n;x,y) = B(P_{n-1};x,y) + xy(1+x)B(P_{n-3};x,y).$$

Note 6.6 Similar recurrences, which seem to be new, are satisfied by $B(C_n;x,y)$. They may be verified using [17, Thms. 7.6.28, 7.7.34]. Let $\lambda_n(x,y) = x^{2k}y^{n-2k-2}(x+y)$ if $n = 3k+2$, and zero otherwise. Then, for $n \geq 5$,

$$B(C_n;x,y) = B(C_{n-1};x,y) + xy(1+x)B(C_{n-3};x,y) - \lambda_n(x,y).$$

Also, for $n \geq 5$,

$$B(C_n; x, y) = B(P_{n-1}; x, y) + 2xy(1+x)B(P_{n-4}; x, y) + x^2 yB(C_{n-3}; x, y).$$

A graph G is called *weakly chordal* if neither G nor its complement contains an induced cycle with five or more vertices. By definition, non-adjacent vertices u and v form a *two-pair* of G if any chordless path joining them has exactly two edges; or equivalently, if the removal of their common neighbors in G leaves them in different connected components. An edge $e = uv$ of G is called a *co-pair edge* if u and v form a two-pair in the complement of G. It is well known that a graph is weakly chordal if and only if each induced subgraph either is a clique or contains a two-pair of the subgraph [15].

Proposition 6.7 *Let G be a weakly chordal graph, $e = uv$ a co-pair edge of G and $k = |N[e]| - 2$. If $N(u) \cap N(v) = \varnothing$, then*

$$B(G; x, y) = B(G \backslash e; x, y) + xy(1+x)^k B(G \backslash N[e]; x, y).$$

Proof Since u and v form a two-pair of $\overline{G}$, it holds that e is not the middle edge of any induced chordless path on four vertices in G. This, together with $N(u) \cap N(v) = \varnothing$, implies that $N(u) \subseteq N(v_i)$ for all $v_i \in N(v) \backslash u$. In fact, let $w \in N(u)$. The path $wuvv_i$ may not be chordless. Since $N(u) \cap N(v) = \varnothing$ implies that $w \notin N(v)$ and $v_i \notin N(u)$, it must hold that $w \in N(v_i)$. Then, by Lemma 3.1,

$$B(G; x, y) = B(G \backslash u; x, y) + (1+x)^{k(v)} \big[B(G \backslash (N(v) \backslash u); x, y) - B(G \backslash N(v); x, y) \big],$$

where $k(v) = |N(v) \backslash u|$. By a similar argument for the graph $G \backslash e$, and taking into account Corollary 2.6(iii),

$$B(G \backslash e; x, y) = B(G \backslash u; x, y) + (1+x)^{k(v)} \big[B(G \backslash (N[v] \backslash u); x, y) - B(G \backslash N[v]; x, y) \big].$$

Now observe that in the graph $G \backslash (N(v) \backslash u)$ the vertex v has degree 1, so, by Corollary 6.4,

$$B(G \backslash (N(v) \backslash u); x, y) = B(G \backslash (N[v] \backslash u); x, y) + xy(1+x)^{k(u)} B(G \backslash N[e]; x, y),$$

where $k(u) = |N(u) \backslash v|$. Putting everything together, and using Corollary 2.6(iii) again, we obtain

$$B(G; x, y) = B(G \backslash e; x, y) + xy(1+x)^{k(u)+k(v)} B(G \backslash N[e]; x, y). \qquad \blacksquare$$

A graph that is both weakly chordal and bipartite is called *chordal bipartite*. Since the family of weakly chordal graphs is closed under the operation of deleting co-pair edges [28], it follows that the family of chordal bipartite graphs is closed under the same operation. Therefore, we have the following corollary from Proposition 6.7.

Corollary 6.8 *Betti numbers of chordal bipartite graphs can be computed recursively, and they do not depend on the characteristic of the field.*

Proof Let G be a chordal bipartite graph, and let $e = uv$ be a co-pair edge of G. Since G is bipartite, the condition $N(u) \cap N(v) = \varnothing$ is trivially verified. So, Proposition 6.7 can be applied. ■

Question 6.9 Can the Betti numbers of weakly chordal graphs be computed recursively? Do they depend on the characteristic of the field?

References

[1] J. Abbott, A. M. Bigatti, and G. Lagoio, *CoCoA-5: A system for doing computations in commutative algebra.* http://cocoa.dima.unige.it

[2] A. Bondy and U.S.R. Murty, *Graph theory*. Graduate Texts in Mathematics, 244, Springer, New York, 2008.

[3] W. Bruns and J. Herzog, *Cohen-Macaulay rings*. Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[4] A. Corso and U. Nagel, *Monomial and toric ideals associated to Ferrers graphs.* Trans. Amer. Math. Soc. **361**(2009), no. 3, 1371–1395. http://dx.doi.org/10.1090/S0002-9947-08-04636-9

[5] T. de Alwis, *Free minimal resolutions.* Comm. Algebra **21**(1993), no. 12, 4575–4585. http://dx.doi.org/10.1080/00927879308824817

[6] ———, *Free minimal resolutions and the Betti numbers of the suspension of an n-agon.* Int. J. Math. Math. Sci. **23**(2000), no. 3, 211–216. http://dx.doi.org/10.1155/S0161171200001563

[7] G. A. Dirac, *On rigid circuit graphs.* Abh. Math. Sem. Univ. Hamburg **25**(1961), 71–76. http://dx.doi.org/10.1007/BF02992776

[8] A. Dochtermann and A. Engström, *Algebraic properties of edge ideals via combinatorial topology.* Electron. J. Combin. **16**(2009), no. 2, Research Paper 2.

[9] D. Ferrarello and R. Fröberg, *The Hilbert series of the clique complex.* Graphs Combin. **21**(2005), no. 4, 401–405. http://dx.doi.org/10.1007/s00373-005-0634-z

[10] C. A. Francisco and A. Van Tuyl, *Sequentially Cohen-Macaulay edge ideals.* Proc. Amer. Math. Soc. **135**(2007), no. 8, 2327–2337. http://dx.doi.org/10.1090/S0002-9939-07-08841-7

[11] M. Ch. Golumbic and M. Lewenstein, *New results on induced matchings.* Discrete Appl. Math. **101**(2000), no. 1, 157–165. http://dx.doi.org/10.1016/S0166-218X(99)00194-8

[12] D. R. Fulkerson and O. A. Gross, *Incidence matrices and interval graphs.* Pacific J. Math. **15**(1965), 835–855. http://dx.doi.org/10.2140/pjm.1965.15.835

[13] H. T Hà and A. Van Tuyl, *Splittable ideals and the resolution of monomial ideals.* J. Algebra **309**(2007), no. 1, 405–425. http://dx.doi.org/10.1016/j.jalgebra.2006.08.022

[14] ———, *Resolutions of square-free monomial ideals via facet ideals: a survey.* In: Algebra, geometry and their interactions, Contemp. Math., 448, American Mathematical Society, Providence, RI, 2007, pp. 91–117.

[15] R. Hayward, C. Hoàng, and F. Maffray, *Optimizing weakly triangulated graphs.* Graphs Combin. **5**(1989), no. 4, 339–349. http://dx.doi.org/10.1007/BF01788689

[16] M. Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes.* In: Ring theory, II, (Proc. Second Cond., Univ. Oklahoma, Norman, Okla., 1975), Lecture Notes in Pure and Appl. Math., 26, Dekker, New York, 1977, pp. 171–223.

[17] S. Jacques, *Betti numbers of graph ideals*. Ph.D. Thesis, University of Sheffield, 2004. arxiv:math/0410107

[18] S. Jacques and M. Katzman, *The Betti numbers of forests.* arxiv:math/0501226v2

[19] M. Katzman, *Characteristic-Independence of Betti numbers of graph ideals.* J. Combin. Theory Ser. A **113**(2006), no. 3, 435–454. http://dx.doi.org/10.1016/j.jcta.2005.04.005

[20] M. Kummini, *Regularity, depth and arithmetical rank of bipartite edge ideals.* J. Algebraic Combin. **30**(2009), no. 4, 429–445. http://dx.doi.org/10.1007/s10801-009-0171-6

[21] J. Martínez–Bernal, C. Rentería, and R.H. Villarreal, *Combinatorics of symbolic Rees algebras of edge ideals of clutters.* Contemp. Math. **555**(2011), 151–164.

[22] E. Miller and B. Sturmfels, *Combinatorial commutative algebra*. Graduate Texts in Mathematics, 227, Springer-Verlag, New York, 2005.

[23] S. Morey and R. H. Villarreal, *Edge ideals: Algebraic and combinatorial properties.* In: Progress in commutative algebra 1, de Gruyter, Berlin, 85–126.

[24] A. Mousivand, *Algebraic properties of product of graphs.* Comm. Algebra **40**(2012), no. 11, 4177–4194. http://dx.doi.org/10.1080/00927872.2011.605408

[25] I. Peeva and M. Stillman, *Open problems on syzygies and Hilbert functions.* J. Commut. Algebra **1**(2009), no. 1, 159–195. http://dx.doi.org/10.1216/JCA-2009-1-1-159

[26] P. Renteln, *The Hilbert series of the face ring of a flag complex.* Graphs Combin. **18**(2002), no. 3, 605–619. http://dx.doi.org/10.1007/s003730200045

[27] A. Schrijver, *Combinatorial optimization.* Algorithms and Combinatorics, 24, Springer-Verlag, Berlin, 2003.
[28] J. P. Spinrad and R. Sritharan, *Algorithms for weakly triangulated graphs.* Discrete Appl. Math. **59**(1995), no. 2, 181–191. http://dx.doi.org/10.1016/0166-218X(93)E0161-Q
[29] N. Terai, *Alexander duality theorem and Stanley-Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998).* Surikaisekikenkyusho Kokyuroku **1078**(1999), 174–184.
[30] A. Van Tuyl, *Sequentially Cohen-Macaulay bipartite graphs: vertex decomposability and regularity.* Arch. Math. **93**(2009), 451–459. http://dx.doi.org/10.1007/s00013-009-0049-9
[31] R. H. Villarreal, *Cohen-Macaulay graphs*. Manuscripta Math. **66**(1990), 277–293. http://dx.doi.org/10.1007/BF02568497
[32] G. Whieldon, Jump sequences of edge ideals, `arXiv:1012.0108v1`, 2010.
[33] R. Woodroofe, *Vertex decomposable graphs and obstructions to shellability.* Proc. Amer. Math. Soc. **137**(2009), no. 10, 3235–3246. http://dx.doi.org/10.1090/S0002-9939-09-09981-X
[34] ———, *Matchings, coverings, and Castelnuovo-Mumford regularity.* J. Commut. Algebra **6**(2014), no. 2, 287–304. http://dx.doi.org/10.1216/JCA-2014-6-2-287

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