

DOMINATED EXTENSIONS OF FUNCTIONALS AND V-CONVEX FUNCTIONS ON CANCELLATIVE CONES

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Let C be a cancellative cone and consider a subcone C_0 of C . We study the natural problem of obtaining conditions on a non negative homogeneous function $\phi : C \rightarrow R^+$ so that for each linear functional f defined in C_0 which is bounded by ϕ , there exists a linear extension to C . In order to do this we assume several geometric conditions for cones related to the existence of special algebraic basis of the linear span of these cones.

1. INTRODUCTION

Let R be the field of real numbers and let R^+ be the set of non-negative real numbers. A cone C is a triple $(C, +, \cdot)$ such that $(C, +)$ is a monoid and the product $\cdot : R^+ \times C \rightarrow C$ satisfies the usual axioms of a linear space when restricted to non negative scalars. A subset B of a real linear space X is said to be algebraically closed when $a + b \in B$ and $\lambda a \in B$ for each pair $a, b \in B$ and $\lambda \in R^+$. It can be proved that each cancellative cone can be identified with an algebraically closed subset of a linear space (see [2, 11]). A non negative function ϕ defined on a cone C is called positively homogeneous if $\phi(ax) = a\phi(x)$ for every $x \in C$ and $a \in R^+$. Such a function is convex provided that $\phi(x + y) \leq \phi(x) + \phi(y)$, $x, y \in C$. In [6] the reader can find several results related to extensions of such kind of functions defined from cancellative cones to linear spaces.

It is possible to find a lot of dominating extension results for functionals defined on cones in the bibliography (see for example [2, 11, 1]). However, these results depend on particular properties of the dominating function ϕ , for which is usual to require convexity. Arguments related to order relations defined in the cones are also used (see [13]). Moreover, the results are often obtained for extended functions ϕ (that is, functions ϕ defined in the extended non negative real line $R^+ \cup \{\infty\}$). In this paper we deal with cancellative cones that satisfy a geometric condition related to certain generating properties of algebraic basis of the linear space that contains the cone. We obtain those conditions

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that a positively homogeneous real function ϕ defined on a cancellative cone C must have so that each linear function defined on a subcone C_0 and dominated by ϕ has a linear extension to the whole C .

Our notation is standard. The natural numbers will be denoted by N . The main references about the Hahn-Banach Theorem for linear spaces and cones are [12, 2, 11]. If B is a subset of a cone C , we define $\text{span}\{B\}$ as the set of positive linear combinations of elements of B , that is, the elements of $\text{span}\{B\}$ would be written as $\lambda_1x_1 + \dots + \lambda_nx_n$, where n is a natural number, $\lambda_i \in R^+$ and $x_i \in B$ for every $i = 1, \dots, n$. It is obvious that $\text{span}\{B\}$ is a subcone of C . Since C is a subset of a linear space X , we can also consider the linear subspace $\text{linspan}\{B\}$ of X defined as the (non necessarily positive) linear combinations of elements of B . If C is a cancellative cone and B is a basis of $\text{linspan}\{C\}$ we shall simply say that B is a basis of C . Thus, we shall say that the dimension of a cone C is n if the dimension of $\text{linspan}\{C\}$ is n , $n \in N \cup \{\infty\}$.

DEFINITION 1: Let C be a cancellative cone and let C_0 be a subcone of C . We say that C_0 is compatible with an algebraic basis $V = \{v_i : i \in I\}$ of $\text{linspan}\{C\}$ (V -compatible for short) if there is a subset J of I such that $V_0 = \{v_i : i \in J\}$ is an algebraic basis of $\text{linspan}\{C_0\}$.

It is well known that each linear space X has an algebraic basis. Moreover, if W is a basis for a subspace S of X , it can be always found a basis V for X such that $W \subset V$ (see [8, Theorem 2.4, Chapter IV]). This means that for each subcone C_0 of a cancellative cone C there exists always a basis V of C such that C_0 is V -compatible.

DEFINITION 2: We say that a cancellative cone C is well generated by an algebraic basis $V = \{v_i : i \in I\}$ of $\text{linspan}\{C\}$ if for every subset J of I and each $i_0 \in I - J$, we can find for every element

$$y \in \text{linspan}\{\{v_i : i \in J\}, v_{i_0}\} \cap C,$$

a representation $y = x + \lambda v_{i_0}$, where $x \in C \cap \text{linspan}\{v_i : i \in J\}$ and $\lambda \in R$.

It is easy to find examples of well generated cones by algebraic basis. For instance, every linear space is well generated by any algebraic basis. Another easy example is the positive cone of R^n , that is well generated by the canonical basis of R^n . However, we can find examples of cones that are not well generated by particular basis. For instance, consider the positive cone $(R^3)^+$ of R^3 and the basis given by the set of vectors $B = \{(1, 1, 0), (1, 0, 1), (0, 0, 1)\}$, where (x, y, z) are the coordinates with respect to the canonical basis. The element $(0, 1, 0)$ of the positive cone of R^3 can not be written as a sum of an element of $\text{linspan}\{(1, 1, 0), (1, 0, 1)\} \cap (R^3)^+$ and an element of $\text{linspan}\{(0, 0, 1)\}$.

2. V -CONVEXITY AND DOMINATED EXTENSIONS OF FUNCTIONALS.

DEFINITION 3: Let C be a cancellative cone. Consider an algebraic basis $V = \{v_i : i \in I\}$ of C . Let C_0 be a V -compatible subcone of C with a basis $V_0 = \{v_i : i \in J\}$,

$J \subset I, J \neq I$. We say that a positively homogeneous function ϕ is V -convex with respect to C_0 if for every $i \in I - J$ and $x, y \in C_0$ such that $x + v_i, y - v_i \in C$,

$$\phi(x + y) \leq \phi(x + v_i) + \phi(y - v_i).$$

It is easy to see that convexity for the function ϕ implies V -convexity with respect to every V -compatible subcone C_0 and each algebraic basis V . The following example shows that the converse is not true, that is, we can find a V -convex function with respect to a one dimensional subcone of a semiplane of R^2 that is not convex.

EXAMPLE 4: Consider the canonical basis $\mathcal{B} = \{e_1, e_2\}$ of R^2 and the cone $C = \text{span}\{e_1, e_2, -e_2\}$. We can consider the subcone $C_0 = \text{span}\{e_1\}$ and the function ϕ_0 given by

$$\phi_0(z) = \lambda + |\mu|, \quad z = \lambda e_1 + \mu e_2, \quad \lambda > 0, \mu \in R,$$

and $\phi_0(\lambda e_2) = \phi_0(-\lambda e_2) = 0$ for every $\lambda \in R^+$.

This function is not convex, since $\phi_0(e_1 + e_2) = 2$ but $\phi_0(e_1) = 1$ and $\phi_0(e_2) = 0$. However, it is \mathcal{B} -convex with respect to C_0 , since for each $\lambda_1, \lambda_2 > 0$,

$$\phi_0(\lambda_1 e_1 + \lambda_2 e_1) = \lambda_1 + \lambda_2 < \lambda_1 + \lambda_2 + 2 = \phi_0(\lambda_1 e_1 + e_2) + \phi_0(\lambda_2 e_1 - e_2),$$

and in the case that $\lambda_1 = 0$ or $\lambda_2 = 0$ the inequalities also hold.

DEFINITION 5: We say that a cone C has the 1-extension property with respect to ϕ if for each one dimensional subcone D of C and each linear function $f : D \rightarrow R$ such that $f \leq \phi$ on D there is a linear extension \bar{f} to C that satisfies $\bar{f}(x) \leq \phi(x)$ for every $x \in C$.

THEOREM 6. *Let C be a cancellative cone. Consider a positively homogeneous function $\phi : C \rightarrow R^+$. If C has the 1-extension property with respect to ϕ , then for every algebraic basis V of C and every V -compatible subcone C_0 , ϕ is V -convex with respect to C_0 .*

PROOF: Let V be an algebraic basis of C . Suppose that ϕ is not V -convex with respect to the V -compatible subcone C_0 . Let $V_0 \subset V$ be a basis of C_0 . Then there are elements $v_i \in V - V_0$ and $x, y \in C_0$ such that $x + v_i, y - v_i \in C$ and $\phi(x + y) > \phi(x + v_i) + \phi(y - v_i)$. We can define the linear function $f : \text{linspan}\{x + y\} \cap C \rightarrow R$ by $f(\lambda(x + y)) := \lambda\phi(x + y)$, $\lambda \in R$. We have that $\text{linspan}\{x + y\} \cap C \subset C_0$. By the 1-extension property with respect to ϕ of C we can extend f to a function $f_0 : C_0 \rightarrow R$ that satisfies $f_0(z) \leq \phi(z)$ for every $z \in C_0$. But for such an extension we have

$$f_0(x) + f_0(y) = f(x + y) = \phi(x + y) > \phi(x + v_i) + \phi(y - v_i),$$

and then

$$f_0(y) - \phi(y - v_i) > \phi(x + v_i) - f_0(x).$$

Let us show that we can not define an extension \bar{f} of the restriction of f_0 to $\text{linspan}\{x, y\} \cap C$ dominated by ϕ to the subcone $\text{linspan}\{x, y, v_i\} \cap C$, which provides a contradiction since there is a dominated extension to the whole C . Suppose that there is such an extension \bar{f} that satisfies $\bar{f}(y - v_i) \leq \phi(y - v_i)$. There must be an α with $\bar{f}(y - v_i) = \bar{f}(y) - \alpha$, and then we obtain

$$\alpha \geq \bar{f}(y) - \phi(y - v_i) > \phi(x + v_i) - \bar{f}(x).$$

Thus,

$$\bar{f}(x + v_i) = \bar{f}(x) + \alpha > \bar{f}(x) + \phi(x + v_i) - \bar{f}(x) = \phi(x + v_i),$$

and hence \bar{f} can not be dominated by ϕ . □

LEMMA 7. *Let C be a cancellative cone and let C_0 be a subcone of C . Let f be a linear function $f : C_0 \rightarrow R$. Then the set $\langle C_0 \rangle = \text{linspan}\{C_0\} \cap C$ is a subcone of C and the extension $f_{\langle C_0 \rangle}$ of f to $\langle C_0 \rangle$ given by $f_{\langle C_0 \rangle}(z) := f(x + z) - f(x)$, where $x \in C_0$ is an element such that $x + z \in C_0$, is a well-defined linear map.*

PROOF: It is clear that $\langle C_0 \rangle$ is a cone, since $\text{linspan}\{C_0\}$ and C are cones. Let us show that $f_{\langle C_0 \rangle}$ is a well-defined linear map. First we show that for every $z \in \langle C_0 \rangle$ there is an element $x \in C_0$ such that $x + z \in C_0$. Indeed, there are elements $u, v \in C_0$ such that $z = u - v$, since C_0 is a subcone and $z \in \text{linspan}\{C_0\}$, and then $z + v = u \in C_0$.

Now, suppose that there are two elements $x, y \in C_0$ such that $x + z, y + z \in C_0$. Then $f(x + z) + f(y) = f(x + z + y) = f(y + z) + f(x)$ and $f(x + z) - f(x) = f(y + z) - f(y)$. Thus, $f_{\langle C_0 \rangle}$ is well-defined. Straightforward calculations show that $f_{\langle C_0 \rangle} : \langle C_0 \rangle \rightarrow R$ is a linear function. □

The following theorem gives conditions for a function ϕ satisfied a dominated extension theorem for every linear function defined on a particular subcone C_0 .

THEOREM 8. *Let C be a cancellative cone. Consider a positively homogeneous function $\phi : C \rightarrow R^+$. Let V be an algebraic basis such that C is well generated by V , and let C_0 be a V -compatible subcone. Then the first of the following statements implies the second one.*

- (1) *The function ϕ is V -convex with respect to each V -compatible subcone that contains C_0 .*
- (2) *For every linear function $f : C_0 \rightarrow R$ such that*

$$f(x + z) - f(x) \leq \phi(z), \quad x, x + z \in C_0, z \in C,$$

there is an extension \bar{f} to C satisfying $\bar{f}(z) \leq \phi(z)$ for every $z \in C$.

PROOF: Let $f : C_0 \rightarrow R$ be a linear function satisfying the condition (2). Consider the subcone $\langle C_0 \rangle$ of C and the extension $f_{\langle C_0 \rangle}$ of f given by Lemma 7. The definition of $f_{\langle C_0 \rangle}$ makes clear that $f_{\langle C_0 \rangle}(z) \leq \phi(z)$ for each $z \in \langle C_0 \rangle$. Moreover, a direct argument

shows that $\langle C_0 \rangle$ is V -compatible if C is well generated by a basis $V = \{v_i : i \in I\}$. Consider a 1-dimensional non trivial extension of $\langle C_0 \rangle$ given by an element $v_i \in V$ such that

$$C_i = \text{linspan}\{v_i, \langle C_0 \rangle\} \cap C \neq \langle C_0 \rangle.$$

For each element z of C_i there is a unique representation as a sum $z = x + \lambda v_i$, $\lambda \in R$, since C is well generated by V . Then we define the extension by mean of the formula $f_i(z) := f_{\langle C_0 \rangle}(x) + \lambda \alpha$, where α must satisfy

$$\alpha \leq \inf\{\phi(x + v_i) - f_{\langle C_0 \rangle}(x) : x \in \langle C_0 \rangle, x + v_i \in C_i\} = \alpha_1$$

and

$$\alpha \geq \sup\{f_{\langle C_0 \rangle}(x) - \phi(x - v_i) : x \in \langle C_0 \rangle, x - v_i \in C_i\} = \alpha_2.$$

Given an $\varepsilon > 0$, we can find $x_1, x_2 \in \langle C_0 \rangle$ such that

$$\phi(x_1 + v_i) - f_{\langle C_0 \rangle}(x_1) < \alpha_1 + \varepsilon$$

and

$$\alpha_2 - \varepsilon < f_{\langle C_0 \rangle}(x_2) - \phi(x_2 - v_i).$$

The following inequalities are consequences of the V -convexity of ϕ and show that there exists such an α , since

$$f_{\langle C_0 \rangle}(x_1) + f_{\langle C_0 \rangle}(x_2) = f_{\langle C_0 \rangle}(x_1 + x_2) \leq \phi(x_1 + x_2) \leq \phi(x_1 + v_i) + \phi(x_2 - v_i),$$

and then

$$\alpha_2 - \varepsilon < f_{\langle C_0 \rangle}(x_2) - \phi(x_2 - v_i) \leq \phi(x_1 + v_i) - f_{\langle C_0 \rangle}(x_1) < \alpha_1 + \varepsilon.$$

Now let us show that $f_i(z) \leq \phi(z)$ for every z of C_i . If $z = x + \lambda v_i$, where $\lambda \geq 0$, we obtain

$$f_i(z) = f_{\langle C_0 \rangle}(x) + \lambda \alpha \leq f_{\langle C_0 \rangle}(x) + \lambda \left(\phi\left(\frac{x}{\lambda} + v_i\right) - f_{\langle C_0 \rangle}\left(\frac{x}{\lambda}\right) \right) \leq \phi(x + \lambda v_i).$$

On the other hand, if $z = x - \mu v_i$ for $\mu \geq 0$,

$$f_i(z) = f_{\langle C_0 \rangle}(x) - \mu \alpha \leq f_{\langle C_0 \rangle}(x) - \mu \left(f_{\langle C_0 \rangle}\left(\frac{x}{\mu}\right) - \phi\left(\frac{x}{\mu} - v_i\right) \right) \leq \phi(x - \mu v_i).$$

We have shown that there is a non trivial extension of the linear function $f_{\langle C_0 \rangle}$ to C_i induced by the element $v_i \in V$. Since C_0 is V -compatible, there is a subset J of I such that $\text{linspan}\{v_i : i \in J\} \cap C = \langle C_0 \rangle$. Consider the set $K = I - J$. If $L \subset K$, we define the subcone C_L of C given by $C_L = \text{linspan}\{L \cup J\} \cap C$. Note that $C_0 \subset \langle C_0 \rangle \subset C_L$ and $\langle C_L \rangle = C_L$ for every $L \subset K$. Take the family of subcones of C given by $\mathcal{F} = \{C_L : L \subset K\}$ and define the family of extensions

$$\overline{\mathcal{F}} = \{(C_L, f_{L,t}) : C_L \in \mathcal{F}, t \in T\},$$

where T is a collection of index such that for each $L \subset K$ and $t \in T$, $f_{L,t} : C_L \rightarrow R$ extends $f_{(C_0)}$ and $f_{L,t}(z) \leq \phi(z)$ for every $z \in C_L$. Note that the well generation of C by V implies that for each L and for each r which is not in $L \cup J$, if $z \in \text{linspan}\{v_r, C_L\} \cap C$ there is a $\lambda \in R$ and an element $x \in C_L$ such that $z = x + \lambda v_r$. We can define an order relation in $\overline{\mathcal{F}}$ in the usual way, that is, $(C_L, f_{L,t}) \leq (C_H, f_{H,s})$ if $C_L \subset C_H$ and $f_{H,s}$ extends $f_{L,t}$. To see that every chain gives an upper bound, it is enough to consider the union of all the elements of the chain and the corresponding natural definition of the extension. Moreover, such a union can also be written as an element of \mathcal{F} . Since we have shown that $\overline{\mathcal{F}}$ is non void, we can apply Zorn's Lemma to find a maximal element $(C_M, f_{M,r})$. The proof finishes with the observation that $C_M = C$. If this is not the case, we could find an element $v_r \in V$ such that $C_M \neq \text{linspan}\{v_r, C_M\} \cap C$. The same argument given for the construction of the one dimensional extension in the beginning of the proof would give a proper extension of $(C_M, f_{M,r})$, which contradicts its maximality. \square

COROLLARY 9. *Let C be a cancellative cone and let $\phi : C \rightarrow R^+$ be a positively homogeneous function. If C has the 1-extension property with respect to ϕ , then for every algebraic basis V such that C is well generated by V , for every V -compatible subcone C_0 and for every linear function $f : C_0 \rightarrow R$ such that*

$$f(x + z) - f(x) \leq \phi(z), \quad x, x + z \in C_0, z \in C,$$

there is a dominated extension \overline{f} to C , that is, $\overline{f}(z) \leq \phi(z)$ for every $z \in C$.

We conclude this paper with an application to the case of linear spaces. If the cone is a linear space it is well generated by any algebraic basis. Note that similar arguments could also be used for cones with the same kind of geometric properties. In this case we obtain, up to a point, an equivalence between V -convexity, the existence of dominated extensions for functions defined on subcones and the 1-extension property.

DEFINITION 10: Let E be a linear space. We say that a positively homogeneous function $\phi : E \rightarrow R^+$ is a Hahn-Banach function for E if for every subspace E_0 of E and each algebraic basis V such that E_0 is V -compatible, then every V -compatible subspace E_c that contains E_0 , ϕ is V -convex with respect to E_c .

COROLLARY 11. *Let E be a linear space and let $\phi : E \rightarrow R^+$ be an homogeneous function. Then the following are equivalent.*

- (1) ϕ is a Hahn-Banach function for E .
- (2) If $x, y, z \in E$ and $z \neq 0$ is not a linear combination of x and y , then

$$\phi(x + y) \leq \phi(x + z) + \phi(y - z).$$

- (3) For every subcone C_0 of E and every linear function $f : C_0 \rightarrow R$ such that

$$f(x + z) - f(x) \leq \phi(z), \quad x, x + z \in C_0, z \in E,$$

there is a linear extension \bar{f} to E dominated by ϕ .

(4) The space E has the 1-extension property with respect to ϕ .

PROOF: The equivalence between (1) and (2) holds from the fact that each linearly independent subset of a linear space can be used to construct an (algebraic) basis for the space (see [8, Theorem 2.4, Chapter IV]). To prove the equivalence of (1) and (3) it is enough to note that the existence of a dominated extension for every linear functional defined in a subcone C_0 of E is a direct consequence of Theorem 8. On the other hand, (3) implies in particular that E has the 1-extension property with respect to ϕ . Finally, (4) \rightarrow (1) is a consequence of Theorem 6. \square

Following [4, 5, 6, 7], an asymmetric norm (quasi-norm in [1, 3]) on a linear space E is a non negative real valued function $\|\cdot\|$ on E such that for all $x, y \in E$ and $\lambda \in \mathbb{R}^+$,

$$(i) \quad \|x\| = \|-x\| = 0 \Leftrightarrow x = 0;$$

$$(ii) \quad \|\lambda x\| = \lambda \|x\|; \text{ and}$$

$$(iii) \quad \|x + y\| \leq \|x\| + \|y\|.$$

The pair $(E, \|\cdot\|)$ is then called an asymmetric normed linear space.

Each asymmetric norm $\|\cdot\|$ on E induces in a natural way a topology on E which has as a open neighbourhood basis for each $x \in E$ the sets of the form $\{y \in E : \|y - x\| < r\}, r > 0$.

Recent applications show that asymmetric norms constitute an efficient tool in the study of some problems in the theory of computational complexity ([10]) and in approximation theory ([9]).

Since every asymmetric norm is a positively homogeneous function, we can obtain a Hahn-Banach type theorem as a consequence of Theorem 8. This will be done with the help of the following result, which can be found in [1].

LEMMA 12. *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be two asymmetric normed linear spaces and let $f : X \rightarrow Y$ be a linear function. Then f is continuous if and only if there is $M > 0$ such that $\|f(x)\|_2 \leq M \|x\|_1$ for all $x \in X$.*

Notice that each asymmetric norm $\|\cdot\|$ defined on a linear space E is a convex function, so it is a Hahn-Banach function on E . Hence we obtain the following corollary.

COROLLARY 13. *Let $(E, \|\cdot\|)$ be an asymmetric normed linear space. Then for every subcone C_0 and every linear function $f : C_0 \rightarrow \mathbb{R}$ such that*

$$f(x + z) - f(x) \leq \|z\|$$

for all $x \in C_0$ and $z \in E$ with $x + z \in C_0$, there exists a continuous linear extension \bar{f} to E .

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