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LIMITS OF PURE STATES, II

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We answer a question raised in an earlier paper concerning the pure state space of a separable C^*-algebra.

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Let A be a unital separable C^*-algebra with state space $S(A)$. Let $P(A)$ denote the set of pure states of A and let $F(A)$ denote the set of factorial states. For $\phi \in S(A)$ let π_ϕ be the associated Gelfand–Naimark–Segal representation of A. It was shown in Archbold [1] that the mapping θ defined by $\theta(\phi) = \ker \pi_\phi$ is a continuous, open surjection from $\overline{F(A)}$ (the weak*-closure of $F(A)$) onto Primal′(A) (the set of proper primal ideals of A). Furthermore, the restriction θ_0 of θ to the pure state space $\overline{P(A)}$ is also surjective (Archbold [2]).

Question 2 in Archbold [2] asks whether θ_0 is open. In certain special cases the answer is affirmative. For example, if A is antiliminal than $\overline{P(A)} = \overline{F(A)}$ (Batty and Archbold [4]) and so $\theta_0 = \theta$. Furthermore, we show that θ_0 is always "almost open" in the sense that the image of any non-empty open set has dense interior. However, θ_0 can fail to be open and we give an example in which A is liminal and the primitive ideal space Prim(A) is Hausdorff.

We begin by recalling from Archbold and Batty [3] that a (closed two-sided) ideal J of A is said to be primal if whenever $n \geq 2$ and $J_1, J_2, \ldots, J_n$ are ideals of A such that $J_1 J_2 \ldots J_n = \{0\}$ then $J_i \subseteq J$ for at least one value of i. In this paper we shall be concerned with the weak topology τ_w on Primal′(A) (see Archbold [1]). A base is given by the family of sets of the form

$$U(F) = \{I \in \text{Primal}'(A) : J \nsubseteq I \text{ for all } J \in F\}$$

where F is a finite set (possibly empty) of ideals of A. When restricted to Prim(A), τ_w coincides with the Jacobson topology. If Prim(A) is Hausdorff then Primal′(A) = Prim(A) (see Archbold and Batty [3, p. 63]).

Theorem. *Let A be a unital separable C^*-algebra and let $\theta_0 : \overline{P(A)} \to \text{Primal}'(A)$ be defined by*

$$\theta_0(\phi) = \ker \pi_\phi \qquad (\phi \in \overline{P(A)}).$$

Let U be any non-empty open subset of $\overline{P(A)}$. Then the interior of $\theta_0(U)$ is dense in $\theta_0(U)$.

Proof. Let $W=U\cap P(A)$, a non-empty open subset of $P(A)$. By Pedersen [6, 4.3.3], $\theta_0(W)$ is a non-empty open subset of $\text{Prim}(A)$. Hence there exists a non-zero ideal J of A such that

$$\theta_0(W)=\{P\in \text{Prim}(A):P\not\supseteq J\}.$$

Define

$$V=\{I\in \text{Primal}'(A):I\not\supseteq J\},$$

a τ_w-open subset of $\text{Primal}'(A)$. Since W is dense in U and θ_0 is continuous (see Archbold [1, Section 2]), we have

$$\bar{V}\supseteq\overline{\theta_0(W)}\supseteq\theta_0(\bar{W})\supseteq\theta_0(U)$$

(where the bars denote closures in the appropriate topologies). It remains only to show that $V\subseteq\theta_0(U)$.

Let $I\in V$. Since $I\not\supseteq J$ there exists a primitive ideal P_1 of A such that $P_1\supseteq I$ and $P_1\not\supseteq J$ (Dixmier [5, 2.9.7(ii)]). Hence $P_1\in\theta_0(W)$ and so there exists $\phi_1\in W$ such that $P_1=\ker\pi_{\phi_1}$. Since A is separable there is a countable family $P_1,P_2,\dots$ of distinct elements of $\text{Prim}\,A$ whose intersection is I (Pedersen [6, 4.3.4]). We shall assume that this family is infinite (in the finite case, a similar but easier argument applies).

For $i\geqq 2$ let ϕ_i be a pure state such that $P_i=\ker\pi_{\phi_i}$. For $n\geqq 1$ let

$$\psi_n=\frac{n-1}{n}\phi_1+\frac{1}{n}\sum_{i=2}^{\infty}2^{-i+1}\phi_i.$$

Since I is primal, it follows from Archbold and Batty [3, Proposition 3.1] that there is a net (π_α) of irreducible representations of A such that $\pi_\alpha\to\pi_{\phi_i}$ for each $i\geqq 1$. By Archbold [2, Theorem 2 ((ii)⇒(iii))], $\psi_n\in\overline{P(A)}$. Since $\|\psi_n-\phi_1\|\leqq 2/n$ and $\phi_1\in U$, there exists N such that $\psi_N\in U$. However,

$$\ker\pi_{\psi_N}=\bigcap_{i=1}^{\infty}\ker\pi_{\phi_i}=\bigcap_{i=1}^{\infty}P_i=I.$$

Hence $I\in\theta_0(U)$ as required. □

We now give an example in which the map θ_0 is not open. Let $M_n(\mathbf{C})$ denote the C^*-algebra of all $n\times n$ complex matrices. Let B be the C^*-subalgebra of $M_6(\mathbf{C})$ consisting of all matrices of the form $\begin{bmatrix} S & 0\\ 0 & T\end{bmatrix}$ where $S\in M_4(\mathbf{C})$ and $T\in M_2(\mathbf{C})$. We define A to be the C^*-algebra of all sequences $x=(x_n)_{n\geq 1}$, where $x_n=\begin{bmatrix} S_n(x) & 0\\ 0 & T_n(x)\end{bmatrix}\in B$, which are convergent to a matrix of the form

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$$\begin{bmatrix} T(x) & 0 & 0 \\ 0 & T(x) & 0 \\ 0 & 0 & T(x) \end{bmatrix}$$

where $T(x) \in M_2(\mathbf{C})$. The algebraic operations in A are defined pointwise and the norm is the supremum norm.

For each $n \geqq 1$ there are irreducible representations π_n and σ_n of A given by $\pi_n(x) = S_n(x)$ and $\sigma_n(x) = T_n(x)$ for $x \in A$. The only other irreducible representation is the representation σ_∞ given by $\sigma_\infty(x) = T(x)$ for $x \in A$. Hence Prim(A) consists of the ideals P_n, Q_n $(n \geqq 1)$ and Q_∞ where $P_n = \ker \pi_n$, $Q_n = \ker \sigma_n$ and

$$Q_\infty = \ker \sigma_\infty = \{x \in A : x_n \to 0 \text{ as } n \to \infty\}.$$

Each P_n and each Q_n is an isolated point in Prim(A). However $P_n \to Q_\infty$ and $Q_n \to Q_\infty$ as $n \to \infty$. In particular, Prim(A) is Hausdorff and so Primal$'(A) = $ Prim(A). We note also that A is unital, separable and liminal.

We define $\phi \in S(A)$ by

$$\phi(x) = \mathrm{tr}(\sigma_\infty(x)) = \mathrm{tr}(T(x)) \qquad (x \in A)$$

where tr is the (unique) tracial state of $M_2(\mathbf{C})$. We shall show that $\phi \in \overline{P(A)}$ and that there exists an open neighbourhood U of ϕ in $\overline{P(A)}$ such that $\theta_0(U)$ is not a neighbourhood of $\theta_0(\phi)$ $(= Q_\infty)$. To see that $\phi \in \overline{P(A)}$, let $\xi = 2^{-1/2}(1, 0, 0, 1) \in \mathbf{C}^4$ and define $\phi_n \in P(A)$ by

$$\phi_n(x) = \langle \pi_n(x)\xi, \xi \rangle \qquad (x \in A).$$

Then

$$\begin{aligned} \phi_n(x) &= \tfrac{1}{2}[(S_n(x))_{11} + (S_n(x))_{14} + (S_n(x))_{41} + (S_n(x))_{44}] \\ &\to \tfrac{1}{2}[T(x)_{11} + 0 + 0 + T(x)_{22}] \end{aligned}$$

as $n \to \infty$. Thus $\phi_n \to \phi$ (weak*) and so $\phi \in \overline{P(A)}$.

Now let U be the complement in $\overline{P(A)}$ of the weak*-closure of the set

$$W = \{\psi \in \overline{P(A)} : \ker \pi_\psi = Q_n \text{ for some } n\}.$$

It suffices to show that $\phi \in U$ for then $Q_\infty = \theta_0(\phi) \in \theta_0(U)$ but $\theta_0(U)$ is not a neighbourhood of Q_∞ since $Q_n \notin \theta_0(U)$ for each n.

We show first of all that $W \subseteq P(A)$. So let $\psi \in \overline{P(A)}$ with $\ker \pi_\psi = Q_n$ for some n. There exists a net (ψ_α) in $P(A)$ such that $\psi_\alpha \to \psi$. Since θ_0 is continuous, $\ker \pi_{\psi_\alpha} \to Q_n$ in Prim(A). Hence $\ker \pi_{\psi_\alpha}$ is eventually equal to Q_n. Regarding ψ as a state of A/Q_n $(\cong M_2(\mathbf{C}))$ in the usual way, we obtain that

$$\psi \in \overline{P(A/Q_n)} = P(A/Q_n),$$

as required.

Let us suppose that $\phi \notin U$. Then there exists a net (ϕ_α) in W such that $\phi_\alpha \to \phi$. Let $Q_{n_\alpha} = \ker \pi_{\phi_\alpha}$. Since ϕ_α is pure, there exists a unit vector $\xi_\alpha \in \mathbf{C}^2$ such that

$$\phi_\alpha(x) = \langle \sigma_{n_\alpha}(x)\xi_\alpha, \xi_\alpha \rangle \qquad (x \in A).$$

By passing to a subnet if necessary we may suppose that the net (ξ_α) is convergent to some unit vector $\xi \in \mathbf{C}^2$. Since $\phi_\alpha \to \phi$ and θ_0 is continuous, $Q_{n_\alpha} \to Q_\infty$ in Prim(A).

Let $x \in A$ and $\varepsilon > 0$. There exists $N \geqq 1$ such that $\|T_n(x) - T(x)\| < \varepsilon/2$ for all $n \geqq N$. Since $Q_{n_\alpha} \to Q_\infty$, there exists α_0 such that $n_\alpha \geqq N$ for all $\alpha \geqq \alpha_0$. By increasing α_0 if necessary, we may assume that

$$\|\xi_\alpha - \xi\| < \varepsilon(1 + 4\|T(x)\|)^{-1}$$

for all $\alpha \geqq \alpha_0$. Then for $\alpha \geqq \alpha_0$ we have

$$\begin{aligned} |\phi_\alpha(x) - \langle T(x)\xi, \xi \rangle)| &= |\langle \mathrm{T}_{n_\alpha}(x)\xi_\alpha, \xi_\alpha \rangle - \langle T(x)\xi, \xi \rangle| \\ &< |\langle T(x)\xi_\alpha, \xi_\alpha \rangle - \langle T(x)\xi, \xi \rangle| + \frac{\varepsilon}{2} \\ &\leqq \|T(x)(\xi_\alpha - \xi)\| \, \|\xi_\alpha\| + \|T(x)\xi\| \|\xi_\alpha - \xi\| + \frac{\varepsilon}{2} \\ &\leqq \varepsilon. \end{aligned}$$

Hence $\phi(x) = \lim \phi_\alpha(x) = \langle T(x)\xi, \xi \rangle$. This shows that ϕ is pure, contradicting the fact that ϕ is defined to be the average of two distinct pure states. This contradiction shows that $\phi \in U$, as required.

We note that in this example

$$\theta_0(U) = \{P_n : n \geqq 1\} \cup \{Q_\infty\}.$$

The interior of $\theta_0(U)$ is $\{P_n : n \geqq 1\}$. This is dense in $\theta_0(U)$ (since $P_n \to Q_\infty$ as $n \to \infty$) as predicted by the theorem.

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