SPECTRAL THEORY FOR THE DIFFERENTIAL EQUATION $Lu = \lambda Mu$

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Introduction. Let L and M be linear ordinary differential operators defined on an interval I, not necessarily bounded, of the real line. We wish to consider the expansion of arbitrary functions in eigenfunctions of the differential equation $Lu = \lambda Mu$ on I. The case where M is the identity operator and L has a self-adjoint realization as an operator in the Hilbert space $L^2(I)$ has been treated in various ways by several authors; an extensive bibliography may be found in **(4)** or **(8)**. A characterization of the self-adjoint realizations of L by means of boundary conditions has been given by Kodaira **(11)** and Coddington **(3)**. The elementary approach used by Coddington and Levinson **(4**; chap. 10**)** has been used by the author in **(1)** to show the existence of eigenfunction expansions in the general case, provided M is a positive, semi-bounded operator. Here, a different existence proof is given, based on the spectral theorem and the theory of direct integrals of von Neumann **(12)**. The method, first used by Gårding **(6)** in the special case mentioned above, can be applied to the case where L and M are elliptic partial differential operators, but the results obtained are not as general as those of Gelfand and Kostyucenko **(9)**, Browder **(2)**, and Gårding **(8)**.

Following the development of Gårding **(7, 8)**, the existence of a Green's function is shown, and the analogue of the formula due to Titchmarsh **(14)** and Kodaira **(11)**, relating the Green's function to the spectral matrix, is obtained. Finally, the self-adjoint realizations are studied, and are characterized by boundary conditions.

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1. The spectral theorem in direct integral form. Let σ be a measure on the real line R, and let $\nu(\lambda)$ be a function defined for all real λ, taking the values $1, 2, \ldots$, and ∞, and measurable with respect to σ. Consider vector-valued functions $F(\lambda) = [F_1(\lambda), F_2(\lambda), \ldots]$ with $\nu(\lambda)$ complex-valued components. Let $L^2(\sigma, \nu)$ be the set of all such functions with measurable components for which the square norm

$$(F, F) = \int_R |F(\lambda)|^2 d\sigma(\lambda), \quad \text{where} \quad |F(\lambda)|^2 = \sum_{k=1}^{\nu(\lambda)} |F_k(\lambda)|^2,$$

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is finite. The equivalence classes of $L^2(\sigma, \nu)$ with respect to functions of vanishing norm form a separable Hilbert space. We identify functions in the same equivalence class, and thus refer to $L^2(\sigma, \nu)$ itself as a Hilbert space, with inner product $(F, G) = \int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda)$, where

$$F(\lambda)\overline{G}(\lambda) = \sum_{k=1}^{\nu(\lambda)} F_k(\lambda)\overline{G}_k(\lambda).$$

Since the value $F(\lambda)$ of F at the point λ is an element of a Hilbert (sequence) space H_λ of dimension $\nu(\lambda)$, we may regard $L^2(\sigma, \nu)$ as a symbolic integral of the H_λ with respect to σ, called a direct integral. The formula

$$F(\lambda)\overline{G}(\lambda) = \sum_{k=1}^{\nu(\lambda)} F_k(\lambda)\overline{G}_k(\lambda)$$

means that each H has been referred to an orthonormal basis. This has been done for convenience only, and other bases will be used in section 3.

Let A be a self-adjoint linear operator on a separable Hilbert space H. Then a form of the spectral theorem due to von Neumann **(12)** states that there is a direct integral $L^2(\sigma, \nu)$ and a unitary mapping U from H to $L^2(\sigma, \nu)$ which diagonalizes A in the sense that UAU^{-1} is multiplication by λ in $L^2(\sigma, \nu)$. More precisely, if D_A is the domain of A, then $U(D_A)$ consists of those $F \in L^2(\sigma,\nu)$ for which $\lambda F(\lambda) \in L^2(\sigma, \nu)$, that is, for which $\int_R \lambda^2 |F(\lambda)|^2 d\sigma(\lambda) < \infty$, and if $F \in U(D_A)$, then $UAU^{-1}F(\lambda) = \lambda F(\lambda)$ for almost all λ. The measure σ is concentrated on the spectrum of A, and the number $\nu(\lambda)$ may be called the multiplicity of the point λ in the spectrum of A. If λ is an eigenvalue of A, then σ has a jump at λ, and there are $\nu(\lambda)$ linearly independent eigenfunctions of A at λ. If $L^2(\sigma_1, \nu_1)$ and $L^2(\sigma_2, \nu_2)$ are two direct integrals which diagonalize A in this manner, then σ_1 and σ_2 are equivalent (have the same null sets), and $\nu_1 = \nu_2$ except on a set of σ-measure zero.

2. The eigenfunction expansion theorem. Let I be an interval on the real line, not necessarily bounded. Let $L^2(I)$ be the set of all complex-valued square integrable functions on I, with inner product $(f, g) = \int_I f(x)\bar{g}(x)dx$. Let L and $\tilde{M}$ be linear ordinary differential operators of orders n and m respectively $(n > m)$, defined by

$$Lu = \sum_{i=0}^{n} p_i(x)u^{(n-i)},\ \tilde{M}u = \sum_{i=0}^{m} q_i(x)u^{(m-i)}.$$

We assume that the p_i and q_i are complex-valued functions of class C^{n-i} and C^{m-i} respectively on I, and that neither p_0 nor q_0 vanishes on any compact subinterval of I.

Let $C^n{}_0(I)$ denote the set of complex-valued functions of class C^n on I which vanish identically outside a compact subinterval of I. We will always assume that L and $\tilde{M}$ are symmetric, $(Lf, g) = (f, Lg)$ and $(\tilde{M}f, g) = (f, \tilde{M}g)$ for all $f, g \in C^n{}_0(I)$. This implies that L and $\tilde{M}$ coincide with their formal adjoints. We will also assume that $\tilde{M}$ is semi-bounded with a positive lower bound, $(\tilde{M}f, f) \geqslant c(f, f)$ for some constant $c > 0$ and all $f \in C^n{}_0(I)$. The symmetry and semi-boundedness of $\tilde{M}$ imply that $[f, g] = (\tilde{M}f, g) = (f, \tilde{M}g)$ may be considered as an inner product on $C^n{}_0(I)$. Let H be the Hilbert space completion of $C^n{}_0(I)$ in this inner product. It is clear that H contains all functions whose derivatives up to order m belong to $L^2(I)$ and which vanish identically outside a compact subinterval of I. It is easy to verify that H can be identified with a linear subset of $L^2(I)$. Considered as an operator in the Hilbert space $L^2(I)$, $\tilde{M}$ has at least one self-adjoint extension. There is a unique self-adjoint extension M of $\tilde{M}$ whose domain D_M is contained in H. The range of M is $L^2(I)$, and M has a bounded inverse mapping $L^2(I)$ into H. This result is due to Friedrichs **(5)**; see also **(13**; pp. 331–334**)**. We consider $M^{-1}L$ as an operator in H with domain $C^n{}_0(I)$. This operator is symmetric since $[M^{-1}Lf, g] = (Lf, g) = (f, Lg) = [f, M^{-1}Lg]$ for $f, g \in C^n{}_0(I)$. We assume that $M^{-1}L$ has a self-adjoint extension A, considered as an operator in H. If M and L have real coefficients, then $M^{-1}L$ is a real operator and always has at least one self-adjoint extension **(13**; p. 329**)**. Also, if L is semi-bounded in $L^2(I)$, $(Lf, f) \geqslant d(f, f)$ for some constant d and all $f \in C^n{}_0(I)$, then $M^{-1}L$ is a semi-bounded operator in H. Decreasing the lower bound if necessary, we may assume $d \leqslant 0$. Since $[f, f] \geqslant c(f, f)$, $d/c[f, f] \leqslant d(f, f)$, and $[M^{-1}Lf, f] = (Lf, f) \geqslant d(f, f) \geqslant d/c[f, f]$ for $f \in C^n{}_0(I)$, and $M^{-1}L$ is semi-bounded. Then by the theorem of Friedrichs cited above, $M^{-1}L$ has at least one self-adjoint extension.

We can now state the main result to be obtained in this section:

THEOREM 1: *Let A be a self-adjoint extension of $M^{-1}L$, considered as an operator in H. The spectral theorem furnishes a direct integral $L^2(\sigma, \nu)$ and a unitary transformation U from H to $L^2(\sigma, \nu)$ which diagonalizes A. Under the conditions imposed above on L and M, this transformation is given by $(Uf)(\lambda) = \int_I Mf(x)\bar{E}(x, \lambda)dx$ for $f \in D_M$ and its inverse by $(U^{-1}F)(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$ for $F \in L^2(\sigma, \nu)$, with the integrals converging to the functions in the norms of the Hilbert spaces $L^2(\sigma, \nu)$ and H respectively. The components of $E(x, \lambda)$ are linearly independent functions for almost all λ, having locally square integrable derivatives with respect to x, and are improper eigenfunctions (not necessarily belonging to H) of the differential equation $Lu = \lambda Mu$ for almost all λ. If λ_0 is an eigenvalue of $Lu = \lambda Mu$, then the components of $E(x, \lambda_0)$ are proper eigenfunctions.*

It is well known **(4**, pp. 190–191**)** that there exists a fundamental solution $k(x, y)$ for the operator L with the following properties:

(i) The function k and its partial derivatives up to total order $(n - 2)$ are continuous on the square IxI. The partial derivatives of orders $(n - 1)$ and n are continuous except on the diagonal $x = y$, and the partial derivatives of order $(n - 1)$ have a jump of $1/p_0(y)$ on $x = y$.

(ii) As a function of x, k satisfies $Lu = 0$ if $x \neq y$.

(iii) If S is any compact subinterval of I and f is any function belonging to $C^n{}_0(I)$, then $f(x) = \int_S k(x, y)Lf(y)dy$ for $x \in S$.

Let S be any compact subinterval of I and let $\phi_S \in C^n{}_0(I)$ be equal to 1 on S. If $f \in C^n{}_0(S)$, then $f(x) = \int_S k(x, y)Lf(y)dy = \int_S k(x, y)\phi_S(y)Lf(y)dy$ for $x \in S$. The function $k(x, \ )\phi_S(\ )$, for any fixed $x \in S$, vanishes identically outside some compact subinterval of I and has square integrable derivatives up to order $n - 1 \geqslant m$. Thus $k(x, \ )\phi_S(\ )$ belongs to H and $f(x) = (Lf, \bar{k}(x, \ )\bar{\phi}_S(\ )) = [Af, \bar{k}(x, \ )\bar{\phi}_S(\ )]$.

Let $L^2(\sigma, \nu)$ be a suitable direct integral and let U be the unitary mapping of H to $L^2(\sigma, \nu)$ which diagonalizes the self-adjoint operator A. The fact that U is unitary is expressed by the Parseval formula,

$$[f, g] = (Uf, Ug) = \int_R (Uf)(\lambda)(U\bar{g})(\lambda)d\sigma(\lambda)$$

for any $f, g \in H$. Here, and in all that follows, an expression of the form $U\bar{g}$, where U is an operator and g is a function, will denote the complex conjugate of Ug. Let $f \in C^n{}_0(S)$, and let g belong to D^S, the set of functions g in D_M such that Mg vanishes identically outside S. We let $F = Uf$, $G = Ug$, $K(x, \ ) = U\{k(x, \ )\phi_S(\ )\}$, $E(x, \lambda) = \lambda K(x, \lambda)$. Then

$$f(x) = [Af, \bar{k}(x, \ )\bar{\phi}_S(\ )] = (UAf, \bar{K}(x, \ )) = (\lambda Uf, \bar{K}(x, \ )) = (F, \bar{E}(x, \ )),$$

or $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$. In addition

$$\begin{aligned}[f, g] = (f, Mg) &= \int_S[\int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)]M\bar{g}(x)dx \\ &= \int_R F(\lambda)[\int_S M\bar{g}(x)E(x, \lambda)dx]d\sigma(\lambda),\end{aligned}$$

the interchange in the order of integration being justified by the absolute convergence of the integral. On the other hand, $[f, g] = \int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda)$, and thus $G(\lambda) = \int_S Mg(x)\bar{E}(x, \lambda)dx$ for almost all λ.

The Parseval formula gives $||k(x, \ )\phi_S(\ )||^2 = \int_R|K(x, \lambda)|^2 d\sigma(\lambda)$, so that $\int_S\int_R|K(x, \lambda)|^2 d\sigma(\lambda)dx < \infty$. By the Fubini theorem, $\int_R\int_S|K(x, \lambda)|^2 dx d\sigma(\lambda) < \infty$, and $\int_S|K(x, \lambda)|^2 dx$ is finite except for λ in a set of σ-measure zero. Then $\int_S|E(x, \lambda)|^2 dx$ is finite if λ is outside this same null set. We can redefine E without changing its equivalence class in $L^2(\sigma, \nu)$ by making $E(x, \lambda) = 0$ when λ is in this null set, and then $\int_S|E(x, \lambda)|^2 dx$ is finite for all λ.

For $g \in C^n{}_0(S)$, we have seen that $(Ug)(\lambda) = \int_S Mg(x)\bar{E}(x, \lambda)dx$. Since $MAg = Lg$, which vanishes identically outside S, $Ag \in D^S$, and the above relation holds with g replaced by Ag, so that $(UAg)(\lambda) = \int_S Lg(x)\bar{E}(x, \lambda)dx$. Since $(UAg)(\lambda) = \lambda(Ug)(\lambda)$ for almost all λ, $\int_S[Lg(x) - \lambda Mg(x)]\bar{E}(x, \lambda)dx = 0$ when λ does not belong to a set N_g of σ-measure zero, with N_g dependent on g. The same is true for a sequence g_j of functions when λ does not belong to the null set

$$N = \bigcup_{j=1}^{\infty} N_{g_j}.$$

We choose the sequence g_j dense in $C^n{}_0(S)$, and then $\int_S[Lg(x) - \lambda Mg(x)]$ $\bar{E}(x, \lambda)dx = 0$ for all $g \in C^n{}_0(S)$ if $\lambda \notin N$. We let $E(x, \lambda) = 0$ if $\lambda \in N$, and then this relation holds for all λ. Thus the components of $E(x, \lambda)$ are weak solutions of $Lu = \lambda Mu$ on S. It follows from a well-known theorem on weak solutions of partial differential equations that the components E_k of $E(x, \lambda)$ have derivatives which are locally square integrable, that each E_k is of class C^n in x after correction on a null set for each λ, and that $LE_k(x, \lambda) = \lambda ME_k(x,\lambda)$ for $k = 1, 2, \ldots, \nu(\lambda)$ and almost all λ. This theorem is easily proved by using the properties of the fundamental solution; see for example **(10)**.

The components of E depend on the compact subinterval S. Let E' be another function with the same properties, corresponding to another subinterval $S' \supset S$. Then $\int_S Mg(x)[\bar{E}_k(x, \lambda) - \bar{E}'_k(x, \lambda)]dx = 0$ for each k and almost all λ, independent of $g \in D^S$. Since $M(D^S)$ is easily seen to be dense in $L^2(S)$, we may let Mg run through a countable dense subset of $L^2(S)$, and it follows that the components of $E(x, \lambda)$ and $E'(x, \lambda)$ are square integrable in x over S and are equal for λ outside some null set P. We set $E(x, \lambda) = E'(x, \lambda) = 0$ for $\lambda \in P$, and then $E(x, \lambda) = E'(x, \lambda)$ for $x \in S$ and all λ. By taking a sequence of compact subintervals S tending to I, we can extend E uniquely to a function defined for all $x \in I$ and all λ. Each component of E is differentiable with respect to x, and its derivatives are locally square integrable. Also, $LE_k(x, \lambda) = \lambda ME_k(x, \lambda)$ for $k = 1, \ldots, \nu(\lambda)$, and the E_k are improper eigenfunctions of $Lu = \lambda Mu$, not necessarily belonging to the space H.

If λ_0 is an eigenvalue of A, then σ has a jump, which we may assume to be a jump of 1, at λ_0. We choose $F = 0$ except at λ_0, and $F_j(\lambda_0) = \delta_{jk}$ for any fixed index $k \leqslant \nu(\lambda_0)$. Then $F \in H$ and $(U^{-1}F)(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda) = E_k(x, \lambda_0)$ belongs to H. Thus, if λ_0 is an eigenvalue of A, the $E_k(x, \lambda_0)$ are proper eigenfunctions of $Lu = \lambda Mu$.

The eigenfunctions $E_k(x, \lambda)[k = 1, \ldots, \nu(\lambda)]$ are linearly independent for almost all λ. To prove this, consider the matrices $Q(\lambda) = (q_{jk}(\lambda)) = (\int_S E_j(x, \lambda)$ $\bar{E}_k(x, \lambda)dx)$, where S is a compact subinterval of I. Let $\mu = \mu(\lambda)$ be the rank of $Q(\lambda)$, with the value $\mu = \infty$ being permitted. Then $\mu(\lambda) \leqslant \nu(\lambda)$, and the $E_k(x, \lambda)$ are linearly independent if and only if $\mu(\lambda) = \nu(\lambda)$. We define the function $F \in L^2(\sigma, \nu)$ as follows. If $\mu(\lambda) = \nu(\lambda)$, we set $F(\lambda) = 0$; otherwise, we set $F_r(\lambda) = 0$ if $r > \mu$ and make $F_r(\lambda)$ proportional to the cofactor of $q_{\mu r}(\lambda)$ in the $\mu \times \mu$ determinant of the $q_{jk}(\lambda)$ for $j, k \leqslant \mu$ if $r \leqslant \mu$. The proportionality factor can always be chosen to be non-zero and such that $F \in L^2(\sigma, \nu)$. Now $F(\lambda)E(x, \lambda) = 0$ for all λ. It follows that $\int_R F(\lambda)\bar{G}(\lambda)d\sigma(\lambda) = 0$ if $G = Ug$ and g is in some D^S. Thus $F = 0$ except for λ in a fixed null set, and the $E_k(x, \lambda)$ are linearly independent except for λ in this null set.

The inversion formulae $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$ and $F(\lambda) = \int_S Mf(x)$ $\bar{E}(x, \lambda)dx$, as well as the Parseval equality $[f, g] = (F, G)$, which have been proved for $f \in C^n{}_0(S)$, can be extended to $f \in D_M$ by a standard density argument. They become $f(x) = \int_R F(\lambda)E(x, \lambda)d\sigma(\lambda)$ and $F(\lambda) = \int_I Mf(x)\bar{E}(x,\lambda)$

dx, with the integrals converging to the functions in the norms of the appropriate Hilbert spaces. These formulae give the expansion of an arbitrary function $f \in D_M$ in eigenfunctions of the differential equation $Lu = \lambda Mu$. This completes the proof of Theorem 1.

Theorem 1 remains valid if L and $\tilde{M}$ are elliptic partial differential operators with sufficiently differentiable coefficients on a domain I in t-dimensional Euclidean space. The only change in the proof is caused by the slightly worse behaviour of the fundamental solution $k(x, y)$ of L. A derivative of order j of $k(x, y)$ is $O(|x - y|^{n-j-t-\epsilon})$ near $x = y$ for any $\epsilon > 0$ (**10**, chap. 3), and thus to ensure that the derivatives of order m are locally square integrable, we must impose the additional condition $n - m > \frac{1}{2}t$ on the orders of L and $\tilde{M}$.

Returning to the case of ordinary differential operators, let $\phi_k(x, \lambda)$ $[k = 1, \ldots, n]$ be a basis of solutions of $Lu = \lambda Mu$, with each ϕ_k analytic in λ for fixed x. For example, we may choose ϕ_k to obey the initial conditions $\phi_k^{(j-1)}(\xi, \lambda) = \delta_{jk}$ $[j, k = 1, \ldots, n]$ for any fixed $\xi \in I$. To prepare for the next section, we now express the eigenfunctions $E_k(x, \lambda)$ in terms of this basis. Since the $E_k(x, \lambda)$ $[k = 1, \ldots, \nu(\lambda)]$ are linearly independent solutions of $Lu = \lambda Mu$, the dimension function $\nu(\lambda) \leqslant n$. We write

$$E_p(x, \lambda) = \sum_{j=1}^{n} r_{pj}\phi_j(x, \lambda) \qquad [p = 1, \ldots, \nu(\lambda)],$$

where the r_{pj} are complex constants. The Parseval equality

$$||f||^2 = ||F||^2 = \int_R \sum_{p=1}^{\nu(\lambda)} |F_p(\lambda)|^2 d\sigma(\lambda)$$

now takes the form

$$||f||^2 = \int_R \sum_{p=1}^{\nu(\lambda)} \sum_{j,k=1}^{n} (Vf)_j(\lambda)(\overline{Vf})_k(\lambda)\bar{r}_{pj}r_{pk}d\sigma(\lambda),$$

where $(Vf)_j(\lambda) = \int_I Mf(x)\bar{\phi}_j(x, \lambda)dx$. We let

$$c_{jk}(\lambda) = \sum_{p=1}^{\nu(\lambda)} \bar{r}_{pj}r_{pk},$$

and then $(c_{jk}(\lambda))$ is a Hermitian positive semi-definite matrix of rank $\nu(\lambda)$. The formulae $d\rho_{jk}(\lambda) = c_{jk}(\lambda)d\sigma(\lambda)$, $\rho_{jk}(0) = 0$, determine an n by n matrix $\rho(\lambda) = (\rho_{jk}(\lambda))$, called a spectral matrix, which is Hermitian, positive semi-definite, and non-decreasing. Let H^* be the Hilbert space of all complex-valued vector functions $F(\lambda) = [F_1(\lambda), \ldots, F_n(\lambda)]$ such that

$$\int_R \sum_{j,k=1}^{n} F_j(\lambda)\bar{F}_k(\lambda)d\rho_{jk}(\lambda) < \infty,$$

with inner product

$$(F, G) = \int_R \sum_{j,k=1}^{n} F_j(\lambda)\bar{G}_k(\lambda)d\rho_{jk}(\lambda).$$

The spectral theorem may be regarded as saying that $(Vf)(\lambda) = \{\int_I Mf(x) \bar{\phi}_j(x, \lambda)dx\}$ defines a unitary mapping V of H onto H^* which diagonalizes A. A straightforward computation gives

$$(V^{-1}F)(x) = \int_R \sum_{j,k=1}^{n} F_j(\lambda)\phi_k(x, \lambda)d\rho_{jk}(\lambda).$$

3. Green's function and the spectral matrix. Let A be a self-adjoint extension of $M^{-1}L$, as in section 2, and let $R_\lambda = (A - \lambda)^{-1}$, for $Im\lambda \neq 0$, be the resolvent of A, a bounded operator in H. Earlier, we made use of the relation $f(x) = \int_S k(x, y)\phi_S(y)Lf(y)dy$ for $x \in S$, where S is a compact subinterval of I and f belongs to $C^n{}_0(S)$. We now modify this in two ways. Instead of using a fundamental solution $k(x, y)$ of L, we use a fundamental solution $k(x, y, \lambda)$ of $L - \lambda M$, and we no longer assume that f vanishes outside a compact subinterval of S. As a result, we have

$$f(x) = \int_S k(x, y, \lambda)\phi_S(y)(L - \lambda M)f(y)dy + u(x)$$

for $x \in S$, where u satisfies $Lu = \lambda Mu$. We apply this relation to $R_\lambda f$ instead of f, obtaining

$$R_\lambda f(x) = \int_S k(x, y, \lambda)\phi_S(y)(L - \lambda M)R_\lambda f(y)dy + u(x).$$

Since $(L - \lambda M)R_\lambda f(y) = Mf(y)$ for any $f \in C^n{}_0(S)$ (which shows also that $R_\lambda f \in C^n(I)$, so that the above relation can be applied to $R_\lambda f$),

$$R_\lambda f(x) = \int_S k(x, y, \lambda)\phi_S(y)Mf(y)dy + u(x).$$

By the Schwarz inequality for the inner product in H, $|R_\lambda f(x) - u(x)| \leqslant ||k(x, \ , \lambda)\phi_S(\ )|| \,.\, ||f||$, so that $R_\lambda f(x) - u(x)$ is a bounded linear functional of f for $x \in S$. Thus for each $x \in S$ and for $Im\lambda \neq 0$, there exists a function $g(x, \ , \lambda) \in H$ such that

$$R_\lambda f(x) - u(x) = [f, \bar{g}(x, \ , \lambda)] = \int_S g(x, y, \lambda)Mf(y)dy$$

for $f \in C^n{}_0(S)$. Now we can write

$$u(x) = \int_S [k(x, y, \lambda)\phi_S(y) - g(x, y, \lambda)]Mf(y)dy,$$

so that

$$|u(x)| \leqslant ||k(x, \ , \lambda)\phi_S(\ ) - g(x, \ , \lambda)|| \,.\, ||f||,$$

and $u(x)$ is a bounded linear functional of f for each fixed $x \in S$. Then $u(x) = \int_S v(x, y, \lambda)Mf(y)dy$ for some $v(x, \ , \lambda) \in H$. Since $Lu = \lambda Mu$ for all $f \in C^n(S)$, we can write

$$u(x) = \sum_{i=1}^{n} c_i\phi_i(x, \lambda),$$

where $\phi_i(x, \lambda)$ $(i = 1, \ldots, n)$ form a basis of solutions of $Lu = \lambda Mu$, and the c_i are constants depending on f. This means that $v(x, y, \lambda)$ has the form

$$v(x, y, \lambda) = \sum_{i=1}^{n} v_i(y)\phi_i(x, \lambda).$$

We define

$$G(x, y, \lambda) = g(x, y, \lambda) + \sum_{i=1}^{n} v_i(y)\phi_i(x, \lambda),$$

so that $G(x, \ , \lambda) \in H$ and

$$R_\lambda f(x) = \int_S G(x, y, \lambda) Mf(y) dy = [f, \bar{G}(x, \ , \lambda)].$$

This function G depends on the interval S but is uniquely determined by S. If S' is another compact subinterval which contains S and G' is the corresponding function, it is easy to see that $G(x, y, \lambda) = G'(x, y, \lambda)$ for $x, y \in S$, $Im\lambda \neq 0$. Thus, by taking a sequence of compact subintervals S tending to I, we can extend G uniquely to a function $G(x, y, \lambda)$ defined for $x, y \in I$, $Im\lambda \neq 0$. This function is called the Green's function of A, and has the following properties:

(i) G is analytic in λ for fixed x, y and $Im\lambda \neq 0$, and has continuous partial derivatives with respect to x up to order $n - 2$ on $I \times I$ for fixed λ with $Im\lambda \neq 0$. The partial derivatives of orders $n - 1$ and n are continuous except on $x = y$, and the partial derivative of order $n - 1$ has a jump of $1/p_0(y)$ on $x = y$ if $Im\lambda \neq 0$.

(ii) $G(y, x, \lambda) = \bar{G}(x, y, \bar{\lambda})$.

(iii) Considered as a function of x, G satisfies $Lu = \lambda Mu$ if $x \neq y$.

(iv) G is uniquely determined by A.

(v) If $f \in C^n_0(I)$, then $f(x) = \int_I G(x, y, \lambda)(L - \lambda M)f(y)dy$.

To verify these properties, we begin by noting that for any $f \in D^S$ (the set of functions in D_M such that Mf vanishes identically outside S),

$$R_\lambda f(x) = \int_S \left[k(x, y, \lambda) + \sum_{i=1}^{n} \phi_i(x, \lambda) v_i(y) \right] Mf(y)\, dy$$
$$= \int_S G(x, y, \lambda) Mf(y)\, dy,$$

and therefore

$$\int_S \left[G(x, y, \lambda) - k(x, y, \lambda) - \sum_{i=1}^{n} \phi_i(x, \lambda) v_i(y) \right] Mf(y)\, dy = 0.$$

This implies, since the functions Mf for f ranging over D^S are dense in $L^2(S)$, that $G(x, y, \lambda)$ has the same differentiability properties with respect to x as

$$k(x, y, \lambda) + \sum_{i=1}^{n} \phi_i(x, \lambda) v_i(y),$$

and thus the same properties as $k(x, y, \lambda)$, since the second term is of class C^n in x. Since $R_\lambda f$ is analytic in λ for $Im\lambda \neq 0$ and all $f \in D_M$ **(13)**, $G(x, y, \lambda)$ is analytic in λ for fixed x, y and $Im\lambda \neq 0$ which completes the proof of (i).

To verify (ii), we note that $(R_\lambda f, g) = (f, R_{\bar{\lambda}} g)$ for $f, g \in D_M$, whence

$$\int_I \int_I G(x, y, \lambda) Mf(y) M\bar{g}(x)\, dydx = \int_I \int_I \bar{G}(y, x, \bar{\lambda}) Mf(y) M\bar{g}(x)\, dxdy.$$

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Since the functions Mf for f ranging over D_M are dense in $L^2(I)$, (ii) follows. Since the difference between two Green's functions would be an eigenfunction of A and the spectrum of A is real, the Green's function of A is unique for $Im\lambda \neq 0$. The property (v) is an immediate consequence of the definition of the resolvent. In view of (i), if $f \in C^n{}_0(S)$, we may apply Green's formula (**4**, p. 86) to L and M separately in (v) to obtain

$$f(x) = \int_S (\overline{L_y} - \overline{\lambda M_y}) G(x, y, \lambda) f(y)\, dy,$$

the subscripts indicating that the differentiations are with respect to y. Then $(L_y - \lambda M_y)\bar{G}(x, y, \lambda) = 0$ for $Im\lambda \neq 0, x \neq y$, and application of (ii) yields property (iii).

Now we express the Green's function in terms of a basis $\phi_k(x, \lambda)$ $[k = 1, \ldots, n]$ of solutions of $Lu = \lambda Mu$. It is easily deduced from the above properties of the Green's function that G may be written

$$G(x, y, \lambda) = \sum_{j,k=1}^{n} P^+{}_{jk}(\lambda)\, \phi_j(x, \lambda) \overline{\phi_k}(y, \bar{\lambda})$$

for $y \geqslant x$ and

$$G(x, y, \lambda) = \sum_{j,k=1}^{n} P^-{}_{jk}(\lambda)\, \phi_j(x, \lambda) \overline{\phi_k}(x, \bar{\lambda})$$

for $y \leqslant x$. The $P_{jk}{}^+$ and $P_{jk}{}^-$ are analytic in λ except possibly on the real axis, and $P^-{}_{kj} = \bar{P}^+{}_{jk}$. We define the matrix $P = (P_{jk})$ by $P_{jk}(\lambda) = \frac{1}{2}[P^+{}_{jk}(\lambda) + P^-{}_{jk}(\lambda)]$. Then each P_{jk} is analytic for $Im\lambda \neq 0$ and $\bar{P}_{jk} = P_{kj}$.

THEOREM 2. (Titchmarsh-Kodaira formula.) *The Green's function of A is related to the spectral matrix ρ associated with the basis of solutions $\phi_k(x, \lambda)$ of $Lu = \lambda Mu$ by the formula*

$$P(\mu) = \int_{-\infty}^{\infty} \frac{d\rho(\lambda)}{(\lambda - \mu)},$$

where the elements of the matrix P are defined as above, and the formula is to be taken in the sense that

$$P(\mu) - \int_{-N}^{N} \frac{d\rho(\lambda)}{(\lambda - \mu)}$$

is analytic across the real axis on the interval $(-N, N)$.

Proof: Let $f \in D_M$. If $F_j(\lambda) = \int_I Mf(x)\bar{\phi}_j(x, \lambda)dx$, then

$$f(x) = \int_R \sum_{j,k=1}^{n} F_j(\lambda)\, \phi_k(x, \lambda) d\rho_{jk}(\lambda),$$

as we have seen in the previous section. If

$$u(x) = \int_R \sum_{j,k=1}^n (\lambda - \mu)^{-1} F_j(\lambda)\phi_k(x, \lambda) d\rho_{jk}(\lambda),$$

it is easy to verify that $Lu - \mu Mu = Mf$, or $u = R_\mu f$. Thus

$$\mu(Vu)_j(\lambda) = \mu \int_I Mu(x)\overline{\phi}_j(x, \lambda)\, dx = \int_I Lu(x)\overline{\phi}_j(x, \lambda)\, dx - \int_I Mf(x)\overline{\phi}_j(x, \lambda)\, dx$$
$$= (VAu)_j(\lambda) - (Vf)_j(\lambda) = \lambda(Vu)_j(\lambda) - (Vf)_j(\lambda),$$

or $(\lambda - \mu)(Vu)_j(\lambda) = (Vf)_j(\lambda)$. Now, the Parseval equality applied to u and f yields

$$[u, f] = \int_R \sum_{j,k=1}^n (Vu)_j(\lambda)(\overline{Vf})_k(\lambda) d\rho_{jk}(\lambda)$$
$$= \int_R \sum_{j,k=1}^n (\lambda - \mu)^{-1} F_j(\lambda)\overline{F}_k(\lambda) d\rho_{jk}(\lambda),$$

and this is equal to

$$\sum_{j,k=1}^n F_j(\mu)\overline{F}_k(\mu) \int_{-N}^N \frac{d\rho_{jk}(\lambda)}{\lambda - \mu}$$

plus a function which is analytic unless μ is real and $|\mu| \geqslant N$. On the other hand, since $u(x) = \int_I G(x, y, \mu) Mf(y) dy$, $[u, f] = \int_I \int_I G(x, y, \mu) M\bar{f}(x) Mf(y) dy dx$. This expression is equal to

$$\sum_{j,k=1}^n P_{jk}(\mu) F_j(\mu) \overline{F}_k(\mu)$$

plus an analytic function. Since f may run through a dense subset of H, which means that F may run through a dense subset of H^*, it follows that the elements of the matrix

$$P(\mu) - \int_{-N}^N \frac{d\rho(\lambda)}{\lambda - \mu}$$

are analytic unless μ is real and $|\mu| \geqslant N$.

Another form of the Titchmarsh-Kodaira formula is

$$\rho(\lambda) = \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \frac{1}{2\pi i} \int_\delta^{\lambda + \delta} [P(\mu + i\epsilon) - P(\mu - i\epsilon)]\, d\mu,$$

with ρ normalized to be continuous from the right and $\rho(0) = 0$, and with the formula interpreted as in Theorem 2 .We write

$$P(\mu + i\epsilon) - P(\mu - i\epsilon) = \int_{-N}^N \left[\frac{1}{\tau - \mu - i\epsilon} - \frac{1}{\tau - \mu + i\epsilon}\right] d\rho(\tau) + w(\mu)$$
$$= 2i \int_{-N}^N \frac{\epsilon d\rho(\tau)}{(\tau - \mu)^2 + \epsilon^2} + w(\mu),$$

where $w(\mu)$ is analytic unless μ is real and $|\mu| \geqslant N$. Then

$$
\begin{aligned}
&\frac{1}{2i} \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \int_{\delta}^{\lambda+\delta} [P(\mu + i\epsilon) - P(\mu - i\epsilon)]\, d\mu \\
&= \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \int_{\delta}^{\lambda+\delta} \int_{-N}^{N} \frac{\epsilon d\rho(\tau) d\mu}{(\tau - \mu)^2 + \epsilon^2} + w_1(\lambda) \\
&= \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \int_{-N}^{N} \left[\int_{\delta}^{\lambda+\delta} \frac{\epsilon d\mu}{(\tau - \mu)^2 + \epsilon^2} \right] d\rho(\tau) + w_1(\lambda) \\
&= \lim_{\delta \to 0+} \lim_{\epsilon \to 0+} \int_{-N}^{N} \left[\tan^{-1}\left(\frac{\lambda + \delta - \tau}{\epsilon} \right) - \tan^{-1}\left(\frac{\delta - \tau}{\epsilon} \right) \right] d\rho(\tau) + w_1(\lambda) \\
&= \pi \lim_{\delta \to 0+} [\rho(\lambda + \delta) - \rho(\delta)] + w_1(\lambda) \\
&= \pi \rho(\lambda) + w_1(\lambda),
\end{aligned}
$$

where $w_1(\lambda)$ is analytic unless λ is real and $|\lambda| \geqslant N$, as desired.

4. Boundary conditions. Let D be the set of functions f in H with continuous derivatives up to order $n - 1$ on I, such that $f^{(n-1)}$ is absolutely continuous on every compact subinterval of I, so that $f^{(n)}$ exists almost everywhere on I, and such that Lf belongs to $L^2(I)$. Let T be the operator in H with domain D defined by $Tf = M^{-1}Lf$ for $f \in D$. We assume that T has at least one self-adjoint restriction. Let R_λ be the resolvent of some self-adjoint restriction of T, so that $R_\lambda f(x) = \int_I G(x, y, \lambda) Mf(y) dy$ for $f \in D_M$, $Im\lambda \neq 0$. Then R_λ is a bounded operator for $Im\lambda \neq 0$, whose adjoint R^*_λ is $R_{\bar{\lambda}}$. Let $\mathfrak{E}(\lambda)$ be the eigenspace of T corresponding to the value λ, the set of all solutions in D of the differential equation $Lu = \lambda Mu$.

LEMMA 1: *T is a closed operator whose domain consists of all $f \in H$ of the form $f = R_\lambda h + w$, where $h \in H$, $w \in \mathfrak{E}(\lambda)$, $Im\lambda \neq 0$.*

Proof: Since R_λ maps H into D and $\mathfrak{E}(\lambda)$ is contained in D, it is clear that every f of this form belongs to D. Conversely, suppose $f \in D$ is given. Let $h = Tf - \lambda f$, $w = f - R_\lambda h$. Then $Tw = Tf - TR_\lambda h = Tf - \lambda R_\lambda h - h = Tf - h - \lambda(h - w) = \lambda w$, and thus $w \in \mathfrak{E}(\lambda)$, while $f = R_\lambda h + w$. When f is written in this way, $Tf - \lambda f = h$. To show that T is a closed operator, take a sequence f_k in D such that

$$f = \lim_{k\to\infty} f_k \quad \text{and} \quad f^* = \lim_{k\to\infty} Tf_k$$

exist. We can write $f_k = R_\lambda(Tf_k - \lambda f_k) + w_k$, and we deduce that

$$w = \lim_{k\to\infty} w_k$$

exists and belongs to $\mathfrak{E}(\lambda)$. Letting $k \to \infty$, we obtain $f = R_\lambda(f^* - \lambda f) + w$, which implies $f \in D$ and $Tf = f^*$. This proves that T is closed.

Since T is closed and its domain D is dense in H, T has a closed adjoint T^* whose domain D^* is dense in H. Also, $T = T^{**} = (T^*)^*$. If $\mathfrak{M}$ is a subspace of H, let $H - \mathfrak{M}$ denote the orthogonal complement of $\mathfrak{M}$ in H.

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LEMMA 2: *D^* consists of all $g \in D$ of the form $g = \bar{R}_\lambda z$, where $z \in H - \mathfrak{E}(\bar{\lambda})$. The adjoint operator T^* is a restriction of T and is closed and symmetric.*

Proof. $g^* = T^*g$ means $[Tf, g] = [f, g^*]$ for every $f \in D$. By Lemma 1, any $f \in D$ may be written $f = R_{\bar{\lambda}}h + w$, with $h \in H, w \in \mathfrak{E}(\bar{\lambda})$, and then $Tf = \bar{\lambda}f + h$. Substituting in the equation $[Tf, g] = [f, g^*]$, we obtain $[\bar{\lambda}f + h, g] = [R_{\bar{\lambda}}h + w, g^*]$, or $[\bar{\lambda}R_{\bar{\lambda}}h + \bar{\lambda}w + h, g] = [R_{\bar{\lambda}}h + w, g^*]$. This is equivalent to

$$[h, \lambda R_{\bar{\lambda}}{}^*g + g - R_{\bar{\lambda}}{}^*g^*] + [w, \lambda g - g^*] = 0$$

for all $h \in H$, $w \in \mathfrak{E}(\bar{\lambda})$. Then $\lambda g - g^* = z$ is orthogonal to $\mathfrak{E}(\bar{\lambda})$ and $g = R_{\bar{\lambda}}{}^*z = \bar{R}_\lambda z$. Since D and D^* are obviously unaltered by complex conjugation, $\bar{g} = R_\lambda \bar{z}$ belongs to D^*, and by Lemma 1, $\bar{g} \in D$, so that $g \in D$. Conversely, if $g = \bar{R}_\lambda z$ with $z \in \mathfrak{E}(\bar{\lambda})$, we let $g^* = -z - \lambda g$, and find that $g \in D^*$, $g^* = T^*g$. Thus D^* is as described in the statement of the lemma, and is contained in D. Now $T \supseteq T^*$, $T = (T^*)^* \supseteq T^*$, and T^* is symmetric.

Since $T = (T^*)^*$, the theory of the Cayley transform implies that $D = D^* \oplus \mathfrak{E}(i) \oplus \mathfrak{E}(-i)$, where $\oplus$ denotes a direct sum. Let the linear spaces $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ have dimensions τ^+ and τ^- respectively. Since the set of solutions of the differential equation $Lu = \lambda Mu$ is a linear space of dimension n, neither of these dimensions can exceed n. The defect index of T^* is (τ^+, τ^-), and T^* has self-adjoint extensions, which are restrictions of T, if and only if $\tau^+ = \tau^-$. We have already assumed that T has at least one self-adjoint restriction. This assumption is equivalent to the assumption $\tau^+ = \tau^- = \tau$, which we now make .We will characterize all the self-adjoint extensions of T^* by boundary conditions. The self-adjoint extensions of T^* are in one to one correspondence with the unitary operators U of $\mathfrak{E}(i)$ onto $\mathfrak{E}(-i)$. Corresponding to any such U there is a self-adjoint extension A of T^* whose domain D_A is the set of all $f \in D$ of the form $f = f^* + (1 - U)f^+$ with $f^* \in D^*, f^+ \in \mathfrak{E}(i)$, where 1 is the identity operator on $\mathfrak{E}(i)$. Conversely, every such A has a domain of this type. Let $y_1, \ldots, y_\tau$ and $z_1, \ldots, z_\tau$ be orthonormal bases for $\mathfrak{E}(i)$ and $\mathfrak{E}(-i)$ respectively. Then every $f^+ \in \mathfrak{E}(i)$ is of the form $f^+ = \Sigma_{j=1}^\tau a_j y_j$, for some constants a_j. The effect of U on f^+ can be represented by a unitary matrix $U = (u_{jk})$;

$$Uf^+ = \sum_{j=1}^{\tau} a_j U y_j = \sum_{j=1}^{\tau} a_j \sum_{k=1}^{\tau} u_{jk} z_k.$$

Thus $f \in D_A$ if and only if

$$f = f^* + \sum_{j=1}^{\tau} a_j g_j,$$

where $f^* \in D^*$ and

$$g_j = y_j - \sum_{k=1}^{\tau} u_{jk} z_k, \qquad [j = 1, \ldots, \tau].$$

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Green's formula **(4**, p. 86**)** is

$$\int_\alpha^\beta [Lf(x)\bar{g}(x) - f(x)L\bar{g}(x)]dx = [fg](\beta) - [fg](\alpha),$$

for $f, g \in D$, where $[\alpha, \beta]$ is any compact subinterval of I. Here $[fg](x)$ is a bilinear form in the derivatives up to order $(n-1)$ of f and g which is non-degenerate for all $x \in I$. It follows easily from Green's formula that $[fg](x)$ is skew-Hermitian, $[fg](x) = -[\bar{g}\bar{f}](x)$. If I is the interval (a, b), then

$$[fg](a) = \lim_{x \to a+0} [fg](x) \quad \text{and} \quad [fg](b) = \lim_{x \to b-0} [fg](x)$$

exist for all $f, g \in D$. We let

$$< fg > = [fg](b) - [fg](a).$$

A homogeneous boundary condition is a condition on $f \in D$ of the form $< f\alpha > = 0$, where α is a fixed function in D. The conditions $< f\alpha_j > = 0$, $[j = 1, \ldots, p]$ are said to be linearly independent if the only set of complex numbers $\gamma_1, \ldots, \gamma_p$ for which

$$\sum_{j=1}^{p} \gamma_j < f\alpha_j > = 0$$

identically in $f \in D$ is $\gamma_1 = \ldots = \gamma_p = 0$. It is easily seen, since $[Tf, g] - [f, T^*g] = < fg > = 0$ for all $f \in D$, $g \in D^*$, that these boundary conditions are linearly independent if and only if the functions $\alpha_1, \ldots, \alpha_p$ are linearly independent (mod D^*). A set of p linearly independent boundary conditions $< f\alpha_j > = 0$ $[j = 1, \ldots, p]$ is said to be self-adjoint if $< \alpha_j\alpha_k > = 0$ for $j, k = 1, \ldots, p$. Two sets of boundary conditions are said to be equivalent if the sets of functions satisfying the two sets of conditions are identical.

THEOREM 3: *If A is a self-adjoint extension of T^* with domain D_A, then there exists a self-adjoint set of τ linearly independent boundary conditions such that D_A is the set of all $f \in D$ satisfying these conditions. Conversely, corresponding to a self-adjoint set of τ linearly independent boundary conditions, there exists a self-adjoint extension of T^* whose domain D_A is the set of all $f \in D$ satisfying these boundary conditions.*

Proof. The proof is very similar to the proof of Theorem 3 in **(3)**. First, suppose that A is a self-adjoint extension of T^* with domain D_A. There exists a unitary matrix $U = (u_{jk})$ such that

$$f = f^* + \sum_{j=1}^{\tau} a_j g_j$$

for all $f \in D_A$, where $f^* \in D^*$ and

$$g_j = y_j - \sum_{k=1}^{\tau} u_{jk} z_k, \qquad [j = 1, \ldots, \tau].$$

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We will show that the conditions $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ are self-adjoint and that $f \in D_A$ if and only if f satisfies these conditions. It is clear that $g_j \in \mathfrak{E}(i) \oplus \mathfrak{E}(-i)$, and thus

$$\sum_{j=1}^{\tau} \gamma_j g_j \in D^* \quad \text{implies} \quad \sum_{j=1}^{\tau} \gamma_j g_j = 0.$$

Then

$$\sum_{j=1}^{\tau} \gamma_j y_j = 0 \quad \text{and} \quad \sum_{j=1}^{\tau} \gamma_j \sum_{k=1}^{\tau} u_{jk} z_k = 0,$$

and since the y_j are linearly independent, this can happen only if $\gamma_1 = \ldots = \gamma_\tau = 0$. It follows that the boundary conditions $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ are linearly independent. It follows from Green's formula that $< y_j y_k > = 2i\delta_{jk}$, $< z_j z_k > = -2i\delta_{jk}$, $< y_j z_k > = 0$ for $j, k = 1, \ldots, \tau$, and thus

$$\begin{aligned} < g_j g_k > &= < y_j y_k > - \sum_{q=1}^{\tau} u_{jq} < z_q y_k > - \sum_{p=1}^{\tau} \bar{u}_{kp} < y_j z_p > \\ &\qquad + \sum_{p,q=1}^{\tau} u_{jq} \bar{u}_{kp} < z_q z_p > \\ &= 2i\delta_{jk} - 2i \sum_{q=1}^{\tau} u_{jq} \bar{u}_{kq}. \end{aligned}$$

This vanishes for $j, k = 1, \ldots, \tau$ because U is a unitary matrix. Thus the boundary conditions are self-adjoint. If $f \in D_A$,

$$f = f^* + \sum_{p=1}^{\tau} a_p g_p, \quad \text{and} \quad < fg_j > = < f^* g_j > + \sum_{p=1}^{\tau} a_p < g_p g_j >.$$

The first term vanishes since $f^* \in D^*$, and the second term vanishes because the boundary conditions are self-adjoint. Thus if $f \in D_A$, f satisfies the boundary conditions. Conversely, suppose $f \in D$ and $< fg_j > = 0$ $[j = 1, \ldots, \tau]$. We can write

$$f = f^* + \sum_{j=1}^{\tau} b_j y_j + \sum_{j=1}^{\tau} c_j z_j$$

for some constants b_j, c_j. Then $< fg_j > = 0$ implies

$$b_j = - \sum_{k=1}^{\tau} \bar{u}_{jk} c_k,$$

or, equivalently,

$$c_p = - \sum_{j=1}^{\tau} u_{jp} b_j,$$

and this yields

$$\begin{aligned} f &= f^* + \sum_{j=1}^{\tau} b_j \Big(y_j - \sum_{k=1}^{\tau} u_{jk} z_k\Big) \\ &= f^* + \sum_{j=1}^{\tau} b_j g_j \in D_A. \end{aligned}$$

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To prove the converse, we assume that $<fg_j> = 0$ $[j = 1, \ldots, \tau]$ is a self-adjoint set of τ linearly independent boundary conditions. It suffices to show that there exists a unitary matrix $U = (u_{jk})$ such that the given boundary conditions are equivalent to a self-adjoint set of the form $<f\tilde{g}_j> = 0$, where

$$\tilde{g}_j = y_j - \sum_{k=1}^{\tau} u_{jk}z_k, \qquad [j = 1, \ldots, \tau].$$

Since $g_j \in D$, there exists a unique set of complex numbers b_{jk}, c_{jk} such that

$$g_j = g^*_j + \sum_{k=1}^{\tau} b_{jk}y_k + \sum_{k=1}^{\tau} c_{jk}z_k,$$

with $g^*_j \in D^*$ $[j = 1, \ldots, \tau]$. The relations

$$< g_j g_k > = 0 \quad \text{imply} \quad \sum_{p=1}^{\tau} (b_{jk}\bar{b}_{kp} - c_{jp}\bar{c}_{kp}) = 0.$$

If the matrices B and C are defined by $B = (b_{jk})$, $C = (c_{jk})$, this may be written $BB^* = CC^*$. The linear independence of the $g_j(\text{mod } D^*)$ is equivalent to the fact that the two sets of functions

$$\tilde{y}_j = \sum_{k=1}^{\tau} b_{jk}y_k, \quad \tilde{z}_j = \sum_{k=1}^{\tau} c_{jk}z_k \qquad [j = 1, \ldots, \tau]$$

are each linearly independent. Suppose there exist constants $\gamma_1, \ldots, \gamma_\tau$, not all zero, such that

$$y = \sum_{j=1}^{\tau} \gamma_j\tilde{y}_j = 0.$$

If

$$z = \sum_{j=1}^{\tau} \gamma_j\tilde{z}_j,$$

then $< zz > = 0$. Since $z \in \mathfrak{E}(-i)$, $[Tz, z] - [z, Tz] = -2i[z, z] = <zz> = 0$, and thus $z = 0$. Now $y = z = 0$ implies that there exist constants $\gamma_1, \ldots, \gamma_\tau$, not all zero, such that

$$\sum_{j=1}^{\tau} \gamma_j g_j = \sum_{j=1}^{\tau} \gamma_j g^*_j + \sum_{j=1}^{\tau} \gamma_j\tilde{y}_j + \sum_{j=1}^{\tau} \gamma_j\tilde{z}_j \equiv 0(\text{mod } D^*),$$

which contradicts the linear independence (mod D^*) of the g_j. Thus the $\tilde{y}_j$ and $\tilde{z}_j$ are linearly independent, which implies that the matrices B and C are non-singular. Let $U = -B^{-1}C$. Then $UU^* = B^{-1}CC^*B^{*-1} = B^{-1}BB^*B^{*-1}$, which is the identity matrix, and thus U is unitary. Now we define

$$\tilde{g}_j = y_j - \sum_{k=1}^{\tau} u_{jk}z_k, \qquad [j = 1, \ldots, \tau].$$

If $B^{-1} = (d_{jk})$, then

$$\sum_{j=1}^{\tau} d_{pj}g_j = \sum_{j=1}^{\tau} d_{pj}g^*_j + y_p - \sum_{k=1}^{\tau} u_{pk}z_k,$$

and

$$\sum_{j=1}^{\tau} \bar{d}_{jp} < fg_j > \; = \; < f\tilde{g}_p > \qquad f \in D, p = 1, \ldots, \tau,$$

Thus f satisfies $< fg_j > = 0$ $[j = 1, \ldots, \tau]$ if and only if $< f\tilde{g}_j > = 0$ $[j = 1, \ldots, \tau]$, and the theorem is proved.

If I is a finite interval $[a, b]$, the author has shown in **(1)** that n boundary conditions are required. It is also shown in **(1)** that if M has real coefficients, so that m is an even integer, $m = 2s$, and these conditions include $f(a) = \ldots = f^{(s-1)}(a) = f(b) = \ldots = f^{(s-1)}(b) = 0$, then H is a space of functions satisfying these $m = 2s$ boundary conditions, and only $(n - m)$ conditions are actually involved in defining the self-adjoint extensions. It may be conjectured that, in general, if I is a finite interval, the definition of H involves m boundary conditions, and $\tau = n - m$. It is not clear how many boundary conditions, if any, are needed to define H if I is an infinite interval.

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