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Dirichlet-type spaces of the unit bidisc and toral 2-isometries

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Abstract. We introduce and study Dirichlet-type spaces $\mathcal{D}(\mu_1, \mu_2)$ of the unit bidisc $\mathbb{D}^2$, where μ_1, μ_2 are finite positive Borel measures on the unit circle. We show that the coordinate functions z_1 and z_2 are multipliers for $\mathcal{D}(\mu_1, \mu_2)$ and the complex polynomials are dense in $\mathcal{D}(\mu_1, \mu_2)$. Further, we obtain the division property and solve Gleason's problem for $\mathcal{D}(\mu_1, \mu_2)$ over a bidisc centered at the origin. In particular, we show that the commuting pair $\mathscr{M}_z$ of the multiplication operators $\mathscr{M}_{z_1}, \mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$ defines a cyclic toral 2-isometry and $\mathscr{M}_z^*$ belongs to the Cowen–Douglas class $\mathbf{B}_1(\mathbb{D}_r^2)$ for some $r > 0$. Moreover, we formulate a notion of wandering subspace for commuting tuples and use it to obtain a bidisc analog of Richter's representation theorem for cyclic analytic 2-isometries. In particular, we show that a cyclic analytic toral 2-isometric pair T with cyclic vector f_0 is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ for some μ_1, μ_2 if and only if $\ker T^*$, spanned by f_0, is a wandering subspace for T.

1 Introduction and preliminaries

The aim of this paper is to obtain a bidisc counter-part of the theory of Dirichlet-type spaces of the open unit disc as presented in [26] (see [8] for a ball counter-part of this theory). Throughout this paper, $\mathbb{D}$ denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$. Recall that Dirichlet-type spaces of $\mathbb{D}$ are model spaces for the class of cyclic analytic 2-isometries (see [26]). Thus to arrive at an appropriate notion of the Dirichlet-type spaces of the unit bidisc $\mathbb{D}^2$, it is helpful to look for function spaces which support the class of 2-isometries naturally associated with $\mathbb{D}^2$. Let us first recall the definition of such 2-isometries.

For a complex Hilbert space $\mathcal{H}$, let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. For a positive integer d, a *commuting d-tuple T on $\mathcal{H}$* is the d-tuple $(T_1, \dots, T_d)$ of operators $T_1, \dots, T_d \in \mathcal{B}(\mathcal{H})$ satisfying $T_iT_j = T_jT_i, 1 \leqslant i \neq j \leqslant d$. Let $T = (T_1, \dots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. We say that $T = (T_1, \dots, T_d)$ is a *toral isometry* if $T_1, \dots, T_d$ are isometries. Following [1, 5, 26], T is said to be a *toral* 2-isometry if

$$I - T_i^* T_i - T_j^* T_j + T_j^* T_i^* T_i T_j = 0, \quad i, j = 1, \dots, d. \tag{1.1}$$

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A toral isometry is necessarily a toral 2-isometry, but the converse is not true (see [5, Example 1]).

To propose a successful analog of Dirichlet-type spaces on $\mathbb{D}^2$, it is helpful to examine examples of toral 2-isometries arising from function spaces. Since the operator of multiplication by the coordinate function on the classical Dirichlet space $\mathcal{D}(\mathbb{D})$ is a 2-isometry, it is natural to seek the classical Dirichlet space of the unit bidisc. Recall that the *Dirichlet space* $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$ of $\mathbb{D}^2$ is given by

$$\Big\{f \in \mathcal{O}(\mathbb{D}^2) : \|f\|^2_{\mathcal{D}(\mathbb{D})\otimes\mathcal{D}(\mathbb{D})} := \sum_{(m,n)\in\mathbb{Z}^2_+} |\hat{f}(m,n)|^2 (m+1)(n+1) < \infty\Big\},$$

where $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on a domain Ω, $\mathbb{Z}_+$ denotes the set of nonnegative integers and $\hat{f}$ denotes the Fourier transform of f. It turns out that if $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are the operators of multiplication by the coordinate functions z_1 and z_2, respectively, on $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$, then the commuting pair $(\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ satisfies (1.1) for $1 \leqslant i = j \leqslant 2$, but it fails to satisfy (1.1) for $1 \leqslant i \neq j \leqslant 2$. This failure may be attributed to the fact that the mapping $(m,n) \mapsto \|z_1^m z_2^n\|^2$ is a polynomial of bi-degree $(1,1)$. Interestingly, there is a "natural" choice $\mathcal{D}(\mathbb{D}^2)$ of the Dirichlet space containing $\mathcal{D}(\mathbb{D}) \otimes \mathcal{D}(\mathbb{D})$ for which the associated pair $(\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a toral 2-isometry:

$$\mathcal{D}(\mathbb{D}^2) = \Big\{f \in \mathcal{O}(\mathbb{D}^2) : \|f\|^2_{\mathcal{D}(\mathbb{D}^2)} := \sum_{(m,n)\in\mathbb{Z}^2_+} |\hat{f}(m,n)|^2 (m+n+1) < \infty\Big\}.$$

The norm $\|\cdot\|_{\mathcal{D}(\mathbb{D}^2)}$ can also be written as follows:

$$\begin{aligned}\|f\|^2_{\mathcal{D}(\mathbb{D}^2)} = \|f\|^2_{H^2(\mathbb{D}^2)} &+ \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 \, dA(z_1) d\theta \\ &+ \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 \, dA(z_2) d\theta, \end{aligned} \tag{1.2}$$

where $d\theta$ (resp. dA) denotes the *normalized Lebesgue arc-length* (*resp. area*) *measure* on $\mathbb{T}$ (resp. $\mathbb{D}$). Recall that the *Hardy space* $H^2(\mathbb{D}^2)$ of the unit bidisc $\mathbb{D}^2$ is the reproducing kernel Hilbert space (see [23] for the definition of the reproducing kernel Hilbert space) associated with the *Cauchy kernel*

$$\kappa(z,w) = \prod_{j=1}^{2} (1 - z_j \overline{w}_j)^{-1}, \quad z = (z_1, z_2), \ w = (w_1, w_2) \in \mathbb{D}^2.$$

It is worth noting that for any $f \in H^2(\mathbb{D}^2)$,

$$\|f\|^2_{H^2(\mathbb{D}^2)} = \sum_{\alpha\in\mathbb{Z}^2_+} |\hat{f}(\alpha)|^2 \tag{1.3}$$

$$= \sup_{0<r<1} \int_{[0,2\pi]^2} |f(re^{i\theta_1}, re^{i\theta_2})|^2 d\theta_1 d\theta_2 \tag{1.4}$$

(see [27, Section 3.4]).

For a nonempty subset Ω of $\mathbb{C}$, let $M_+(\Omega)$ denote the set of finite positive Borel measures on Ω. Let $P_\mu(w)$ denote the *Poisson integral* $\int_{\mathbb{T}} \frac{1-|w|^2}{|w-\zeta|^2} d\mu(\zeta)$ of the measure $\mu \in M_+(\mathbb{T})$. For future reference, we record the following consequence of the Fubini's theorem (see [28, Theorem 8.8]) and the fact that the mapping $r \mapsto \int_{\mathbb{T}} |f(z, re^{i\theta})|^2 d\theta$ is increasing.

Lemma 1.1 *For $f \in \mathcal{O}(\mathbb{D}^2)$ and $\mu \in M_+(\mathbb{D})$, the extended real-valued mapping $\phi(r) = \int_{\mathbb{T}} \int_{\mathbb{D}} |f(z, re^{i\theta})|^2 d\mu(z) d\theta, r \in (0,1)$, is increasing.*

The formula (1.2) together with Richter's notion of Dirichlet-type spaces (see [26, Section 3]) motivates us to the following:

Definition 1.2 For $\mu_1, \mu_2 \in M_+(\mathbb{T})$ and $f \in \mathcal{O}(\mathbb{D}^2)$, *the Dirichlet integral $D_{\mu_1,\mu_2}(f)$* of f is given by

$$\begin{aligned} D_{\mu_1,\mu_2}(f) = \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ + \sup_{0<r<1} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta. \end{aligned}$$

If either μ_1 or μ_2 is 0, then the *Dirichlet-type space* $\mathcal{D}(\mu_1, \mu_2)$ is the space of functions $f \in H^2(\mathbb{D}^2)$ satisfying $D_{\mu_1,\mu_2}(f) < \infty$. Otherwise, we set $\mathcal{D}(\mu_1, \mu_2) = \{f \in \mathcal{O}(\mathbb{D}^2) : D_{\mu_1,\mu_2}(f) < \infty\}$.

Before we define a norm on the Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$, we present a 2-variable analog of [26, Lemma 3.1].

Lemma 1.3 *For $\mu_1, \mu_2 \in M_+(\mathbb{T}), \mathcal{D}(\mu_1, \mu_2) \subseteq H^2(\mathbb{D}^2)$.*

Proof By the definition of $\mathcal{D}(\mu_1, \mu_2)$, we may assume that both measures μ_1 and μ_2 are nonzero. Note that

$$P_\mu(w) \geqslant \frac{\mu(\mathbb{T})}{4}(1 - |w|^2), \quad \mu \in M_+(\mathbb{T}),\ w \in \mathbb{D}. \tag{1.5}$$

Thus, for any $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n}\, z_1^m z_2^n \in \mathcal{D}(\mu_1, \mu_2)$,

$$\begin{aligned} &\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\overset{(1.5)}{\geqslant} \frac{\mu_1(\mathbb{T})}{4} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} |a_{m,n}|^2 m^2 r^{2n} \int_{\mathbb{D}} |z_1^{m-1}|^2 (1 - |z_1|^2) dA(z_1) \\ &= \frac{\mu_1(\mathbb{T})}{4} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} |a_{m,n}|^2 \frac{m r^{2n}}{m+1}. \end{aligned}$$

A similar estimate using (1.5) gives

$$\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \geqslant \frac{\mu_2(\mathbb{T})}{4} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|^2 \frac{n r^{2m}}{n+1}.$$

Since $f \in \mathcal{D}(\mu_1, \mu_2)$,

$$\sup_{0<r<1} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} |a_{m,n}|^2 \frac{m r^{2n}}{m+1} < \infty, \quad \sup_{0<r<1} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} |a_{m,n}|^2 \frac{n r^{2m}}{n+1} < \infty.$$

It is now easy to see using the monotone convergence theorem (see [28, Theorem 1.26]) that f belongs to $H^2(\mathbb{D}^2)$. ∎

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In view of Lemmas 1.1 and 1.3, the Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ can be endowed with the norm

$$\|f\|^2_{\mathcal{D}(\mu_1,\mu_2)} = \|f\|^2_{H^2(\mathbb{D}^2)} + \lim_{r\to 1^-} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ + \lim_{r\to 1^-} \int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta.$$

We see that $\mathcal{D}(\mu_1, \mu_2)$ is a reproducing kernel Hilbert space (see Lemma 3.1).

The present paper is devoted to the study of Dirichlet-type spaces with efforts to understand the bidisc counter-part of the work carried out in [26]. Before we state the main results of this paper, we need some definitions.

For a positive integer d, let Ω be a domain in $\mathbb{C}^d$, and let $\mathscr{H}$ be a Hilbert space such that $\mathscr{H} \subseteq \mathcal{O}(\Omega)$. A function $\varphi : \Omega \to \mathbb{C}$ is said to be a *multiplier* of $\mathscr{H}$ if $\varphi f \in \mathscr{H}$ for every $f \in \mathscr{H}$. For a nonempty subset U of Ω, we say that *Gleason's problem can be solved for $\mathscr{H}$ over U* if for every $f \in \mathscr{H}$ and $\lambda = (\lambda_1, \dots, \lambda_d) \in U$, there exist functions $g_1, \dots, g_d$ in $\mathscr{H}$ such that

$$f(z) = f(\lambda) + \sum_{j=1}^{d} (z_j - \lambda_j) g_j(z), \quad z = (z_1, \dots, z_d) \in \Omega.$$

We say that *Gleason's problem can be solved for $\mathscr{H}$* if Gleason's problem can be solved for $\mathscr{H}$ over Ω (the reader is referred to [31] for a solution of Gleason's problem for Bergman and Bloch spaces of the unit ball). It turns out that Gleason's problem can be solved for $H^2(\mathbb{D}^d)$ (see Remark 5.2).

Definition 1.4 Let Ω be a domain in $\mathbb{C}^d$, and let $\mathscr{H}$ be a Hilbert space such that $\mathscr{H} \subseteq \mathcal{O}(\Omega)$. We say that $\mathscr{H}$ has the *j-division property*, $j = 1, \dots, d$, if $\frac{f(z)}{z_j - \lambda_j}$ defines a function in $\mathscr{H}$ whenever $\lambda \in \Omega$, $f \in \mathscr{H}$ and $\{z \in \Omega : z_j = \lambda_j\}$ is contained in $Z(f)$, the zero set of f. If $\mathscr{H}$ has j-division property for every $j = 1, \dots, d$, then we say that $\mathscr{H}$ has the *division property*.

In case of $d = 1$, this property appeared in [4, Definition 1.1]. One of the main results of this paper shows that $\mathcal{D}(\mu_1, \mu_2)$ has the division property. In what follows, we require a generalization of the notion of the wandering subspace introduced by Halmos (see [20, p. 103]).

Definition 1.5 Let $T = (T_1, \dots, T_d)$ be a commuting d-tuple on $\mathcal{H}$. A closed subspace $\mathcal{W}$ of $\mathcal{H}$ is said to be *wandering* for T if for every $i = 1, \dots, d$,

$$\prod_{j=1}^{d} T_j^{\alpha_j} \mathcal{W} \perp \prod_{j=1}^{d} T_j^{\beta_j} \mathcal{W}, \quad \alpha_j, \beta_j \in \mathbb{Z}_+,\ j = 1, \dots, d,\ \alpha_i = 0,\ \beta_i \neq 0.$$

Remark 1.6 If $d = 1$, then $\mathcal{W}$ is a wandering subspace for T if and only if $\mathcal{W} \perp T^k(\mathcal{W})$ for every integer $k \geqslant 1$. In particular, $\ker T^*$ is a wandering subspace for any $T \in \mathcal{B}(\mathcal{H})$. Moreover, if $T = (T_1, \dots, T_d)$ is a commuting d-tuple such that $T_j^* T_i = T_i T_j^*, 1 \leqslant i \neq j \leqslant d$, then $\ker T^* = \cap_{j=1}^{d} \ker T_j^*$ is a wandering subspace for T.

It follows from Remark 1.6 that the space spanned by the constant function 1 is a wandering subspace for the multiplication 2-tuple $\mathscr{M}_z$ on $H^2(\mathbb{D}^2)$. Interestingly, this fact extends to the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ (see Corollary 3.12).

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Recall that a commuting d-tuple $T = (T_1, \dots, T_d)$ on $\mathcal{H}$ is *cyclic* with *cyclic vector* $f_0 \in \mathcal{H}$ if $\bigvee \{T^\alpha f_0 : \alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d\} = \mathcal{H}$, where $\bigvee$ denotes the closed linear span and $T^\alpha = \prod_{j=1}^d T_j^{\alpha_j}$. For later purpose, we state the following property of cyclic tuples (see [3, Proposition 1.1]):

(1.6) *If T is cyclic, then for any $\omega \in \mathbb{C}^d$,* $\dim \ker(T^* - \omega)$ *is at most* 1,

where $\ker S = \cap_{j=1}^d \ker S_j$ for the d-tuple $S = (S_1, \dots, S_d)$ and dim stands for the Hilbert space dimension. A commuting d-tuple T on $\mathcal{H}$ has the *wandering subspace property* if $\mathcal{H} = \bigvee_{\alpha \in \mathbb{Z}_+} T^\alpha(\ker T^*)$. Following [15, p. 56], we say that a commuting d-tuple $T = (T_1, \dots, T_d)$ on $\mathcal{H}$ is *analytic* if

$$\bigcap_{k=0}^{\infty} \sum_{\alpha \in \Gamma_k} T^\alpha \mathcal{H} = \{0\},$$

where, for $k \in \mathbb{Z}_+, \Gamma_k := \{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{Z}_+^d : \alpha_1 + \cdots + \alpha_d = k\}$. Note that if T is analytic, then $T_1, \dots, T_d$ are analytic.

Let Ω be a domain in $\mathbb{C}^d$. For a positive integer n, let $\mathbf{B}_n(\Omega)$ denote the set of all commuting d-tuples T on $\mathcal{H}$ satisfying the following conditions:

- for every $\omega = (\omega_1, \dots, \omega_d) \in \Omega$, the map $D_{T-\omega}(x) = ((T_j - \omega_j)x)_{j=1}^d$ from $\mathcal{H}$ into $\mathcal{H}^{\oplus d}$ has closed range and $\dim \ker(T - \omega) = n$,
- the subspace $\bigvee_{\omega \in \Omega} \ker(T - \omega)$ of $\mathcal{H}$ equals $\mathcal{H}$.

We call the set $\mathbf{B}_n(\Omega)$ the *Cowen–Douglas class of rank n with respect to* Ω (refer to [10, 13] for the basic theory of Cowen–Douglas class).

2 Statements of main theorems

The following three theorems collect several basic properties of Dirichlet-type spaces $\mathcal{D}(\mu_1, \mu_2)$.

Theorem 2.1 *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, we have the following statements*:

(i) *the coordinate functions z_1, z_2 are multipliers of $\mathcal{D}(\mu_1, \mu_2)$,*
(ii) *the polynomials are dense in $\mathcal{D}(\mu_1, \mu_2)$,*
(iii) *for nonnegative integers k, l and a polynomial p in z_1 and z_2,*

$$\begin{aligned} \|z_1^k z_2^l p\|^2_{\mathcal{D}(\mu_1,\mu_2)} &= \|p\|^2_{\mathcal{D}(\mu_1,\mu_2)} + k \int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta) d\theta \\ &\quad + l \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta) d\theta. \end{aligned} \tag{2.1}$$

Theorem 2.2 *For $\mu_1, \mu_2 \in M_+(\mathbb{T}), \mathcal{D}(\mu_1, \mu_2)$ has the division property.*

Theorem 2.3 *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, Gleason's problem can be solved for $\mathcal{D}(\mu_1, \mu_2)$ over $\mathbb{D}_r^2$ for some $r \in (0,1]$.*

Here $\mathbb{D}_r^2$ denotes the bidisc $\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < r, |z_2| < r\}$, where r is a positive real number. Unlike the one variable situation, we do not know whether Gleason's problem can be solved for $\mathcal{D}(\mu_1, \mu_2)$ over the unit bidisc. It is worth noting that not all facts about Dirichlet-type spaces of the unit disc have successful counterparts in the bidisc case. For example, the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ on $\mathcal{D}(\mu_1, \mu_2)$ fails to

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be essentially normal (see Corollary 3.13). Moreover, the verbatim analog of the model theorem [26, Theorem 5.1] does not hold true (see Remark 2.5).

The following result asserts that $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ is a canonical model for analytic 2-isometries T for which $\ker T^*$ is a cyclic wandering subspace.

Theorem 2.4 (A representation theorem) *Let $T = (T_1, T_2)$ be a commuting pair on $\mathcal{H}$. Then the following statements are equivalent:*

(i) *T is a cyclic analytic toral 2-isometry with cyclic vector $f_0 \in \ker T^*$ and $\ker T^*$ is a wandering subspace for T,*
(ii) *T is a cyclic toral 2-isometry with cyclic vector $f_0 \in \ker T^*$, T^* belongs to $\mathbf{B}_1(\mathbb{D}^2_r)$ for some $r \in (0,1]$ and $\ker T^*$ is a wandering subspace for T,*
(iii) *there exist $\mu_1, \mu_2 \in M_+(\mathbb{T})$ such that T is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.*

Remark 2.5 By [25, Theorem 1], any analytic 2-isometry T on $\mathcal{H}$ has the wandering subspace property. This result fails even for analytic toral isometric d-tuples if $d > 1$. Indeed, if $a \in \mathbb{D}^2 \setminus \{(0,0)\}$, then the restriction of $\mathscr{M}_z$ to $\{f \in H^2(\mathbb{D}^2) : f(a) = 0\}$ is a toral isometry without the wandering subspace property. This may be seen by imitating the argument of [6, Example 6.8] with the only change that the application of [19, Theorem 4.3] is replaced by that of [19, Corollary 4.6]. This example also shows that the assumption that the cyclic vector f_0 belongs to $\ker T^*$ in (i) can not be dropped from Theorem 2.4. Also, by Theorem 2.1(ii), the cyclicity of T in (ii) of Theorem 2.4 can not be relaxed.

Theorems 2.1, 2.3, and 2.4 provide bidisc analogs of [26, Theorems 3.6, 3.7, and 5.1], respectively. Also, Theorem 2.2 presents a counterpart of the fact that Dirichlet-type spaces on the unit disc have the division property (see [26, Corollary 3.8] and [24, Lemma 2.1]). The proofs of these results and their consequences are presented in Sections 3–6 (see Corollaries 3.8, 3.9, 3.12, 3.13, 4.6, 5.5, 5.6, 6.2, 6.6). In the final short section, we discuss the spectral picture of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$ and raise some related questions.

3 Proof of Theorem 2.1 and its consequences

We need several lemmas to prove Theorem 2.1.

Lemma 3.1 *The Dirichlet-type space $\mathcal{D}(\mu_1, \mu_2)$ is a reproducing kernel Hilbert space. If $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is the reproducing kernel of $\mathcal{D}(\mu_1, \mu_2)$, then for any $r \in (0,1)$, $\bigvee\{\kappa(\cdot, w) : |w| < r\} = \mathcal{D}(\mu_1, \mu_2)$ and $\kappa(\cdot, 0) = 1$.*

Proof We borrow an argument from the proof of [14, Theorem 1.6.3]. Let $\{f_n\}_{n \geqslant 0}$ be a Cauchy sequence in $\mathcal{D}(\mu_1, \mu_2)$. Since $H^2(\mathbb{D}^2)$ is complete (see [27, p. 53]), there exists a $f \in H^2(\mathbb{D}^2)$ such that $\|f_n - f\|^2_{H^2(\mathbb{D}^2)} \to 0$ as $n \to \infty$. Moreover, since $H^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space, for every $j = 1, 2$, $\partial_j f_n$ converges compactly to $\partial_j f$ on $\mathbb{D}^2$. Also, since $\{f_n\}_{n \geqslant 0}$ is bounded in $\mathcal{D}(\mu_1, \mu_2)$, by Lemma 1.1, there exists an $M > 0$ such that for every integer $n \geqslant 0$ and $r \in (0,1)$,

$$\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_1 f_n(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta < M,$$
$$\int_{\mathbb{T}} \int_{\mathbb{D}} |\partial_2 f_n(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta < M.$$

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By Fatou's lemma (see [28, Lemma 1.28]), for any $r \in (0,1)$,

$$\begin{aligned}(3.1) \qquad &\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\leqslant \liminf_n \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f_n(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \leqslant M.\end{aligned}$$

Similarly, one can see that

$$\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 f(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \leqslant M, \quad r \in (0,1).$$

This shows that $f \in \mathcal{D}(\mu_1, \mu_2)$. We may now argue as in (3.1) (with f replaced by $f_n - f$ and f_n replaced by $f_n - f_m$) and use Fatou's lemma to conclude that

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 (f_n - f)(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &\leqslant \liminf_m \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 (f_n - f_m)(z_1, re^{i\theta})|^2 P_{\mu_1}(z_1)\, dA(z_1) d\theta.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 (f_n - f)(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta \\ &\leqslant \liminf_m \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 (f_n - f_m)(re^{i\theta}, z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2) d\theta.\end{aligned}$$

These two estimates combined with Lemma 1.1 yield

$$D_{\mu_1,\mu_2}(f_n - f) \leqslant \liminf_m D_{\mu_1,\mu_2}(f_n - f_m), \quad n \geqslant 0.$$

This shows that $\{f_n\}_{n\geqslant 0}$ converges to f in $\mathcal{D}(\mu_1, \mu_2)$. Finally, since $H^2(\mathbb{D}^2)$ is a reproducing kernel Hilbert space, so is $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 1.3).

To see the "moreover" part, note that for any $f \in \mathcal{D}(\mu_1, \mu_2)$, by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$,

$$\langle f, 1\rangle_{\mathcal{D}(\mu_1,\mu_2)} = \langle f, 1\rangle_{H^2(\mathbb{D}^2)} = f(0) = \langle f, \kappa(\cdot, 0)\rangle_{\mathcal{D}(\mu_1,\mu_2)},$$

and hence $\kappa(\cdot, 0) = 1$. The rest follows from the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$ together with an application of the identity theorem. ∎

Although we do not need in this section, the full strength of the following lemma (cf. [18, Theorem 4.2]), we include it for later usage.

Lemma 3.2 *Let $f : \mathbb{D}^2 \to \mathbb{C}$ be a holomorphic function. For $r \in (0,1)$ and $\theta \in [0, 2\pi]$, consider the holomorphic function $f_{r,\theta}(w) = f(w, re^{i\theta}), w \in \mathbb{D}$. If $f \in H^2(\mathbb{D}^2)$, then $f_{r,\theta} \in H^2(\mathbb{D})$ for every $r \in (0,1)$ and $\theta \in [0, 2\pi]$. Moreover,*

$$(3.2) \qquad \sup_{0<r<1} \int_0^{2\pi} \|f_{r,\theta}\|^2_{H^2(\mathbb{D})} d\theta = \|f\|^2_{H^2(\mathbb{D}^2)}, \quad f \in H^2(\mathbb{D}^2).$$

Proof The proof relies on the formula (1.3). First, note that

$$(3.3) \qquad \sum_{m=0}^{\infty} |\hat{f}_{r,\theta}(m)|^2 = \sum_{m=0}^{\infty} \Big| \sum_{n=0}^{\infty} \hat{f}(m,n) r^n e^{in\theta} \Big|^2.$$

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If $f \in H^2(\mathbb{D}^2)$, then applying the Cauchy–Schwarz inequality to (3.3) gives that $f_{r,\theta} \in H^2(\mathbb{D})$ for every $r \in (0,1)$ and $\theta \in [0, 2\pi]$. Moreover, integrating both sides of (3.3) with respect to θ over $[0, 2\pi]$ and taking supremum over $r \in (0,1)$ yields (3.2). ∎

Remark 3.3 We note that

$$\text{if } f \in \mathcal{D}(\mu_1, \mu_2), \text{ then for every } r \in (0,1),\ f(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$$
$$\text{and } f(re^{i\theta}, \cdot) \in \mathcal{D}(\mu_2) \text{ for almost every } \theta \in [0, 2\pi]. \tag{3.4}$$

To see this, note that for any holomorphic function $f : \mathbb{D}^2 \to \mathbb{C}$,

$$D_{\mu_1,\mu_2}(f) = \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta, \tag{3.5}$$

and hence, if $f \in \mathcal{D}(\mu_1, \mu_2)$, then by Lemma 1.1, $\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta$ and $\int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta$ are finite for every $r \in (0,1)$. One may now apply Lemma 3.2 to complete the verification of (3.4).

It turns out that the operator $\mathscr{M}_{z_j}$ of multiplication by the coordinate functions z_j, $j = 1, 2$, defines a bounded linear operator on $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 3.4 *The coordinate functions z_1, z_2 are multipliers of $\mathcal{D}(\mu_1, \mu_2)$.*

Proof By (3.4), for any $f \in \mathcal{D}(\mu_1, \mu_2)$ and $r \in (0,1)$, $f(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$ for a.e. $\theta \in [0, 2\pi]$. By [26, Theorem 3.6], the operator $\mathscr{M}_w$ of multiplication by the coordinate function w on $\mathcal{D}(\mu_1)$ is bounded and satisfies

$$\|\mathscr{M}_w f(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \leqslant \|\mathscr{M}_w\| \|f(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \text{ for a.e. } \theta \in [0, 2\pi]. \tag{3.6}$$

Since $\mathscr{M}_w^* \mathscr{M}_w \geqslant I$, $\|\mathscr{M}_w\| \geqslant 1$. Fix now $f \in \mathcal{D}(\mu_1, \mu_2)$. By Lemma 1.3, $f \in H^2(\mathbb{D}^2)$, and hence $z_1 f \in H^2(\mathbb{D}^2)$. By (3.5) (two applications),

$$\begin{aligned}
D_{\mu_1,\mu_2}(z_1 f) &= \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_1}((z_1 f)(\cdot, re^{i\theta}))d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}((z_1 f)(re^{i\theta}, \cdot))d\theta \\
&\leqslant \lim_{r\to 1^-} \int_{\mathbb{T}} \|\mathscr{M}_w f(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta \\
&\overset{(3.6)}{\leqslant} \|\mathscr{M}_w\|^2 \lim_{r\to 1^-} \int_{\mathbb{T}} \|f(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} d\theta + \lim_{r\to 1^-} \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta \\
&\leqslant \|\mathscr{M}_w\|^2 \lim_{r\to 1^-} \int_{\mathbb{T}} \|f(\cdot, re^{i\theta})\|^2_{H^2(\mathbb{D})} d\theta + \|\mathscr{M}_w\|^2 D_{\mu_1,\mu_2}(f) \\
&\overset{(3.2)}{=} \|\mathscr{M}_w\|^2 \|f\|^2_{\mathcal{D}(\mu_1,\mu_2)}.
\end{aligned}$$

Similarly, one can see that for some $c_2 \geqslant 1$,

$$D_{\mu_1,\mu_2}(z_2 f) \leqslant c_2 \|f\|^2_{\mathcal{D}(\mu_1,\mu_2)}, \quad f \in \mathcal{D}(\mu_1, \mu_2).$$

This completes the proof. ∎

The following is a bidisc analog of Richter's formula (see [26, proof of Theorem 4.1], [8, Theorem 1.3]).

Lemma 3.5 *For nonnegative integers k, l and a polynomial p in the complex variables z_1 and z_2, we have the formula (2.1).*

Proof By (3.4) (see also (3.5)) and [26, proof of Theorem 4.1],

$$\begin{aligned}
&\|z_1^k z_2^l p\|^2_{\mathcal{D}(\mu_1,\mu_2)}\\
&= \|z_1^k z_2^l p\|^2_{H^2(\mathbb{D}^2)} + \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_1}(w^k p(w, re^{i\theta}))d\theta\\
&+ \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_2}(w^l p(re^{i\theta}, w))d\theta\\
&= \|p\|^2_{H^2(\mathbb{D}^2)} + \lim_{r\to 1^-}\int_{\mathbb{T}}\Big(D_{\mu_1}(p(w, re^{i\theta})) + k\int_{\mathbb{T}}|p(e^{i\eta}, re^{i\theta})|^2 d\mu_1(\eta)\Big)d\theta\\
&+ \lim_{r\to 1^-}\int_{\mathbb{T}}\Big(D_{\mu_2}(p(re^{i\theta}, w)) + l\int_{\mathbb{T}}|p(re^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta)\Big)d\theta\\
&= \|p\|^2_{\mathcal{D}(\mu_1,\mu_2)} + k\int_{\mathbb{T}^2}|p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta)d\theta + l\int_{\mathbb{T}^2}|p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta)d\theta,
\end{aligned}$$

where we used Lemma 1.1 and the monotone convergence theorem. ∎

For $R = (R_1, R_2) \in (0,1)^2$ and $f \in \mathcal{O}(\mathbb{D}^2)$, let $f_R(z) = f(R_1 z_1, R_2 z_2)$. To get the polynomial density in $\mathcal{D}(\mu_1, \mu_2)$, we need the following inequality.

Lemma 3.6 *For any $R = (R_1, R_2) \in (0,1)^2$ and $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$D_{\mu_1,\mu_2}(f_R) \leqslant D_{\mu_1,\mu_2}(f).$$

Proof By (3.5) and [29, Proposition 3],

$$\begin{aligned}
D_{\mu_1,\mu_2}(f_R) &= \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_1}(f_R(\cdot, re^{i\theta}))d\theta + \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_2}(f_R(re^{i\theta}, \cdot))d\theta\\
&\leqslant \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, rR_2 e^{i\theta}))d\theta + \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_2}(f(rR_1 e^{i\theta}, \cdot))d\theta.
\end{aligned}$$

This, combined with Lemma 1.1, yields

$$D_{\mu_1,\mu_2}(f_R) \leqslant \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta + \lim_{r\to 1^-}\int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta}, \cdot))d\theta.$$

An application of (3.5) now completes the proof. ∎

Here is a key step in deducing the density of polynomials in $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 3.7 *For any $f \in \mathcal{D}(\mu_1, \mu_2)$,*

$$\lim_{R_1, R_2\to 1^-} D_{\mu_1,\mu_2}(f - f_R) = 0.$$

Proof The proof is an adaptation of that of [14, Theorem 7.3.1] to the present situation. For $R = (R_1, R_2) \in (0,1)^2$, by the Parallelogram law (which holds for any seminorm) and Lemma 3.6,

$$\begin{aligned}
D_{\mu_1,\mu_2}(f - f_R) + D_{\mu_1,\mu_2}(f + f_R) &= 2(D_{\mu_1,\mu_2}(f) + D_{\mu_1,\mu_2}(f_R))\\
&\leqslant 4D_{\mu_1,\mu_2}(f).
\end{aligned} \tag{3.7}$$

We claim that

$$\liminf_{R_1, R_2\to 1^-} D_{\mu_1,\mu_2}(f + f_R) \geqslant 4D_{\mu_1,\mu_2}(f). \tag{3.8}$$

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To see this, fix $r \in (0,1)$. By Fatou's lemma,

$$\begin{aligned}
&\liminf_{R_1,R_2\to 1^-}\Big(\int_{\mathbb{T}} D_{\mu_1}((f+f_R)(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}((f+f_R)(re^{i\theta},\cdot))d\theta\Big)\\
&\geqslant 4\Big(\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta},\cdot))d\theta\Big). \qquad (3.9)
\end{aligned}$$

On the other hand, by Lemma 1.1,

$$\begin{aligned}
&D_{\mu_1,\mu_2}(f+f_R)\\
&\geqslant \int_{\mathbb{T}} D_{\mu_1}((f+f_R)(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}((f+f_R)(re^{i\theta},\cdot))d\theta.
\end{aligned}$$

After taking lim inf on both sides (one by one) and applying (3.9), we get

$$\begin{aligned}
&\liminf_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f+f_R)\\
&\geqslant 4\Big(\int_{\mathbb{T}} D_{\mu_1}(f(\cdot, re^{i\theta}))d\theta + \int_{\mathbb{T}} D_{\mu_2}(f(re^{i\theta},\cdot))d\theta\Big).
\end{aligned}$$

Letting $r \to 1^-$ on the right-hand side now yields (3.8) (see (3.5)). Finally, note that by (3.7),

$$\limsup_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f-f_R) \leqslant 4D_{\mu_1,\mu_2}(f) - \liminf_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f+f_R),$$

and hence by (3.8), we get

$$\limsup_{R_1,R_2\to 1^-} D_{\mu_1,\mu_2}(f-f_R) = 0,$$

which completes the proof. ∎

We now complete the proof of Theorem 2.1.

Proof (Proof of Theorem 2.1) Parts (i) and (iii) are Lemmas 3.4 and 3.5, respectively. To see (ii), let $f \in \mathcal{D}(\mu_1,\mu_2)$ and $\varepsilon > 0$. It suffices to check that there exists a polynomial p in z_1 and z_2 such that $\|f-p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon$. It is easy to see using Lemma 3.7 that there exists an $R = (R_1, R_2) \in (0,1)^2$ such that

$$\|f - f_R\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon/2. \qquad (3.10)$$

Since f_R is holomorphic in an open neighborhood of $\overline{\mathbb{D}}^2$, there exists a polynomial p such that

$$\|\partial^\alpha f_R - \partial^\alpha p\|_{\infty,\overline{\mathbb{D}}^2} < \frac{\sqrt{\varepsilon}}{4\sqrt{M}}, \quad \alpha \in \{(0,0),(1,0),(0,1)\},$$

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where $M = \max\left\{\int_{\mathbb{D}} P_{\mu_j}(w)dA(w) : j = 1, 2\right\} + 1$. This together with the fact that the norm on $H^2(\mathbb{D}^2)$ is dominated by the $\|\cdot\|_{\infty,\overline{\mathbb{D}}^2}$ shows that $\|f_R - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon/2$. Combining this with (3.10) yields $\|f - p\|_{\mathcal{D}(\mu_1,\mu_2)} < \varepsilon$, which completes the proof. ∎

The following provides a ground to discuss operator theory on $\mathcal{D}(\mu_1, \mu_2)$.

Corollary 3.8 *For $j = 1, 2$, let $\mathscr{M}_{z_j}$ denote the operator of multiplication by the coordinate function z_j. Then the commuting pair $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ on $\mathcal{D}(\mu_1, \mu_2)$ is a cyclic toral 2-isometry with cyclic vector* 1.

Proof Note that by Theorem 2.1(i) and the closed graph theorem, $\mathscr{M}_z$ defines a pair of bounded linear operators $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$. By Theorem 2.1(ii), $\mathscr{M}_z$ is cyclic with cyclic vector 1. Finally, the fact that $\mathscr{M}_z$ is a toral 2-isometry may be derived from (ii) and (iii) of Theorem 2.1. ∎

Let $\kappa : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ denote the reproducing kernel of $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 3.1).

Corollary 3.9 *For any $w \in \mathbb{D}^2$, $\ker(\mathscr{M}_z - w) = \{0\}$ and $\ker(\mathscr{M}_z^* - w)$ is the one-dimensional space spanned by $\kappa(\cdot, \overline{w})$.*

Proof Since $\mathcal{D}(\mu_1, \mu_2)$ is contained in the space of complex-valued holomorphic functions on $\mathbb{D}^2$, the pair $\mathscr{M}_z$ has no eigenvalue. By Theorem 2.1, $\mathscr{M}_z$ is cyclic, and hence, for any $w \in \mathbb{C}^2$, the dimension of $\ker(\mathscr{M}_z^* - w)$ is at most 1 (see (1.6)). If $w \in \mathbb{D}^2$, then by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 3.1), $\kappa(\cdot, \overline{w}) \in \ker(\mathscr{M}_z^* - w)$. Since $1 \in \mathcal{D}(\mu_1, \mu_2)$, once again by the reproducing property of $\mathcal{D}(\mu_1, \mu_2)$, $\kappa(\cdot, \overline{w}) \neq 0$. ∎

Before we state the next application of Theorem 2.1, we need a formula for the inner-product of monomials in $\mathcal{D}(\mu_1, \mu_2)$.

Lemma 3.10 *For $\mu \in M_+(\mathbb{T})$ and $j \geq 0$, let $\hat{\mu}(j) = \int_{\mathbb{T}} \zeta^{-j} d\mu(\zeta)$. Then*

$$\langle z_1^m z_2^n, z_1^p z_2^q \rangle_{\mathcal{D}(\mu_1,\mu_2)} = \begin{cases} 0, & \text{if } m \neq p,\ n \neq q, \\ \min\{n, q\}\hat{\mu}_2(q - n), & \text{if } m = p,\ n \neq q, \\ \min\{m, p\}\hat{\mu}_1(p - m), & \text{if } m \neq p,\ n = q, \\ 1 + m\hat{\mu}_1(0) + n\hat{\mu}_2(0), & \text{if } m = p,\ n = q. \end{cases} \tag{3.11}$$

In particular, the monomials are orthogonal in $\mathcal{D}(\mu_1, \mu_2)$ if and only if μ_1 and μ_2 are nonnegative multiples of the Lebesgue measure on $\mathbb{T}$.

Proof Fix nonnegative integers m, n, p, q. By the polarization identity,

$$\begin{aligned} &\langle z_1^m z_2^n, z_1^p z_2^q \rangle_{\mathcal{D}(\mu_1,\mu_2)} = \langle z_1^m z_2^n, z_1^p z_2^q \rangle_{H^2(\mathbb{D}^2)} \\ &+ \lim_{r\to1^-} \int_{\mathbb{T}} r^{n+q} e^{i(n-q)\theta} \int_{\mathbb{D}} \partial_1(z_1^m)\,\overline{\partial_1(z_1^p)} P_{\mu_1}(z_1)\, dA(z_1) d\theta \\ &+ \lim_{r\to1^-} \int_{\mathbb{T}} r^{m+p} e^{i(m-p)\theta} \int_{\mathbb{D}} \partial_2(z_2^n)\,\overline{\partial_2(z_2^q)} P_{\mu_2}(z_2)\, dA(z_2) d\theta. \end{aligned}$$

Since $\langle z_1^m z_2^n, z_1^p z_2^q \rangle_{H^2(\mathbb{D}^2)} = \delta(m, p)\delta(n, q)$ with $\delta(\cdot,\cdot)$ denoting the Kronecker delta of two variables, (3.11) may be deduced from the following formula for the inner-product of the Dirichlet-type space $\mathcal{D}(\mu)$ (see [22, Equation (3.2)]):

$$\langle z^r, z^s \rangle_{\mathcal{D}(\mu)} = \delta(r, s) + \min\{r, s\}\hat{\mu}(s - r), \quad r, s \in \mathbb{Z}_+ \tag{3.12}$$

(this formula may also be derived directly using [28, Theorem 11.9]). The "In particular" part follows from the Weierstrass approximation theorem and Riesz representation theorem. ∎

Remark 3.11 Assume that μ_1, μ_2 are nonzero. It is easy to see using (3.11) and (3.12) that $\|f\|_{\mathcal{D}(\mu_1,\mu_2)} = \|f\|_{\mathcal{D}(\mu_1)\otimes\mathcal{D}(\mu_2)}$ holds for all monomials f if and only if at least one of μ_1 and μ_2 is the zero measure. In particular, $\mathcal{D}(\mu_1, \mu_2) \neq \mathcal{D}(\mu_1) \otimes \mathcal{D}(\mu_2)$, in general.

The following is a consequence of (3.11) (see Definition 1.5).

Corollary 3.12 *For $\mu_1, \mu_2 \in M_+(\mathbb{T})$, the subspace of $\mathcal{D}(\mu_1, \mu_2)$ spanned by the constant function* 1 *is a wandering subspace for $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$.*

A bounded linear operator T on a Hilbert space is *essentially normal* if $T^*T - TT^*$ is a compact operator. An essentially normal operator is said to be *essentially unitary* if $T^*T - I$ is compact. Unlike the case of one variable Dirichlet-type spaces (see [7, Proposition 2.21]), $\mathcal{D}(\mu_1, \mu_2)$ does not support essentially normal multiplication 2-tuple $\mathscr{M}_z$.

Corollary 3.13 *The multiplication operators $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ on $\mathcal{D}(\mu_1, \mu_2)$ are never essentially normal.*

Proof By Corollary 3.8, the multiplication 2-tuple $\mathscr{M}_z$ is a toral 2-isometry. In particular, $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are 2-isometries. If these are essentially normal, then the image of $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ in the Calkin algebra is a normal 2-isometry, and hence $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are essentially unitary (since a normal 2-isometry, being invertible, is a unitary). It follows that $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are Fredholm. In view of Atkinson's theorem (see [9, Theorem XI.2.3]), it suffices to check that the kernels of $\mathscr{M}^*_{z_1}$ and $\mathscr{M}^*_{z_2}$ are of infinite dimension. To see this, fix a nonnegative integer j. Note that by (3.11),

$$\langle \mathscr{M}^*_{z_1} z_2^j, z_1^p z_2^q \rangle = \langle z_2^j, z_1^{p+1} z_2^q \rangle = 0, \quad p, q \in \mathbb{Z}_+,$$

and hence by the linearity of the inner-product and the density of the polynomials in $\mathcal{D}(\mu_1, \mu_2)$ (see Theorem 2.1(ii)), we obtain $\mathscr{M}^*_{z_1} z_2^j = 0$. Similarly, one can check that $z_1^j \in \ker \mathscr{M}^*_{z_2}$, completing the proof. ∎

4 Proof of Theorem 2.2 and a consequence

We begin the proof of Theorem 2.2 with the following special case.

Lemma 4.1 *The Hardy space $H^2(\mathbb{D}^2)$ has the division property.*

Proof For $j = 1, 2$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, let $f \in H^2(\mathbb{D}^2)$ be such that $\{z \in \mathbb{D}^2 : z_j = \lambda_j\} \subseteq Z(f)$. Let $w = (w_1, w_2) \in \mathbb{D}^2$. If $w_j \neq \lambda_j$, then clearly $g_j(z) = \frac{f(z)}{z_j - \lambda_j}$ defines a holomorphic function in a neighborhood of w. If $w_j = \lambda_j$, then since $\{z \in \mathbb{D}^2 : z_j = \lambda_j\} \subseteq Z(f)$, f as a function of z_j has a removable singularity at w_j, and hence by Hartogs' separate analyticity theorem (see [27, pp. 1–2]), g_j as above extends holomorphically in a neighborhood of w. This shows that g_j is holomorphic on $\mathbb{D}^2$.

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To see that $g_j \in H^2(\mathbb{D}^2)$, note that for $r \in (|\lambda_j|, 1)$ and $\theta_1, \theta_2 \in [0, 2\pi]$,

$$\frac{|f(re^{i\theta_1}, re^{i\theta_2})|}{|re^{i\theta_j} - \lambda_j|} \leqslant \frac{|f(re^{i\theta_1}, re^{i\theta_2})|}{r - |\lambda_j|}.$$

Since $f \in H^2(\mathbb{D}^2)$, it may now be deduced from (1.4) that $g_j \in H^2(\mathbb{D}^2)$. ■

Remark 4.2 One may argue as above to see that for any positive integer d, the Hardy space $H^2(\mathbb{D}^d)$ has the division property.

We also need the following fact essentially noticed in [26].

Lemma 4.3 *For any $\mu \in M_+(\mathbb{T})$, $\mathcal{D}(\mu)$ has the division property.*

Proof For $\lambda \in \mathbb{D}$, let $g \in \mathcal{D}(\mu)$ be such that $g(\lambda) = 0$. Note that g is orthogonal to $\kappa(\cdot, \lambda)$. Since $\ker(\mathscr{M}_z^* - \overline{\lambda})$ is spanned by $\kappa(\cdot, \lambda)$ and the range of $\mathscr{M}_z - \lambda$ is closed (see [26, Corollary 3.8]), there exists $f \in \mathcal{D}(\mu)$ such that $g = (z - \lambda)f$, which completes the proof. ■

Proof (Proof of Theorem 2.2) For $\lambda \in \mathbb{D}$, assume that $(z_j - \lambda)h \in \mathcal{D}(\mu_1, \mu_2)$ for some $j = 1, 2$. Thus

$$(z_j - \lambda)h \in H^2(\mathbb{D}^2), \tag{4.1}$$

$$D_{\mu_1,\mu_2}((z_j - \lambda)h) < \infty. \tag{4.2}$$

Since the arguments for the cases $j = 1, 2$ are similar, we only treat the case when $j = 1$. It follows from Lemma 4.1 and (4.1) that $h \in H^2(\mathbb{D}^2)$. Applying (3.4) to (4.2) gives

$$D_{\mu_1}((z_1 - \lambda)h(\cdot, re^{i\theta})) < \infty, \quad r \in (0,1), \ \theta \in \Omega_r, \tag{4.3}$$

where Ω_r is a Lebesgue measurable subset of $[0, 2\pi]$ such that $[0, 2\pi] \backslash \Omega_r$ is of measure 0. For $r \in (0, 1)$ and $\theta \in \Omega_r$, consider $f_{r,\theta} : \mathbb{D} \to \mathbb{C}$ defined by

$$f_{r,\theta}(w) = (w - \lambda)h(w, re^{i\theta}), \quad w \in \mathbb{D}.$$

By (4.1) and Lemma 3.2, $f_{r,\theta}$ belongs to $H^2(\mathbb{D})$. Hence, by (4.3), $f_{r,\theta}$ belongs to $\mathcal{D}(\mu_1)$. Hence, by Lemma 4.3, $h(\cdot, re^{i\theta}) \in \mathcal{D}(\mu_1)$. Since

$$\|h(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)} \leqslant \|wh(\cdot, re^{i\theta})\|_{\mathcal{D}(\mu_1)}$$

(see [26, Theorem 3.6]), by the reverse triangle inequality,

$$\|f_{r,\theta}\|^2_{\mathcal{D}(\mu_1)} \geqslant (1 - |\lambda|)^2 \|h(\cdot, re^{i\theta})\|^2_{\mathcal{D}(\mu_1)} \geqslant (1 - |\lambda|)^2 D_{\mu_1}(h(\cdot, re^{i\theta})).$$

Integrating both sides over $[0, 2\pi]$ yields

$$\begin{aligned}(1 - |\lambda|)^2 \int_0^{2\pi} D_{\mu_1}(h(\cdot, re^{i\theta}))d\theta &\leqslant \int_0^{2\pi} \|f_{r,\theta}\|^2_{H^2(\mathbb{D})} d\theta + \int_0^{2\pi} D_{\mu_1}(f_{r,\theta})d\theta \\ &\leqslant \|(z_1 - \lambda)h\|^2_{H^2(\mathbb{D}^2)} + D_{\mu_1,\mu_2}((z_1 - \lambda)h),\end{aligned}$$

where we used (3.2). Taking supremum over $r \in (0, 1)$ gives now

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_1}(h(\cdot, re^{i\theta}))d\theta < \infty.$$

Also, since $h \in H^2(\mathbb{D}^2)$, it now suffices to check that

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}(h(re^{i\theta}, \cdot))d\theta < \infty. \tag{4.4}$$

Note that by (4.2),

$$\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}\Big(((z_1 - \lambda)h)(re^{i\theta}, \cdot)\Big)d\theta < \infty.$$

However, for any $s \in (|\lambda|, 1)$,

$$\begin{aligned}
&\sup_{0<r<1} \int_0^{2\pi} D_{\mu_2}\Big(((z_1 - \lambda)h)(re^{i\theta}, \cdot)\Big)d\theta \\
\geqslant &\int_0^{2\pi} D_{\mu_2}\Big(((z_1 - \lambda)h)(se^{i\theta}, \cdot)\Big)d\theta \\
\geqslant &(s - |\lambda|)^2 \int_0^{2\pi} \int_{\mathbb{D}} |\partial_2 h(se^{i\theta}, w)|^2 P_{\mu_2}(w)dA(w)d\theta.
\end{aligned}$$

Applying Lemma 1.1 and letting $s \uparrow 1^-$ now yields (4.4). ∎

Before we present an application of Theorem 2.2, let us recall some facts from the multivariate spectral theory (see [11, 12, 30]). Let $T = (T_1, T_2)$ be a commuting pair on $\mathcal{H}$ and set $D_T(x) = (T_1x, T_2x), x \in \mathcal{H}$. Note that

$$\text{if } D_T^* D_T \text{ is Fredholm, then } D_T \text{ has closed range.} \tag{4.5}$$

Indeed, if $D_T^* D_T$ is Fredholm, then D_T is left-Fredholm, and hence we obtain (4.5). To define the Taylor spectrum, we consider the following complex:

$$K(T, \mathcal{H}) : \{0\} \xrightarrow{0} \mathcal{H} \xrightarrow{B_2} \mathcal{H} \oplus \mathcal{H} \xrightarrow{B_1} \mathcal{H} \xrightarrow{0} \{0\}, \tag{4.6}$$

where the *boundary maps* B_1 and B_2 are given by

$$B_2(h) := (T_2h, -T_1h), \quad B_1(h_1, h_2) := T_1h_1 + T_2h_2.$$

Note that $K(T, \mathcal{H})$ is a complex, that is, $B_1 \circ B_2 = 0$. Let $H^k(T)$ denote the kth cohomology group in $K(T, \mathcal{H}), k = 0, 1, 2$. Following [30] (resp. [11]), we say that T is *Taylor-invertible* (resp. *Fredholm*) if $H^k(T) = \{0\}$ (resp. $\dim H^k(T) < \infty$) for $k = 0, 1, 2$. The *Taylor spectrum* $\sigma(T)$ and the *essential spectrum* $\sigma_e(T)$ are given by

$$\begin{aligned}
\sigma(T) &= \{\lambda \in \mathbb{C}^2 : T - \lambda \text{ is not Taylor-invertible}\}, \\
\sigma_e(T) &= \{\lambda \in \mathbb{C}^2 : T - \lambda \text{ is not Fredholm}\}.
\end{aligned}$$

The *Fredholm index* $\operatorname{ind}(T)$ of a Fredholm commuting 2-tuple T on $\mathcal{H}$ is the *Euler characteristic* of the Koszul complex $K(T, \mathcal{H})$, that is,

$$\operatorname{ind}(T) := \dim H^0(T) - \dim H^1(T) + \dim H^2(T). \tag{4.7}$$

As an application of the division property, we now show that we always have exactness at the middle stage of the Koszul complex of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. First, a general fact.

Lemma 4.4 *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of complex-valued holomorphic functions on the unit bidisc $\mathbb{D}^2$. Assume that $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a commuting pair on $\mathscr{H}$. If $\mathscr{H}$ has the division property, then for every $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, the Koszul complex of $\mathscr{M}_z - \lambda = (\mathscr{M}_{z_1} - \lambda_1, \mathscr{M}_{z_2} - \lambda_2)$ is exact at the middle stage (see (4.6)).*

Proof Note that $\mathscr{H}$ has the division property if and only if for $j = 1, 2$, we have the following property:

$$\text{for any holomorphic function } h : \mathbb{D}^2 \to \mathbb{C} \text{ and } \lambda \in \mathbb{D}^2,$$
$$\text{if } (z_j - \lambda_j)h \in \mathscr{H}, \text{ then } h \in \mathscr{H}. \tag{4.8}$$

We first assume that (4.8) holds for $j = 2$. To see that the Koszul complex of $\mathscr{M}_z - \lambda$ is exact at the middle stage, let $g, h \in \mathscr{H}$ be such that

$$(z_2 - \lambda_2)g(z_1, z_2) = (z_1 - \lambda_1)h(z_1, z_2), \quad (z_1, z_2) \in \mathbb{D}^2. \tag{4.9}$$

Letting $z_2 = \lambda_2$, we obtain $(w - \lambda_1)h(w, \lambda_2) = 0$ for every $w \in \mathbb{D}$. It follows that $h(\cdot, \lambda_2) = 0$ on $\mathbb{D}$. Since $h : \mathbb{D}^2 \to \mathbb{C}$ is holomorphic, there exists a holomorphic function $k : \mathbb{D}^2 \to \mathbb{C}$ such that

$$h(z_1, z_2) = (z_2 - \lambda_2)k(z_1, z_2), \quad (z_1, z_2) \in \mathbb{D}^2 \tag{4.10}$$

(in case of $\lambda_2 = 0$, this can be seen using the power series for h; the general case can be dealt now by replacing $h(z_1, z_2)$ by $h(z_1, \varphi(z_2))$, where φ is the automorphism of $\mathbb{D}$ which takes λ_2 to 0). Since $h \in \mathscr{H}$, by (4.8), $k \in \mathscr{H}$. We now combine (4.9) with (4.10) to obtain

$$\begin{aligned}(z_2 - \lambda_2)g(z_1, z_2) &= (z_1 - \lambda_1)h(z_1, z_2) \\ &= (z_1 - \lambda_1)(z_2 - \lambda_2)k(z_1, z_2), \quad z \in \mathbb{D}^2.\end{aligned}$$

This gives $g(z_1, z_2) = (z_1 - \lambda_1)k(z_1, z_2), z \in \mathbb{D}^2$. This together with (4.10) shows that $\mathscr{M}_z - \lambda$ is exact at the middle stage. We may also obtain the same conclusion in case (4.8) holds for $j = 1$. Indeed, one may proceed as above with the only change that the roles of λ_1 and λ_2 are interchanged (e.g. (4.9) is evaluated at $z_1 = \lambda_1$). ∎

Remark 4.5 Let Ω be a bounded domain in $\mathbb{C}^2$. One may imitate the first part of the proof of Lemma 4.1 to show that there exists a holomorphic function $k : \Omega \to \mathbb{C}$ satisfying (4.10). This gives an analog of Lemma 4.4 for arbitrary bounded domains.

The following is a consequence of Theorem 2.2 and Lemma 4.4.

Corollary 4.6 *For every $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$, the Koszul complex of the 2-tuple $\mathscr{M}_z - \lambda = (\mathscr{M}_{z_1} - \lambda_1, \mathscr{M}_{z_2} - \lambda_2)$ on $\mathcal{D}(\mu_1, \mu_2)$ is exact at the middle stage (see (4.6)).*

5 Proof of Theorem 2.3 and its consequences

We begin with a lemma, which is a variant of [17, Lemma 4.14]. We include its proof for the sake of completeness.

Lemma 5.1 *For a domain Ω of $\mathbb{C}^2$, let $\mathscr{H}$ be the reproducing kernel Hilbert space of complex-valued holomorphic functions associated with the kernel $\kappa : \Omega \times \Omega \to \mathbb{C}$. Assume that the constant function 1 belongs to $\mathscr{H}$, the multiplication operators*

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$\mathscr{M}_{z_1}, \mathscr{M}_{z_2}$ are bounded on $\mathscr{H}$ and the commuting 2-tuple $\mathscr{M}_z$ is cyclic. For $w \in \Omega$, Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$ if and only if

$$D^*_{\mathscr{M}_z^* - \overline{w}} \text{ has closed range.} \tag{5.1}$$

In particular, Gleason's problem can be solved for $\mathscr{H}$ over $\Omega \backslash \sigma_e(\mathscr{M}_z)$.

Proof Let $w \in \Omega$, and let $f \in \mathscr{H}$. By the reproducing kernel property of $\mathscr{H}$,

$$f - f(w) \in \{c\kappa(\cdot, w) : c \in \mathbb{C}\}^{\perp}. \tag{5.2}$$

However, since $\mathscr{M}_z$ is cyclic, $\dim \ker(\mathscr{M}_z^* - \overline{w}) \leqslant 1$ for every $w \in \mathbb{C}^2$ (see (1.6)). As $1 \in \mathscr{H}$, we have $\kappa(\cdot, w) \neq 0$, and hence

$$\{c\kappa(\cdot, w) : c \in \mathbb{C}\} = \ker(\mathscr{M}_z^* - \overline{w}) = \ker D_{\mathscr{M}_z^* - \overline{w}}.$$

It now follows from (5.2) that

$$f - f(w) \in (\ker D_{\mathscr{M}_z^* - \overline{w}})^{\perp} = \overline{\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})}. \tag{5.3}$$

Also, it is easy to see that

$$\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}}) = \{(z_1 - w_1)g_1 + (z_2 - w_2)g_2 : g_1, g_2 \in \mathscr{H}\}. \tag{5.4}$$

If (5.1) holds, then it now follows from (5.3) that

$$f - f(w) \in \{(z_1 - w_1)g_1 + (z_2 - w_2)g_2 : g_1, g_2 \in \mathscr{H}\},$$

and hence Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$. Conversely, if Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$, then by (5.3), any function in $\overline{\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})}$ is of the form $f - f(w)$ for some $f \in \mathscr{H}$, and hence by (5.4), it belongs to $\operatorname{ran}(D^*_{\mathscr{M}_z^* - \overline{w}})$. This completes the proof of the equivalence.

To see the remaining part, let $w = (w_1, w_2) \in \Omega \backslash \sigma_e(\mathscr{M}_z)$. Since $D_S^* D_S = S_1^* S_1 + S_2^* S_2$ for any commuting pair $S = (S_1, S_2)$, by [11, Corollary 3.6], the operator $D^*_{\mathscr{M}_z^* - \overline{w}} D_{\mathscr{M}_z^* - \overline{w}}$ is Fredholm, and hence by (4.5), $D_{\mathscr{M}_z^* - \overline{w}}$ has closed range. Hence, by the closed-range theorem (see [9, Theorem VI.1.10]), we obtain (5.1) completing the proof. ■

Remark 5.2 Let $\mathscr{M}_z$ be the multiplication 2-tuple on the Hardy space $H^2(\mathbb{D}^2)$ of the unit bidisc $\mathbb{D}^2$. Since $\sigma_e(\mathscr{M}_z) \cap \mathbb{D}^2 = \varnothing$ (see [11, Theorem 5(c)]), by Lemma 5.1, Gleason's problem can be solved for $H^2(\mathbb{D}^2)$.

The following lemma provides a situation in which the division property ensures a solution to Gleason's problem.

Lemma 5.3 *Let $\mathscr{H}$ be a reproducing kernel Hilbert space of complex-valued holomorphic functions on the unit bidisc $\mathbb{D}^2$, and let $w = (w_1, w_2) \in \mathbb{D}^2$. Assume that $\mathscr{H}$ has the division property and $\mathscr{M}_z = (\mathscr{M}_{z_1}, \mathscr{M}_{z_2})$ is a commuting pair on $\mathscr{H}$. If, for every $f \in \mathscr{H}$, either $f(\cdot, w_2)$ or $f(w_1, \cdot)$ belongs to $\mathscr{H}$, then Gleason's problem can be solved for $\mathscr{H}$ over $\{w\}$.*

Proof For $f \in \mathscr{H}$, assume that $f(w_1, \cdot) \in \mathscr{H}$. Thus $f - f(w_1, \cdot) \in \mathscr{H}$. Hence, if $h : \mathbb{D}^2 \to \mathbb{C}$ is a holomorphic function such that

$$f(z_1, z_2) - f(w_1, z_2) = (z_1 - w_1)h(z_1, z_2), \quad z_1, z_2 \in \mathbb{D}, \tag{5.5}$$

by the division property for $\mathscr{H}$, we have $h \in \mathscr{H}$. Also, since $f(w_1,\cdot) \in \mathscr{H}$, one may argue as above to see that there exists $k \in \mathscr{H}$ satisfying

$$f(w_1,z_2) - f(w_1,w_2) = (z_2 - w_2)k(z_1,z_2), \quad z_1,z_2 \in \mathbb{D}.$$

This, combined with (5.5), completes the proof in this case. Similarly, one can deal with the case in which $f(\cdot,w_2) \in \mathscr{H}$. ∎

We also need the following fact of independent interest.

Lemma 5.4 *For every $f \in \mathcal{D}(\mu_1,\mu_2)$, the slice functions $f(\cdot,0)$ and $f(0,\cdot)$ belong to $\mathcal{D}(\mu_1,\mu_2)$. Moreover, the mappings $f \mapsto f(\cdot,0)$ and $f \mapsto f(0,\cdot)$ from $\mathcal{D}(\mu_1,\mu_2)$ into itself are contractive homomorphisms.*

Proof If $f \in \mathcal{D}(\mu_1,\mu_2)$, then

$$\begin{aligned}&\int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_1 f(z_1,0)|^2 P_{\mu_1}(z_1)\, dA(z_1)d\theta \\ &+ \int_{\mathbb{T}}\int_{\mathbb{D}} |\partial_2 f(0,z_2)|^2 P_{\mu_2}(z_2)\, dA(z_2)d\theta \leqslant D_{\mu_1,\mu_2}(f).\end{aligned}$$

Since the mappings $f \mapsto f(\cdot,0)$ and $f \mapsto f(0,\cdot)$ from $H^2(\mathbb{D}^2)$ into itself are contractive homomorphisms, the desired conclusions may be deduced from the estimate above. ∎

Proof (Proof of Theorem 2.3) Since $\mathcal{D}(\mu_1,\mu_2)$ has the division property (see Theorem 2.2), by Lemmas 5.3 and 5.4, Gleason's problem can be solved for $\mathcal{D}(\mu_1,\mu_2)$ over $\{(0,0)\}$. Hence, by Lemma 5.1 (which is applicable since $\mathscr{M}_z$ on $\mathcal{D}(\mu_1,\mu_2)$ is cyclic; see Theorem 2.1), $D^*_{\mathscr{M}^*_z}$ has closed range (see (5.1)). It is now easy to see using Corollaries 3.9 and 4.6 that $\mathscr{M}_z$ is Fredholm. Since the essential spectrum is a closed subset of $\mathbb{C}^2$ not containing $(0,0)$, there exists $r > 0$ such that $\mathbb{D}^2_r \subseteq \mathbb{C}^2\backslash\sigma_e(\mathscr{M}_z)$. Another application of Lemma 5.1 now completes the proof. ∎

The following fact is implicit in the proof of Theorem 2.3.

Corollary 5.5 *The commuting 2-tuple $\mathscr{M}^*_z$ on $\mathcal{D}(\mu_1,\mu_2)$ belongs to $\mathbf{B}_1(\mathbb{D}^2_r)$ for some $r \in (0,1]$.*

Proof This may be deduced from Theorem 2.3, Lemma 3.1, Corollary 3.9, and Lemma 5.1 (see (5.1)). ∎

We conclude this section with the following corollary describing the cokernels of the multiplication operators $\mathscr{M}_{z_j}$, $j = 1,2$, on $\mathcal{D}(\mu_1,\mu_2)$.

Corollary 5.6 *For $1 \leqslant i \neq j \leqslant 2$, $\ker \mathscr{M}^*_{z_j} = \bigvee\{z_i^k : k \geqslant 0\}$.*

Proof As observed in the proof of Corollary 3.13,

$$\{p(z_i) : p \in \mathbb{C}[w]\} \subseteq \ker \mathscr{M}^*_{z_j}, \quad 1 \leqslant i \neq j \leqslant 2. \tag{5.6}$$

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To see the reverse inclusion, let $f \in \ker \mathscr{M}_{z_1}^*$. By Theorem 2.1, there exists a sequence $\{p_n\}_{n\geqslant 1}$ of complex polynomials in z_1, z_2 converging to f. By Lemma 5.4, $f(0,\cdot) \in \mathcal{D}(\mu_1,\mu_2)$, and $\{p_n(0,\cdot)\}_{n\geqslant 1}$ converges to $f(0,\cdot)$. Hence, by (5.6), $f(0,\cdot) \in \ker \mathscr{M}_{z_1}^*$. Thus $f - f(0,\cdot) \in \ker \mathscr{M}_{z_1}^*$. However, there exists a holomorphic function $h : \mathbb{D}^2 \to \mathbb{C}$ such that

$$f(z_1,z_2) - f(0,z_2) = z_1 h(z_1,z_2), \quad z_1, z_2 \in \mathbb{D}. \tag{5.7}$$

By Theorem 2.2, we have $h \in \mathcal{D}(\mu_1,\mu_2)$. It now follows from (5.7) that $\mathscr{M}_{z_1} h \in \ker \mathscr{M}_{z_1}^*$. Since $\mathscr{M}_{z_1}^* \mathscr{M}_{z_1}$ is invertible, we must have $h = 0$, and hence, by (5.7), $f = f(0,\cdot)$, or equivalently, f belongs to the closure of $\{p(z_2) : p \in \mathbb{C}[w]\}$. Similarly, one can check that $\ker \mathscr{M}_{z_2}^*$ is equal to the closure of $\{p(z_1) : p \in \mathbb{C}[w]\}$. ■

6 Proof of Theorem 2.4 and its consequences

The proof of Theorem 2.4 relies on revealing the structure of toral 2-isometries T with $\ker T^*$ as a wandering subspace (see Definition 1.5).

Lemma 6.1 *Let $T = (T_1, T_2)$ be a toral 2-isometry. Then the following statements are true:*

(i) *for any integers $k, l \geqslant 0$,*

$$\begin{aligned} T_1^{*k} T_2^{*l} T_2^l T_1^k &= T_1^{*k} T_1^k + T_2^{*l} T_2^l - I \\ &= k T_1^* T_1 + l T_2^* T_2 - (k + l - 1) I, \end{aligned} \tag{6.1}$$

(ii) *for $f_0 \in \ker T^*$, assume that*

$$\langle T_1^m f_0,\ T_1^p T_2^q f_0 \rangle = 0, \quad q \geqslant 1, m, p \geqslant 0, \tag{6.2}$$

$$\langle T_2^n f_0,\ T_1^p T_2^q f_0 \rangle = 0, \quad p \geqslant 1, n, q \geqslant 0. \tag{6.3}$$

Then we have the following:

$$\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0 \rangle = \begin{cases} 0, & \text{if } m \neq p,\ n \neq q, \\ \langle T_2^n f_0,\ T_2^q f_0 \rangle, & \text{if } m = p,\ n \neq q, \\ \langle T_1^m f_0,\ T_1^p f_0 \rangle, & \text{if } m \neq p,\ n = q, \\ \|T_1^m f_0\|^2 + \|T_2^n f_0\|^2 - \|f_0\|^2, & \text{if } m = p,\ n = q. \end{cases}$$

Proof (i) To see (6.1), we proceed by strong induction on $k + l, k, l \geqslant 0$. Clearly, (6.1) holds for $0 \leqslant k + l \leqslant 1$. Assume that (6.1) holds for integers $k, l \geqslant 0$ such that $0 \leqslant k + l \leqslant n$. Note that for $k \geqslant 1$ and $l \leqslant n - 1$, by the induction hypothesis,

$$\begin{aligned} T_1^* (T_1^{*k} T_2^{*l} T_2^l T_1^k) T_1 &= T_1^{*k+1} T_1^{k+1} + T_1^* T_2^{*l} T_2^l T_1 - T_1^* T_1 \\ &= T_1^{*k+1} T_1^{k+1} + T_2^{*l} T_2^l - I. \end{aligned}$$

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Similarly, for $k \leqslant n-1$ and $l \geqslant 1$, (6.1) holds. This completes the induction argument. The remaining identity in (i) now follows from the known fact that for any 2-isometry S, we have

$$S^{*k}S^k = k(S^*S - I) + I, \quad k \geqslant 0 \tag{6.4}$$

(this known fact can be seen by induction on $k \geqslant 1$).

(ii) Let m, n, p, q be integers such that $m \neq p$ and $n \neq q$. Consider the case when $m < p$ and $n < q$. Since $f_0 \in \ker T^*$, we have

$$\begin{aligned}
&\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle \\
&= \langle T_2^{*n} T_1^{*m} T_1^m T_2^n f_0,\ T_1^{p-m} T_2^{q-n} f_0\rangle \\
&\overset{(6.1)}{=} \langle T_1^{*m} T_1^m f_0,\ T_1^{p-m} T_2^{q-n} f_0\rangle + \langle T_2^{*n} T_2^n f_0,\ T_1^{p-m} T_2^{q-n} f_0\rangle \\
&= \langle T_1^m f_0,\ T_1^p T_2^{q-n} f_0\rangle + \langle T_2^n f_0,\ T_2^q T_1^{p-m} f_0\rangle,
\end{aligned}$$

which, by (6.2) and (6.3), is equal to 0. Since the inner-product is conjugate linear, $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = 0$ when $p < m$ and $q < n$. Consider the case when $m < p$ and $q < n$. Arguing as above, we have

$$\begin{aligned}
&\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle \\
&= \langle T_2^{*q} T_1^{*m} T_1^m T_2^q T_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle \\
&\overset{(6.1)\&(6.3)}{=} \langle T_1^{*m} T_1^m T_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle + \langle T_2^{*q} T_2^q T_2^{n-q} f_0,\ T_1^{p-m} f_0\rangle \\
&= \langle T_1^m T_2^{n-q} f_0,\ T_1^p f_0\rangle + \langle T_2^n f_0,\ T_2^q T_1^{p-m} f_0\rangle,
\end{aligned}$$

which, by (6.2) and (6.3), is equal to 0. Once again, by the conjugate-symmetry, $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = 0$ when $p < m$ and $n < q$.

If $m = p$ and $n \neq q$, then one may argue as above using (6.4) (by making cases $n < q$ and $q < n$) to show that $\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = \langle T_2^n f_0,\ T_2^q f_0\rangle$. Similarly, one may derived the formula in case of $m \neq p$ and $n = q$. Finally, if $m = p$ and $n = q$, then the formula follows from (6.1). ∎

Proof (Proof of Theorem 2.4) (i)⇒(iii) Fix $j \in \{1, 2\}$. Since T is analytic, so is T_j. Thus T_j is an analytic 2-isometry. Consider the T_j-invariant subspace $\mathcal{H}_j := \bigvee\{T_j^k f_0 : k \geqslant 0\}$ and note that $T_j|_{\mathcal{H}_j}$ is a cyclic analytic 2-isometry. Hence, by [26, Theorem 5.1], there exist a finite positive Borel measure μ_j on $\mathbb{T}$ and a unitary map $V_j : \mathcal{H}_j \to \mathcal{D}(\mu_j)$ such that

$$V_j f_0 = 1, \quad V_j T_j = \mathscr{M}_w^{(j)} V_j, \tag{6.5}$$

where $\mathscr{M}_w^{(j)}$ denotes the operator of multiplication by the coordinate function w on $\mathcal{D}(\mu_j)$. We contend that the map given by

$$U(T_1^k T_2^l f_0) = z_1^k z_2^l, \quad k, l \geqslant 0$$

extends to an unitary from $\mathcal{H}$ onto $\mathcal{D}(\mu_1, \mu_2)$. Since $\mathcal{H} = \vee\{T_1^k T_2^l f_0 : k, l \geqslant 0\}$ and $\mathcal{D}(\mu_1, \mu_2) = \vee\{z_1^k z_2^l : k, l \geqslant 0\}$, it suffices to check that

$$\langle T_1^m T_2^n f_0,\ T_1^p T_2^q f_0\rangle = \langle z_1^m z_2^n,\ z_1^p z_2^q\rangle_{\mathcal{D}(\mu_1,\mu_2)}, \quad m, n, p, q \geqslant 0. \tag{6.6}$$

Note that for any integers $m, n \geqslant 0$, by (6.5),

$$\begin{aligned}\langle T_j^m f_0, T_j^n f_0\rangle &= \langle V_j T_j^m f_0, V_j T_j^n f_0\rangle_{\mathcal{D}(\mu_j)}\\ &= \langle (\mathscr{M}_w^{(j)})^m V_j f_0, (\mathscr{M}_w^{(j)})^n V_j f_0\rangle_{\mathcal{D}(\mu_j)}\\ &= \langle w^m, w^n\rangle_{\mathcal{D}(\mu_j)}\\ &= \langle z_j^m, z_j^n\rangle_{\mathcal{D}(\mu_1,\mu_2)}.\end{aligned}$$

Since (6.2) and (6.3) hold (as $\ker T^*$ is a wandering subspace for T), combining this with Lemma 6.1(ii) yields (6.6), which completes the proof.

(iii)⇒(ii) This follows from Corollaries 3.8, 3.12, and 5.5.

(ii)⇒(i) It suffices to check that T is analytic. By Oka–Grauert's theorem (see [21, p. 71, Corollary 2.17], [16, p. 3]), every holomorphic vector bundle on a bidisc is holomorphically trivial. Combining this with the proof of [16, Theorem 4.5] shows that if $T^* \in \mathbf{B}_1(\mathbb{D}_r^2)$, then T is unitarily equivalent to the multiplication 2-tuple $\mathscr{M}_z$ on a reproducing kernel Hilbert space of scalar-valued holomorphic functions on $\mathbb{D}_r^2$. Since $\mathscr{M}_z$ is analytic, T is analytic. ∎

The conclusion of Theorem 2.4 can be rephrased as follows.

Corollary 6.2 *A cyclic analytic toral 2-isometric 2-tuple on $\mathcal{H}$ is unitarily equivalent to the multiplication pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1,\mu_2)$ if and only if* $\ker T^*$ *is wandering subspace for T spanned by a cyclic vector for T.*

Remark 6.3 Let $\mathcal{D}$ denote the Dirichlet space (that is, the Dirichlet-type space associated with the Lebesgue measure on the unit circle), and let $\mathscr{M}_w$ be the operator of multiplication by w on $\mathcal{D}$. It is easy to see that the commuting pair $T = (\mathscr{M}_w, \mathscr{M}_w)$ is a cyclic analytic toral 2-isometry on $\mathcal{D}$. Note that $\ker T^* = \ker \mathscr{M}_w^*$ is spanned by 1 and it is not a wandering subspace for T. It is evident that T is not unitarily equivalent to the multiplication pair $\mathscr{M}_z$ on $\mathcal{D}(\mu_1,\mu_2)$ for any $\mu_1, \mu_2 \in M_+(\mathbb{T})$.

The following is a 2-variable analog of [26, Theorem 5.2].

Theorem 6.4 *For $j = 1,2$, let $\mu_1^{(j)}, \mu_2^{(j)} \in M_+(\mathbb{T})$. Then the multiplication 2-tuple $\mathscr{M}_z^{(1)}$ on $\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)})$ is unitarily equivalent to the multiplication 2-tuple $\mathscr{M}_z^{(2)}$ on $\mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ if and only if $\mu_j^{(1)} = \mu_j^{(2)}$, $j = 1,2$.*

Proof Suppose there is a unitary operator $U : \mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)}) \to \mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$ such that

$$\mathscr{M}_{z_j}^{(2)} U = U \mathscr{M}_{z_j}^{(1)}, \quad j = 1,2. \tag{6.7}$$

Since the joint kernel of the adjoint of multiplication tuples is spanned by 1, by (6.7), U must map 1 to some constant of modulus 1. After multiplying U by a unimodular constant, if required, we may assume that $U1 = 1$. It now follows from (6.7) that U is identity on polynomials. By Lemma 3.5 (applied twice), we obtain for any polynomial p in two variables,

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$$\int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1^{(1)}(\eta) d\theta = \int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1^{(2)}(\eta) d\theta,$$
$$\int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2^{(1)}(\eta) d\theta = \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2^{(2)}(\eta) d\theta.$$

It is easy to see that for any polynomial p in one variable,

$$\int_{\mathbb{T}} |p(e^{i\eta})|^2 d\mu_j^{(1)}(\eta) = \int_{\mathbb{T}} |p(e^{i\eta})|^2 d\mu_j^{(2)}(\eta), \quad j = 1, 2.$$

Combining polarization identity with the uniqueness of the trigonometric moment problem yields the desired uniqueness. ■

Remark 6.5 One may use Lemma 3.5 and argue as in [26, Theorem 6.2] to obtain the following fact: *For* $j = 1, 2$, let $\mu_1^{(j)}, \mu_2^{(j)} \in M_+(\mathbb{T})$. *Then*

$$\mathcal{D}(\mu_1^{(1)}, \mu_2^{(1)}) \subseteq \mathcal{D}(\mu_1^{(2)}, \mu_2^{(2)})$$

if and only if $\mu_j^{(2)} \ll \mu_j^{(1)}$ *and the Radon–Nikodým derivative* $d\mu_j^{(2)}/d\mu_j^{(1)} \in L^\infty(\mathbb{T})$, $j = 1, 2$. We leave the details to the reader.

We conclude this section with an application to toral isometries.

Corollary 6.6 *Let* $T = (T_1, T_2)$ *be a cyclic analytic toral isometry with cyclic vector* $f_0 \in \ker T^*$. *Then the following statements are equivalent:*

(i) $\ker T^*$ *is a wandering subspace for* T,
(ii) T *is unitarily equivalent to* $\mathscr{M}_z$ *on* $H^2(\mathbb{D}^2)$,
(iii) T *is doubly commuting, that is,* $T_j^* T_i = T_i T_j^*, 1 \leqslant i \neq j \leqslant 2$.

Proof (i)⇒(ii) By Theorem 2.4, there exist $\mu_1, \mu_2 \in M_+(\mathbb{T})$ such that T is unitarily equivalent to $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. Since T is a toral isometry, $\mathscr{M}_z$ is also a toral isometry. It now follows from (2.1) that for every polynomial p in two variables,

$$\int_{\mathbb{T}^2} |p(e^{i\eta}, e^{i\theta})|^2 d\mu_1(\eta) d\theta = 0, \qquad \int_{\mathbb{T}^2} |p(e^{i\theta}, e^{i\eta})|^2 d\mu_2(\eta) d\theta = 0.$$

One may now argue as in the proof of Theorem 6.4 to conclude that $\mu_1 = 0$ and $\mu_2 = 0$. This yields (ii).

The implication (ii)⇒(iii) is a routine verification, while the implication (iii)⇒(i) is recorded in Remark 1.6. ■

7 Concluding remarks

We conclude the paper with a brief discussion on the spectral picture of the multiplication 2-tuple $\mathscr{M}_z$ on $\mathcal{D}(\mu_1, \mu_2)$. We claim that

$$\sigma(\mathscr{M}_z) = \overline{\mathbb{D}}^2, \tag{7.1}$$

$$\sigma_e(\mathscr{M}_z) \subseteq \overline{\mathbb{D}}^2 \backslash \Omega \tag{7.2}$$

for some open set Ω in $\mathbb{C}^2$ containing $\big(\mathbb{D} \times \{0\}\big) \cup \big(\{0\} \times \mathbb{D}\big)$. To see (7.1), note that by [12, Theorem 4.9], for any commuting pair $T = (T_1, T_2)$, $\sigma(T) \subseteq \sigma(T_1) \times \sigma(T_2)$. Since the spectrum of any 2-isometry is contained in $\overline{\mathbb{D}}$ (see [2, Lemma 1.21]) and

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both $\mathscr{M}_{z_1}$ and $\mathscr{M}_{z_2}$ are 2-isometries (see Corollary 3.8), we obtain $\sigma(\mathscr{M}_z) \subseteq \overline{\mathbb{D}}^2$. Also, by Corollary 3.9, $\mathbb{D}^2 \subseteq \sigma_p(\mathscr{M}_z^*) \subseteq \sigma(\mathscr{M}_z^*)$. Since $\sigma(\mathscr{M}_z^*) = \{\overline{z} : z \in \sigma(\mathscr{M}_z)\}$, we have the inclusion $\mathbb{D}^2 \subseteq \sigma(\mathscr{M}_z)$. Finally, since the Taylor spectrum is closed (see [12, Corollary 4.2]), we obtain (7.1). On the other hand, an examination of the proof of Theorem 2.3 (using the full power of Lemma 5.3 together with Lemma 5.4) shows that

$$\big(\mathbb{D} \times \{0\}\big) \cup \big(\{0\} \times \mathbb{D}\big) \subseteq \mathbb{C}^2 \backslash \sigma_e(\mathscr{M}_z).$$

Since the essential spectrum is a closed subset of the Taylor spectrum, (7.2) now follows from (7.1). The natural question arises *whether the unit bidisc lies in the complement of the essential spectrum of* $\mathscr{M}_z$ (there are interesting examples of toral 2-isometries supporting this possibility; see [5, Proposition 5(iii)]). If this question has an affirmative answer, then $\sigma_e(\mathscr{M}_z) = \partial(\mathbb{D}^2)$. Indeed, if $\lambda \in \partial(\mathbb{D}^2) \backslash \sigma_e(\mathscr{M}_z)$, then there exist two sequences in $\mathbb{D}^2$ and $\mathbb{C}^2 \backslash \overline{\mathbb{D}}^2$ converging to λ, which together with the continuity of the Fredholm index (see (4.7)) leads to a contradiction. This in turn leads to an improvement of Corollary 5.5 providing a bidisc analog of [26, Corollary 3.8] and also solves Gleason's problem for $\mathcal{D}(\mu_1, \mu_2)$ (see Lemma 5.1).

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