Shintani lifts of nearly holomorphic modular forms

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Abstract. In this paper, we compute the Fourier expansion of the Shintani lift of nearly holomorphic modular forms. As an application, we deduce modularity properties of generating series of cycle integrals of nearly holomorphic modular forms.

1 Introduction

For a discriminant $d \in \mathbb{Z}$, let $\mathcal{Q}_d$ be the set of binary quadratic forms of discriminant d, which is acted on by the group $\Gamma := \mathrm{SL}_2(\mathbb{Z})$ with finitely many orbits. When $d < 0$, each $\lambda \in \mathcal{Q}_d$ gives rise to a CM points z_λ in the upper half-plane $\mathcal{H}$. The values of the j-function

$$j(z) := \frac{1}{q} + 744 + 196{,}884q + \cdots, \qquad q := \mathbf{e}(z) := e^{2\pi i z}$$

at such CM points are called *singular moduli*, and they are algebraic numbers generating certain abelian extensions, e.g., ring class fields, of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ by the theory of complex multiplication. The paper [Za] proved the surprising result that the dth trace of the normalized function $J(z) := j(z) - 744$ is the $|d|$th Fourier coefficient of a weakly holomorphic modular form g of weight $\frac{3}{2}$.

When $d > 0$, each $\lambda = [A, B, C] \in \mathcal{Q}_d$ gives rise to a geodesic

$$c_\lambda := \left\{ z \in \mathcal{H} : A|z|^2 + B\mathfrak{R}(z) + C = 0 \right\}$$

on $\mathcal{H}$. If d is not a perfect square, then the stabilizer Γ_λ of λ in Γ is infinite and $c(\lambda) := \Gamma_\lambda \backslash c_\lambda$ is a closed cycle on the modular curve $Y = \Gamma \backslash \mathcal{H}$. Instead of values, one can consider integrals of modular forms along these cycles, and study the properties (e.g., modularity) of their generating series. This idea lies in the basis of the construction of modular forms of half-integral weight in [Sn].

The non-holomorphic Eisenstein series of weight 2, defined as

$$E_2^*(z) := 1 - 24 \sum_{n \geq 1} \sigma_1(n)q^n - \frac{3}{\pi y}, \qquad y := \mathfrak{I}z,$$

offers an elegant example. For a fixed fundamental discriminant $\Delta < 0$, let χ_Δ be the genus character from, e.g., Section 1.2 of [GKZ] (with $N = 1$), which takes $\lambda \in \mathbb{Z}^3$ to

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(1.1) $$\chi_\Delta(\lambda) := \begin{cases} \left(\frac{\Delta}{n}\right), & \text{if } \lambda \in \mathcal{Q}_d \text{ with } \Delta \mid d, \ (n, \Delta) = 1, \text{ and } \lambda \text{ represents } n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any fundamental discriminant $D < 0$ co-prime to Δ, we have the formula

(1.2) $$\sum_{\lambda \in \Gamma \backslash \mathcal{Q}_{\Delta D}} \chi_\Delta(\lambda) \int_{c(\lambda)} E_2^*(z) dz = -12 H(-\Delta) H(-D),$$

where $H(n)$ is the Hurwitz class number considered in [HZ]. This twisted cycle integral (generalizing the classical integral, in which the character is trivial) is also the $|D|$th Fourier coefficient of $12H(-\Delta)$ times the weight $\frac{3}{2}$ mock modular form studied in loc. cit. In fact, this equality holds for any discriminant $D < 0$ after suitably regularizing the left-hand side (see Corollary 1.12 of [ANS][1]). The modular completion of this mock modular form is a harmonic Maass form in the sense of [BF1], whose image under the differential operator $\xi_{3/2}$ (see (2.1)), also known as the shadow of the mock modular form, is a multiple of the Jacobi theta series of weight $\frac{1}{2}$.

Note that for a fundamental discriminant $D > 0$, the twisted trace of singular moduli

(1.3) $$A(D, -\Delta) := \frac{1}{\sqrt{D}} \sum_{\lambda \in \Gamma \backslash \mathcal{Q}_{\Delta D}, \ \lambda \gg 0} \frac{\chi_\Delta(\lambda)}{|\Gamma_\lambda|} J(z_\lambda)$$

(again generalizing the usual trace, with no character) is the Dth Fourier coefficient of the weakly holomorphic modular form $f_{-\Delta} = q^\Delta + O(q)$ of weight $\frac{1}{2}$ from [Za]. This coefficient is the same with $J = j - 744$ replaced by j when $D\Delta$ is not a square.

While searching for analogues of the result from [Za] mentioned above, Duke, Imamoğlu, and Tóth studied the generating series of cycle integrals of the j-function in [DIT], and showed that it is a mock modular form of weight $\frac{1}{2}$ whose shadow is the weight $\frac{3}{2}$ form g from [Za]. Furthermore, it is the first member of a family of mock modular forms with weakly holomorphic shadows of weight $\frac{3}{2}$.

Using Serre duality, it is easy to see that there is a unique mock modular form $\tilde{f}_{-\Delta}$ of weight $\frac{3}{2}$ and level 4 in Kohnen's plus space with shadow $\frac{3}{2\pi} f_{-\Delta}$ and Fourier expansion

$$\tilde{f}_{-\Delta}(z) = 48|\Delta| H(-\Delta) + O(q^3).$$

From the result in [DIT], it is natural to ask about ways to construction $\tilde{f}_{-\Delta}$. This was first done by Jeon, Kang, and Kim in [JKK1] using Maass–Poincaré series. The sequel [JKK2] expressed its Fourier coefficients, using the same approach as in [DIT], as cycle integrals of sesqui-harmonic modular forms of weight zero.

In [BFI], Bruinier, Funke, and Imamoğlu obtained another proof of the main result of [DIT] by applying a theta lift, which also gave a geometric interpretation of the Fourier coefficients with square indices. This idea was used by Alfes-Neumann and Schwagenscheidt in [ANS] to construct $\tilde{f}_{-\Delta}$ as the holomorphic part of the Shintani theta lift of a harmonic Maass form $\tilde{J}$ of weight 2, which expresses the Fourier coefficients of $\tilde{f}_{-\Delta}$ as the twisted cycle integrals of $\tilde{J}$. Our first result is another

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[1]The different sign comes from the opposite orientation that they use—compare the formula on page 14 with our equation (2.17).

expression of the Fourier coefficients of $\tilde{f}_{-\Delta}$ in terms of cycle integrals of nearly holomorphic modular forms.

Theorem 1.1 *Let $\Delta < 0$ be a fixed fundamental discriminant. For any discriminant $D < 0$, the twisted regularized cycle integral*

$$\mathrm{Tr}_{\Delta,D}\left(J \cdot E_2^*\right) \coloneqq \sum_{\lambda \in \Gamma \backslash \mathcal{Q}_{\Delta D}} \chi_\Delta(\lambda) \int_{c(\lambda)}^{\mathrm{reg}} J(z) E_2^*(z) dz,$$

with the regularization defined as in equation (4.6), *is the $|D|$th Fourier coefficient of $\tilde{f}_{-\Delta}$.*

To prove Theorem 1.1, we will follow the theta lift approach as in [ANS, BF1, BFI], and use the theta kernel with the same archimedean Schwartz function as in [Sh]. (this is also the case $n = 1$ of the theta function from [Ze2]). We shall apply it to nearly holomorphic modular forms, and compute the resulting Fourier expansions. Recall that a real-analytic modular form f on $\mathcal{H}$ with at most linear exponential growth near the cusps is called *nearly holomorphic* if it can be presented as

$$(1.4) \qquad f(z) = \sum_{l=0}^{p} \frac{f_l(z)}{y^l} \qquad \text{with} \qquad f_l : \mathcal{H} \to \mathbb{C} \text{ holomorphic for } 0 \le l \le p$$

for some $p \in \mathbb{N}$, which is called the *depth* of f if f_p is not identically zero. In other words, it is annihilated by the operator L_z^{p+1}, where L_z is the lowering operator defined in (2.1). We denote the space of such modular forms of weight κ with respect to Γ by $\widetilde{M}_\kappa^!$, and use the superscript $\le p$ to mean the subspace of forms with depth at most p. Since these differential operators commute with the slash operators, the condition of being nearly holomorphic is purely archimedean, and can be defined for any weight, Fuchsian group, character, representation, or multiplier system.

Nearly holomorphic modular forms of depth 0 are just weakly holomorphic, and the Fourier expansions of their Shintani lifts have been computed in [ANS, BFI, BGK, Sh]. In particular, Shintani lifts of weakly holomorphic forms are holomorphic. Moreover, the main result of [ANS] shows that the Shintani lift of harmonic weak Maass forms without a special constant term is harmonic, and with this constant term, the Laplacian operator takes the lift to a unary theta function. This "sesqui-harmonicity" is also visible in the zeroth member Z_+ of the family of modular forms from [DIT]. We will show in Corollary 4.4 that when $0 \le p < k$, the Shintani lift of a nearly holomorphic modular form is also nearly holomorphic, and establish results analogous to the harmonic case (with or without constant terms) (see Theorem 1.2 in the Introduction, as well as Theorem 4.3, Proposition 4.5, and Remark 4.6 for the general statement). One could perhaps try to give another proof of the nearly holomorphic lift result by using the isomorphism

$$(1.5) \qquad \bigoplus_{0 \le j \le p} M_{2k-2j}^! \to \widetilde{M}_{2k}^{!,\le p}, \quad (f_j)_{j=0}^p \mapsto \sum_{j=0}^p R_{2k-2j}^j f_j,$$

described in, e.g., [MR, Ze3, Ze7], and analyzing the effects of raising operators on theta kernels. Here, R_{2k-2j}^j is the iterated raising operator defined in (2.2).

When $p \ge k$, the map from equation (1.5) is not surjective, and misses some nearly holomorphic modular forms from the right-hand side. A particular example is the

form $J \cdot E_2^*$ in Theorem 1.1, or just E_2^* itself. The Fourier expansions of their Shintani lifts do not follow from applying differential operators to known results, and are the main concern of this paper. In Theorem 4.3, we give the complete Fourier expansion of their Shintani lift. For the rest of the introduction, though, we will consider a special case of this result in level 1, which we now present.

Given $\lambda = [A, B, C] \in \mathcal{Q}_d$, denote $\lambda(z) := Az^2 + Bz + C$. Suppose that $f \in \widetilde{M}_{2k}^{!, \leq p}$ expands as

$$(1.6) \qquad f(z) = \sum_{l=0}^{p} \sum_{n \in \mathbb{Z}} c(n, l) q^n y^{-l}.$$

Given $d \in \mathbb{Z}$ that is not a square, we define, for k even, the trace

$$(1.7) \qquad \operatorname{Tr}_d(f) := \sum_{\lambda \in \Gamma \backslash \mathcal{Q}_d} \begin{cases} \frac{2}{|\Gamma_\lambda|} f(z_\lambda), & k = 0, \ d < 0, \\ \int_{c(\lambda)} f(z) \lambda(z)^{k-1} dz, & d > 0, \ \sqrt{d} \notin \mathbb{Z} \end{cases}$$

(we shall not use the negative d case when $k > 0$). If $d = r^2 > 0$ with $r \in \mathbb{N}$, then we have $\Gamma \backslash \mathcal{Q}_d = \{\pm[0, r, j] | 0 \leq j < r - 1\}$ and we define the trace as

$$\operatorname{Tr}_d(f) := 2 \lim_{T \to \infty} \sum_{j=0}^{r-1} \int_{\frac{(j,r)^2}{r^2} T^{-1}}^{T} f\left(-\frac{j}{r} + iy\right)(riy)^{k-1} i \, dy + (2ir)^k$$
$$\times \sum_{\substack{0 \leq l \leq p \\ n \leq 0, \ r|n}} c(n, l) \phi_n(k - l, T; 2\pi),$$

where the function ϕ_n is defined in equation (4.4). Finally, for $d = 0$ and even $k > 0$, we set

$$(1.8) \qquad \operatorname{Tr}_0(f) := c(0, 0)\zeta(1 - k) = -c(0, 0)\frac{B_k}{k},$$

where $\zeta(s)$ is the Riemann zeta function and B_k is the kth Bernoulli number.

We can now state the Fourier expansion of the Shintani lift of $f \in \widetilde{M}_{2k}^{!, \leq p}$ for even $k > 0$. For odd $k \in \mathbb{N}$, one can obtain a similar result with twisted cycle integrals as in equation (1.2).

Theorem 1.2 *Let $f \in \widetilde{M}_{2k}^{!, \leq p}$ have the expansion from equation (1.6), and suppose that $0 < k \in \mathbb{N}$ is even and $c(0, k) = 0$. The following expansion defines a real-analytic modular form of weight $k + \frac{1}{2}$ and level 4 in Kohnen's plus space:*

$$\sum_{\substack{d \in \mathbb{N} \\ d \equiv 0, 1 (\mathrm{mod}\ 4)}} \sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\operatorname{Tr}_d(L_z^{2b} f)}{(16\pi y)^b b!} q^d$$

$$+ \sqrt{2\pi} \sum_{\substack{0 > d \in \mathbb{Z} \\ d \equiv 0, 1 (\mathrm{mod}\ 4)}} \sum_{l=k}^{p} \frac{h_l(2\sqrt{2\pi|d|y})}{(2\sqrt{2\pi|d|y})^l} \cdot \frac{\operatorname{Tr}_d(R_{2k-2l}^{l-k} L_z^l f)}{2^{l-k}(l-k)!} \cdot \frac{q^d}{|d|^{\frac{1-k}{2}}}$$

$$+ \sum_{\substack{l=k-1 \\ l \text{ odd}}}^{p} \frac{\left(\frac{l-1}{2}\right)!(-2)^{\frac{l-1}{2}} B_{l+1-k}(-1)^{\frac{k}{2}} c(0,l)}{(8\pi y)^{l/2}(2\pi)^{k-l-\frac{1}{2}}(l+1-k)!}$$

$$- \sqrt{8\pi} \sum_{\substack{0 < r \in \mathbb{N} \\ k \le l \le p \\ n < 0, \ r|n}} \frac{J_l(2r\sqrt{2\pi y})}{(2r\sqrt{2\pi y})^l} \cdot \frac{(2\pi n)^{l-k}}{(-1)^{\frac{k}{2}}} \cdot \frac{l!c(n,l)}{(l-k)!} r^k q^{r^2},$$

where the special functions h_l and J_l are defined in equations (3.17) and (3.31), respectively. When $p < k$, it is nearly holomorphic, of depth $\lfloor \frac{p}{2} \rfloor$. Otherwise, its image under the differential operator $\xi_{k+1/2-2\lfloor p/2 \rfloor} L^{\lfloor p/2 \rfloor}$ is nearly holomorphic of weight $2\lfloor \frac{p}{2} \rfloor - k + \frac{3}{2}$ and depth $2\lfloor \frac{p}{2} \rfloor - k + 1$.

Remark 1.3 Theorem 1.2 holds also for $k = 0$, once one adds to the expansion $2\sqrt{y}$ times the constant

$$\int_Y f(z)d\mu(z) := \lim_{T \to \infty} \int_{Y_T} f(z)d\mu(z).$$

Remark 1.4 In the setting of Theorem 1.2, the generating series $\sum_d \mathrm{Tr}_d(f)q^d$ defines, when $p < k$, a quasi-modular form of weight $k + \frac{1}{2}$ and depth $\lfloor \frac{p}{2} \rfloor$. Note that the constant term $c(0, k - 1)$ appearing in the third sum in Theorem 1.2 vanishes when $p = k - 1$, since it is a multiple of the constant term of the weight 2 weakly holomorphic form $L^{k-1}f$ at the unique cusp of the modular curve of level 1. For $p \ge k$, this series can be completed to a such a modular form using these special functions (see Remark 4.7).

The key ingredient to the calculation of the Fourier expansion in [ANS, BF2, BFI, BFIL] is the construction of a rapidly decaying antiderivatives of the Schwartz function used to construct the theta kernel. Such singular Schwartz functions are important also in evaluating singular theta lifts and constructing Green currents for special divisors on orthogonal and unitary Shimura varieties (see, e.g., [FH]).

In our case, we need not only the first antiderivative, but also the higher-order antiderivatives. For the first antiderivative, we can build it from the error function (see equation (3.4)). Surprisingly, the higher-order derivatives h_v, defined in equation (3.17), turn out to be combinations of the Gaussian and the error function with polynomial coefficients P_v and Q_v. These polynomials, which are defined in equation (3.6), are closely related to the Hermite polynomials, and are of independent interest.

The paper is organized as follows. After recalling some basic notions in Section 2, we devote Section 3 to study the properties of the polynomials P_v and Q_v and of related special functions, including their Fourier transforms, asymptotic behaviors, and certain lattice sum evaluations. Then, in Section 4, we complete the computations of the orbital integrals and the proof of the main theorem (Theorem 4.3), as well as its implications for Theorems 1.1 and 1.2.

2 Isotropic lattices and modular forms

This section introduces the notions and notation that are required for the rest of the paper. We follow the setup of [ANS, BF1, BFI, BFIL] and others.

2.1 Differential operators on modular forms

For $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{R})$ and an element z of $\mathcal{H} := \{z = x + iy \in \mathbb{C} | y > 0\}$, denote $j(M, z) := cz + d$. Let $\mathrm{Mp}_2(\mathbb{R})$ denote the metaplectic double cover of $\mathrm{SL}_2(\mathbb{R})$, and let $\mathrm{Mp}_2(\mathbb{Z})$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{Mp}_2(\mathbb{R})$. We write elements of $\mathrm{Mp}_2(\mathbb{R})$ as pairs (M, ϕ), with $M \in \mathrm{SL}_2(\mathbb{R})$ and ϕ a holomorphic function on $\mathcal{H}$ such that $\phi(z)^2 = j(M, z)$.

Given a representation ρ of a finite index subgroup $\Gamma \subseteq \mathrm{Mp}_2(\mathbb{Z})$ on a finite-dimensional complex vector space V, a function $f : \mathcal{H} \to V$ is called *modular of weight* $\kappa \in \frac{1}{2}\mathbb{Z}$ *and representation* ρ if the functional equation

$$f \mid_\kappa (M, \phi)(z) := \phi(z)^{-2\kappa} f(Mz) = \rho(M, \phi) f(z)$$

holds for every element $(M, \phi) \in \Gamma$. Let $\mathcal{A}^!_\kappa(\Gamma, \rho)$ (resp. $\mathcal{A}_\kappa(\Gamma, \rho)$) denote the space of such functions that are real-analytic with at most exponential (resp. polynomial) growth near the cusps. It contains the subspaces $\widetilde{M}^!_\kappa(\Gamma, \rho)$, $M^!_\kappa(\Gamma, \rho)$, $M_\kappa(\Gamma, \rho)$, and $S_\kappa(\Gamma, \rho)$ of nearly holomorphic, weakly holomorphic, holomorphic, and cusp forms, respectively. We shall omit ρ from the notation when it is trivial.

For a half-integer κ, we define

(2.1)
$$R_{z,\kappa} = R_\kappa := 2i\partial_z + \frac{\kappa}{y}, \qquad L = L_z := -2iy^2\partial_{\overline{z}}, \qquad \xi_\kappa = 2iy^\kappa \overline{\partial_{\overline{z}}} = y^{\kappa-2}\overline{L_z}, \qquad \text{and}$$

$$\Delta_\kappa := -R_{\kappa-2}L_z = -\xi_{2-\kappa}\xi_\kappa = -4y^2\partial_z + \partial_{\overline{z}} + 2i\kappa y\partial_{\overline{z}} = -y^2(\partial_x^2 + \partial_y^2) - \kappa y(\partial_y - i\partial_x),$$

which are the raising operator of weight κ, the weight lowering operator, the ξ-operator of weight κ from [BF1], and the Laplacian operator of weight κ, respectively. For $n \in \mathbb{N}$, we write

(2.2)
$$R^n_\kappa := R_{\kappa+2n-2} \circ R_{\kappa+2n-4} \circ \cdots \circ R_{\kappa+2} \circ R_\kappa$$

for the iterated raising operator.

These differential operators preserve modularity, in the sense that

$$R_\kappa \mathcal{A}^!_\kappa(\Gamma, \rho) \subseteq \mathcal{A}^!_{\kappa+2}(\Gamma, \rho), \; L_z \mathcal{A}^!_\kappa(\Gamma, \rho) \subseteq \mathcal{A}^!_{\kappa-2}(\Gamma, \rho), \text{ and } \Delta_\kappa \mathcal{A}^!_\kappa(\Gamma, \rho) \subseteq \mathcal{A}^!_\kappa(\Gamma, \rho),$$

whereas for ξ_κ, which involves complex conjugation, we have

(2.3) $\quad \xi_\kappa \mathcal{A}^!_\kappa(\Gamma, \rho) \subseteq \mathcal{A}^!_{2-\kappa}(\Gamma, \overline{\rho})$, where $\overline{\rho}$ is the complex conjugate representation.

It is known that L_z, R_κ, and Δ_κ preserve near holomorphicity, with L_z decreasing the depth by 1, and R_κ and Δ_κ increasing it by at most 1. For more on these modular forms, including their relations with quasi-modular forms and Shimura's vector-valued modular forms, see [MR, Ze3, Ze7].

Around a given point $w = s + it \in \mathcal{H}$, the natural local coordinate is

(2.4) $\qquad \zeta = A_w(z) := \frac{z-w}{z-\overline{w}} \in \{\zeta \in \mathbb{C} | |\zeta| < 1\}$, $\qquad$ with $\qquad 1 - A_w(z) = \frac{2it}{z-\overline{w}}$

for any $z \in \mathcal{H}$, which also satisfies

$$
\left| A_{\gamma w}(\gamma z) \right| = \left| \frac{\overline{j(\gamma, w)} A_w(z)}{j(\gamma, w)} \right| = \left| A_w(z) \right|
$$

for every z and w in $\mathcal{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$. The expansion of a holomorphic modular form f of weight $\kappa \in \mathbb{Z}$ is given by Proposition 17 of [BGHZ][2] as

(2.5)

$$
f(z) = \left(\frac{2it}{z - \overline{w}} \right)^{\kappa} \sum_{n=0}^{\infty} R_{\kappa}^n f(w) \frac{t^n A_w(z)^n}{n!} = \left(1 - A_w(z) \right)^{\kappa} \sum_{n=0}^{\infty} R_{\kappa}^n f(w) \frac{t^n A_w(z)^n}{n!}.
$$

Note that the proof of equation (2.5) makes no use of the modularity of f, so that this expansion is valid for every holomorphic function f. We shall need a formula extending equation (2.5) to nearly holomorphic modular forms.

Lemma 2.1 *For $f \in \widetilde{M}_{\kappa}^{!}(\Gamma, \rho)$ and a point $w = s + it \in \mathcal{H}$, we have the expansion*

$$
f(z) = \left(1 - A_w(z) \right)^{\kappa} \sum_{l=0}^{p} \frac{\left(1 - \overline{A_w(z)} \right)^l}{t^l \left(1 - \left| A_w(z) \right|^2 \right)^l} \sum_{n=0}^{\infty} R_{\kappa-l}^n f_l(w) \frac{t^n A_w(z)^n}{n!}.
$$

Proof We write $f(z)$ as in equation (1.4), and express each f_l via equation (2.5), but with κ replaced by $\kappa - l$. Recalling from Lemma 5.1 of [Ze4] that y equals $\frac{t(1-|A_w(z)|^2)}{|1-A_w(z)|^2}$, we get

$$
f(z) = \sum_{l=0}^{p} \frac{\left(1 - A_w(z) \right)^{\kappa-l} \sum_{n=0}^{\infty} R_{\kappa-l}^n f_l(w) \frac{t^n A_w(z)^n}{n!}}{t^l \left(1 - \left| A_w(z) \right|^2 \right)^l} \left| 1 - A_w(z) \right|^{2l}.
$$

Expanding $\left| 1 - A_w(z) \right|^2$ yields the desired result. This proves the lemma. ∎

For any $\varepsilon > 0$, we will denote the pre-image of the ball of radius ε in $\mathbb{C}$ under A_w by $B_{\varepsilon}(w)$ with the natural orientation on its boundary. We shall later need the limit value of the following integral, which is determined as follows.

Corollary 2.2 *Let f and w be as in Lemma 2.1, and take an integer μ. Then*

$$
\lim_{\varepsilon \to 0} \int_{\partial B_{\varepsilon}(w)} \frac{f(z)}{\left(1 - A_w(z) \right)^{\kappa-2}} A_w(z)^{\mu} dz = \begin{cases} -\frac{4\pi t^{|\mu|}}{(|\mu|-1)!} R_{\kappa}^{|\mu|-1} f(w), & \mu < 0, \\ 0, & \mu \geq 0. \end{cases}
$$

Proof The result follows from substituting in $\zeta = A_w(z)$, and thus $dz = \frac{2it}{(1-\zeta)^2} d\zeta$, inside Lemma 2.1. This proves the corollary. ∎

We will carry out some integrations of modular forms on $\mathcal{H}$, with respect to the invariant measure $d\mu(z) := \frac{dz \wedge d\overline{z}}{-2iy^2} = \frac{dx\,dy}{y^2}$. The following standard consequence of Stokes' theorem will be useful for evaluating some of these integrals (see, e.g., Proposition 4.1.1 of [L]).

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[2]Note that there is a small typo there, where the expansion in $4\pi yw$ should be in its additive inverse $-4\pi yw$.

Lemma 2.3 *Let $\mathcal{R}$ be a connected domain in $\mathcal{H}$ whose boundary $\partial\mathcal{R}$ is a piecewise smooth path in $\mathcal{H}$ (positively oriented), and assume that f, g, and G are real-analytic functions on $\mathcal{R}$ such that $g = -L_z G$. Then we have the equality*

$$\int_{\mathcal{R}} f(z)g(z)d\mu(z) = \oint_{\partial\mathcal{R}} f(z)G(z)dz + \int_{\mathcal{R}} L_z f(z)G(z)d\mu(z).$$

2.2 Lattices producing modular curves

Let $V := M_2(\mathbb{Q})^0$ be the signature $(2,1)$ quadratic space of trace zero matrices over $\mathbb{Q}$ with quadratic form $Q(\lambda) := -N\det\lambda$ for some $0 < N \in \mathbb{Q}$. Then $G := \mathrm{Spin}(V) \cong \mathrm{SL}_2$, and the symmetric space of G is the space of oriented negative definite lines in $V_{\mathbb{R}} := M_2(\mathbb{R})^0$. We can identify $\mathcal{H}$ with the connected component of this symmetric space that contains the line spanned by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ as a positive generator, via the map taking $z \in \mathcal{H}$ to $\mathbb{R}Z^{\perp}(z)$ with

$$Z^{\perp}(z) := \frac{1}{\sqrt{N}y}\begin{pmatrix} x & -|z|^2 \\ 1 & -x \end{pmatrix}, \quad \text{and we set} \quad Z(z) := \frac{1}{\sqrt{N}}\begin{pmatrix} z & -z^2 \\ 1 & -z \end{pmatrix}$$

$$= \frac{1}{\sqrt{N}}\begin{pmatrix} z \\ 1 \end{pmatrix}(1 \ -z).$$

It is easy to check that $\gamma \cdot Z^{\perp}(z) = Z^{\perp}(\gamma z)$ and $\gamma \cdot Z(z) = j(\gamma, z)^2 Z(\gamma z)$ for every $\gamma \in \mathrm{SL}_2(\mathbb{R})$.

Given $\lambda \in V_{\mathbb{R}}$ with $Q(\lambda) = -\xi^2 < 0$, we know that $\lambda = \xi Z^{\perp}(z_{\lambda})$ for some $z_{\lambda} = x_{\lambda} + iy_{\lambda} \in \mathcal{H}$, with $\mathrm{sgn}(\xi) = -\mathrm{sgn}(\lambda, Z^{\perp}(z_{\lambda}))$. In this case, Lemma 4.2 of [Ze4] proves the equalities

(2.6)
$$\left(\lambda, Z^{\perp}(z)\right) = -2\xi\cosh d(z,z_{\lambda}) = -2\xi\left(\frac{|z-z_{\lambda}|^2}{2yy_{\lambda}}+1\right) = -2\xi\frac{1+|A_{z_{\lambda}}(z)|^2}{1-|A_{z_{\lambda}}(z)|^2},$$

$$\left(\lambda, Z(z)\right) = -2\xi\frac{(z-z_{\lambda})(z-\overline{z_{\lambda}})}{2y_{\lambda}} = \frac{4\xi y_{\lambda}A_{z_{\lambda}}(z)}{\left(1-A_{z_{\lambda}}(z)\right)^2},$$

where $d(z,z_{\lambda})$ is the hyperbolic distance between z and z_{λ}.

Fix an even, integral lattice $L \subseteq V$, with its dual $L^* := \mathrm{Hom}(L,\mathbb{Z})$ viewed as a subgroup of V containing L, and $D_L := L^*/L$ the associated finite quadratic module. We denote $\Gamma = \Gamma_L \subseteq G(\mathbb{Q}) = \mathrm{SL}_2(\mathbb{Q})$ the inverse image of the discriminant kernel[3] of L, and set $Y := Y_L := \Gamma\backslash\mathcal{H}$ to be the associated (open) modular curve, with the projection map $\pi : \mathcal{H} \to Y$. For every $h \in D_L$ and $m \in \mathbb{Z} + Q(h)$, we denote

(2.7)
$$L_{m,h} := \{\lambda \in L + h | Q(\lambda) = m\}.$$

Typical examples can be found in [BO], [AE], [LZ], or [Ze6]; e.g.,

(2.8)
$$L = \left\{\begin{pmatrix} -B & C \\ -A & B \end{pmatrix} \Big| A, B, C \in \mathbb{Z}\right\} \quad \text{with} \quad Q = -\det \quad \text{and} \quad \Gamma = \mathrm{SL}_2(\mathbb{Z}).$$

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[3]For convenience, we shall henceforth assume that $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ (see Remark 2.4).

Furthermore, let ρ_L be the Weil representation associated with L, in which $\mathrm{Mp}_2(\mathbb{Z})$ operates on the vector space $\mathbb{C}[D_L]$, with the canonical basis $\{\mathfrak{e}_h\}_{h\in D_L}$ (see [Bo, Sch, Str, Zel] and others).

2.3 Cusps and geodesics

The Baily–Borel completion $\mathcal{H}^*$ of $\mathcal{H}$ is obtained by adding the set $\mathrm{Iso}(V) \cong \mathbb{P}^1(\mathbb{Q})$ of isotropic lines in V. Let $\ell_\infty \in \mathrm{Iso}(V)$ be the line spanned by $u_\infty := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, and given $\ell \in \mathrm{Iso}(V)$, we take an element $\sigma_\ell \in \mathrm{SL}_2(\mathbb{Z})$ such that $\ell = \sigma_\ell \ell_\infty$, and set $u_\ell := \sigma_\ell u_\infty$. If $\Gamma_\ell \subseteq \Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is the stabilizer of ℓ, then there exists $\alpha_\ell \in \mathbb{N}$, called the *width* of the cusp ℓ, such that

$$(2.9) \qquad \sigma_\ell^{-1}\Gamma_\ell\sigma_\ell = \left\{ \pm \left(\begin{smallmatrix} 1 & n\alpha_\ell \\ 0 & 1 \end{smallmatrix}\right) \middle| n \in \mathbb{Z} \right\}.$$

Let $0 < \beta_\ell \in \mathbb{Q}$ be such that

$$(2.10) \qquad L \cap \ell = \mathbb{Z}\beta_\ell u_\ell, \quad \text{and set} \quad \varepsilon_\ell := \frac{\alpha_\ell}{\beta_\ell}.$$

When $(L+h) \cap \ell \neq \varnothing$, we define $0 \leq k_{\ell,h} < \beta_\ell$ to be the unique number such that

$$(2.11) \qquad (L+h) \cap \ell = (\mathbb{Z}\beta_\ell + k_{\ell,h})u_\ell, \quad \text{and set} \quad \omega_{\ell,h} := \frac{k_{\ell,h}}{\beta_\ell} + \mathbb{Z} \in \mathbb{R}/\mathbb{Z}.$$

All these parameters are constant on Γ-orbits.

Near the cusp associated with $\ell \in \mathrm{Iso}(V)$, we work with the coordinates

$$(2.12) \qquad z_\ell = x_\ell + iy_\ell := \sigma_\ell^{-1}z \quad \text{and} \quad q_\ell(z_\ell) := \mathbf{e}\left(\frac{z_\ell}{\alpha_\ell}\right).$$

For $\varepsilon > 0$, we define the neighborhood $B_\varepsilon(\ell) := \left\{z \in \mathcal{H} \middle| |q_\ell(z_\ell)| < \varepsilon\right\}$ of the cusp ℓ. The set

$$(2.13) \qquad \mathcal{H}_T := \mathcal{H} \backslash \bigcup_{\ell \in \mathrm{Iso}(V)} B_{e^{-2\pi T}}(\ell), \qquad \text{for} \quad T > 1,$$

is Γ-invariant, and $Y_T := \Gamma\backslash\mathcal{H}_T$ is a truncated modular curve, with a fundamental domain[4]

$$(2.14) \qquad \mathcal{F}_T(L) := \bigcup_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \sigma_\ell \mathcal{F}_T^{\alpha_\ell}, \qquad \text{where} \qquad \mathcal{F}_T^\alpha := \bigcup_{j=0}^{\alpha-1} \left(\begin{smallmatrix} 1 & j \\ 0 & 1 \end{smallmatrix}\right)\mathcal{F}_T$$

is composed, for $\alpha \in \mathbb{N}$, of α translations of

$$\mathcal{F}_T := \left\{z = x + iy \in \mathcal{H} \middle| |x| \leq \tfrac{1}{2}, \ |z| \geq 1, \ y \leq T\right\}.$$

Remark 2.4 The assumption $\Gamma = \Gamma_L \subseteq \mathrm{SL}_2(\mathbb{Z})$ is satisfied for the large family of lattices from [Ze6], but not for every lattice L in V. However, the only place where we use the assumption that $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is in the form of the fundamental domain from equation (2.14), with $T > 1$ being a sufficient bound, and in the integrality of the parameter α_ℓ from equation (2.9). Since none of these facts are used in any proof below, our results hold equally well for more general lattices.

---

[4]The fundamental domain actually depends on a choice of representatives for $\Gamma \backslash \mathrm{Iso}(V)$, but we suppress it from the notation since this choice does not affect the results later.

An element $\lambda \in V_{\mathbb{R}}$ with $Q(\lambda) > 0$ defines a geodesic

$$(2.15) \quad c_\lambda := \left\{ z \in \mathcal{H} \,\middle|\, \left(\lambda, Z^\perp(z) \right) = 0 \right\} \subseteq \mathcal{H}, \qquad \text{as well as} \qquad c(\lambda) := \Gamma_\lambda \backslash c_\lambda \subseteq Y,$$

where Γ_λ is the stabilizer of λ in Γ. For $\lambda_0 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \in V$, we orient the geodesic $c_{\lambda_0} = (0, i\infty)$ to go up, and transfer this to an orientation on c_λ and $c(\lambda)$ for each such $\lambda \in V_{\mathbb{R}}$ via the action of $\mathrm{SL}_2(\mathbb{R})$. We have the following well-known dichotomy.

Lemma 2.5 *Let $\lambda \in V$ be such that $m = Q(\lambda) > 0$. If $m \in N \cdot (\mathbb{Q}^\times)^2$, then Γ_λ is the trivial subgroup $\{\pm I\}$, and the geodesic c_λ connects two cusps in $\mathbb{P}^1(\mathbb{Q})$. Otherwise, the image of Γ_λ in $\mathrm{SO}^+(V) \cong \mathrm{PSL}_2(\mathbb{Q})$ is infinite cyclic.*

In the first case in Lemma 2.5, we call λ *split-hyperbolic*. For $m \in \mathbb{Q}$ and for $\lambda \in V$, we then set, by a slight abuse of notation,

$$(2.16)$$
$$\iota(m) := \begin{cases} 1, & \text{if } \sqrt{m/N} \in \mathbb{Q}^\times, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \iota(\lambda) := \begin{cases} 1, & \text{if } \lambda \text{ is split-hyperbolic,} \\ 0, & \text{otherwise.} \end{cases}$$

If $\iota(\lambda) = 1$, then $\lambda^\perp$ is spanned by ℓ_λ and $\ell_{-\lambda}$ in $\mathrm{Iso}(V)$, which correspond to where c_λ ends and begins, respectively. If $Q(\lambda) = m$, then we have

$$(2.17)$$
$$\sigma_{\ell_\lambda}^{-1} \lambda = \sqrt{\tfrac{m}{N}} \left(\begin{smallmatrix} 1 & -2r_\lambda \\ 0 & -1 \end{smallmatrix} \right) \quad \text{for some} \quad r_\lambda \in \mathbb{Q}, \quad \text{with} \quad r_\lambda + \alpha_{\ell_\lambda} \mathbb{Z} \in \mathbb{Q}/\alpha_{\ell_\lambda} \mathbb{Z} \quad \text{canonical.}$$

The canonical image $r_\lambda + \alpha_{\ell_\lambda} \mathbb{Z} \in \mathbb{Q}/\alpha_{\ell_\lambda} \mathbb{Z}$ from equation (2.17), which we shall henceforth still denote by just r_λ, is called the *real part* of c_λ, and it is constant on Γ-orbits.

For $\ell \in \mathrm{Iso}(V)$, $m \geq 0$, and $h \in D_L$, we set, for later use, the symbol

$$(2.18) \quad \iota_\ell(m, h) := \begin{cases} 1, & \text{there exists } \lambda \in L_{m,h} \cap \ell^\perp, \text{ positively oriented if } m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, for $m > 0$, we have $\iota_\ell(m, h) = 1$ if and only if $\ell = \ell_\lambda$ for some $\lambda \in L_{m,h}$. The additive subgroup $L \cap \ell$ acts on $L_{m,h} \cap \ell^\perp$, and we have the following standard result.

Lemma 2.6 (Lemma 3.1 of [BFIL]) *For $\ell \in \mathrm{Iso}(V)$, $0 < m \in \mathbb{Q}$, and $h \in D_L$ such that $\iota_\ell(m, h) = 1$, the natural map $L_{m,h} \cap \ell^\perp \to (L_{m,h} \cap \ell^\perp)/(L \cap \ell)$ factors through $\Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp)$. For every $\lambda \in L_{m,h} \cap \ell^\perp$, there are $2\sqrt{\tfrac{m}{N}} \varepsilon_\ell$ pre-images of $\lambda + (L \cap \ell)$ in $\Gamma_\ell \backslash (L_{m,h} \cap \ell^\perp)$, namely the images of $\{\lambda + j\beta_\ell u_\ell \mid 0 \leq j \leq 2\sqrt{\tfrac{m}{N}} \varepsilon_\ell - 1\}$ modulo Γ_ℓ.*

Remark 2.7 The number $2\sqrt{\tfrac{m}{N}} \varepsilon_\ell$ from Lemma 2.6 is therefore integral. Moreover, for ℓ, m, and h as in Lemma 2.6, take some positively oriented $\lambda \in L_{m,h} \cap \ell^\perp$, and let r_λ be as in equation (2.17). Then we have

$$\left\{ r_\mu \,\middle|\, \mu \in L_{m,h} \cap \ell^\perp \text{ positively oriented} \right\} = r_\lambda + \tfrac{\beta_\ell}{2} \sqrt{\tfrac{N}{m}} \mathbb{Z} \,\middle/\, \alpha_{\ell_\lambda} \mathbb{Z} \subseteq \mathbb{Q}/\alpha_{\ell_\lambda} \mathbb{Z},$$

a set of $2\sqrt{\tfrac{m}{N}} \varepsilon_\ell$ evenly spaced elements of $\mathbb{Q}/\alpha_{\ell_\lambda} \mathbb{Z}$.

2.4 Schwartz forms, theta functions, and Shintani lifts

Given $k \in \mathbb{N}$, we can define the Schwartz function

$$(2.19) \qquad \tilde{\varphi}_k(\lambda; \tau, z) := (\lambda, Z(z))^k \mathbf{e}\left[Q(\lambda)\tau + (\lambda, Z^\perp(z))^2 \frac{iv}{2}\right]$$

for $\lambda \in V_\mathbb{R}$ and $\tau = u + iv \in \mathcal{H}$, and construct the vector-valued theta function

$$(2.20) \qquad \Theta_{k,L}(\tau, z) := \sum_{h \in D_L} \Theta_{k,L,h}(\tau, z)\mathfrak{e}_h, \qquad \Theta_{k,L,h}(\tau, z) := \sqrt{v} \sum_{\lambda \in L+h} \tilde{\varphi}_k(\lambda; \tau, z).$$

Theorem 4.1 of [Bo] implies that for fixed $z \in \mathcal{H}$, we have $\Theta_{k,L}(\tau, z) \in \mathcal{A}_{k+\frac{1}{2}}(\mathrm{Mp}_2(\mathbb{Z}), \rho_L)$, whereas for fixed $\tau \in \mathcal{H}$, it is easy to verify that $\Theta_{k,L,h}(\tau, z) \in \mathcal{A}_{-2k}(\Gamma)$ for every $h \in D_L$. After collecting terms, we can use equation (2.7) to rewrite

$$(2.21)$$
$$\Theta_{k,L,h}(\tau, z) = \sqrt{v} \sum_{m \in \mathbb{Z}+Q(h)} \left[\sum_{\lambda \in L_{m,h}} (\lambda, Z(z))^k e^{-\pi v(\lambda, Z^\perp(z))^2}\right] q_\tau^m, \qquad q_\tau := \mathbf{e}(\tau).$$

Recall that the (probabilists') Hermite polynomials are defined as

$$(2.22) \quad \mathrm{He}_n(\xi) := (-1)^n e^{\xi^2/2}\left(\frac{d}{d\xi}\right)^n e^{-\xi^2/2} = \left(\xi - \frac{d}{d\xi}\right)^n \cdot 1 = \sum_{b=0}^{\lfloor n/2 \rfloor} \frac{(-1)^b n!}{b!(n-2b)!2^b} \xi^{n-2b}.$$

Then, for $\ell \in \mathrm{Iso}(V)$ and $k \in \mathbb{N}$, one defines the unary theta function

$$(2.23)$$
$$\Theta_{k,\ell}(\tau) := \sum_{\lambda \in (L^* \cap \ell^\perp)/(L^* \cap \ell)} \frac{\mathrm{He}_k\left(\sqrt{2\pi v}(\sigma_\ell^{-1}\lambda, \Im(Z(i)))\right)}{(2\pi v)^{k/2}} q_\tau^{Q(\lambda)} \sum_{\substack{h \in D_L \\ h+(L^* \cap \ell)/(L \cap \ell) = \lambda}} \mathfrak{e}_h$$
$$= \sum_{h \in D_L} \sum_{\substack{0 \le m \in \mathbb{Z}+Q(h) \\ \iota(m)=1}} a(\Theta_{k,\ell}, m, h, v) q_\tau^m \mathfrak{e}_h \in \mathcal{A}_{k+\frac{1}{2}}(\mathrm{Mp}_2(\mathbb{Z}), \rho_L), \qquad \text{with}$$

$$a(\Theta_{k,\ell}, m, h, v) := \frac{\mathrm{He}_k\left(2\sqrt{2\pi m v}\right)}{(2\pi v)^{k/2}} \begin{cases} (\iota_\ell(m, h) + (-1)^k \iota_\ell(m, -h)), & m > 0, \\ \iota_\ell(0, h), & m = 0. \end{cases}$$

Remark 2.8 The theta functions $\Theta_{Sh}(\tau, z)$ and $\Theta_{\ell,k}(\tau)$ from equations (4.1) and (4.2) of [ANS] correspond to $(-\sqrt{N}/y^2)^{k+1}\overline{\Theta_{k+1,L}(\tau, z)}$, $(-i\sqrt{N})^k\overline{\Theta_{k,\ell}(\tau)}$, and equation (2.23), respectively, in our setting.

The function $\Theta_{k,\ell}$ from equation (2.23) appears in the asymptotic expansion of the theta kernel $\Theta_{k,L}$ from (2.20), as is given in the following result. It is essentially part (2) of Proposition 4.2 of [ANS], which refers to Theorem 5.2 of [Bo] for the proof, and can also be proved using properties of appropriate variants of the lattice sums from equation (3.34) below.

Lemma 2.9 *Given $\ell \in \mathrm{Iso}(V)$, there exists a constant $C_\ell > 0$ such that*

$$(\Theta_{k,L} \mid_{2k,z} \sigma_\ell)(\tau, z_\ell) = \frac{i^k y_\ell^{k+1}}{\sqrt{N}\beta_\ell} \Theta_{k,\ell}(\tau) + O(e^{-C_\ell y_\ell^2}) \quad \text{as} \quad y_\ell \to \infty.$$

For $f \in \mathcal{A}_{2k}^!(\Gamma)$, we follow [ANS, BFI] (among others) to define its *regularized Shintani lift*, using the fundamental domain from equation (2.14), to be the theta integral

(2.24)

$$\mathcal{I}_{k,L}(\tau, f) := \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \mathrm{CT}_{s=0} \left[\lim_{T \to \infty} \int_{\mathcal{F}_T^{\alpha_\ell}} (f \mid_{2k} \sigma_\ell)(z_\ell)(\Theta_{k,L} \mid_{-2k} \sigma_\ell)(\tau, z_\ell) y_\ell^{-s} d\mu(z_\ell) \right],$$

which is an element of $\mathcal{A}_{k+\frac{1}{2}}(\mathrm{Mp}_2(\mathbb{Z}), \rho_L)$. When the constant term of f at every cusp is zero, the integral converges absolutely and no regularization is necessary (see Proposition 4.1 of [BF2]).

3 Special functions

In this section, we construct singular Schwartz functions that serve as higher-order antiderivatives of the ones appearing in the theta kernel. This requires the study of several families of polynomials and a few types of special functions and their properties, including their Fourier transforms and lattice sums. All of these expressions show up in the evaluation of the various parts of the Shintani lift of nearly holomorphic modular forms in Section 4.

3.1 Familiar functions

Let $\mathrm{g}(\xi)$ denote the Gaussian $e^{-\xi^2/2}$. For $\xi > 0$, it has the antiderivative

(3.1) $\quad -\dfrac{\sqrt{\pi}}{\sqrt{2}} \cdot \mathrm{erfc}\left(\dfrac{\xi}{\sqrt{2}}\right) = -\displaystyle\int_\xi^\infty e^{-w^2/2} dw = -\int_{\xi^2/2}^\infty e^{-s} \dfrac{ds}{\sqrt{2s}} = -\dfrac{1}{\sqrt{2}} \Gamma\left(\dfrac{1}{2}, \dfrac{\xi^2}{2}\right),$

where erfc is the complementary error function and $\Gamma(\mu, t)$ is the *incomplete Gamma function* defined as

$$\Gamma(\mu, t) := \int_t^\infty e^{-s} s^\mu \frac{ds}{s}$$

for $t > 0$. If $0 < \mu \in \mathbb{N}$, then this formula is well defined for every $t \in \mathbb{R}$, and for $\mu \in \mathbb{Z}$, it is meaningful for $t < 0$ as follows: If $\mu = 0$, then the integral is defined using the Cauchy principal value, and for smaller μ, we employ repeated integration by parts. The explicit formulae are given by

(3.2)

$$\Gamma(\mu, t) = \begin{cases} e^{-t}(\mu-1)! \sum_{a=0}^{\mu-1} \frac{t^a}{a!} = e^{-t} t^\mu \left(1 - \frac{d}{dt}\right)^{\mu-1} \frac{1}{t}, & \text{when } 0 < \mu \in \mathbb{N} \text{ and } t \in \mathbb{R}, \\ \frac{(-1)^\mu}{|\mu|!} \left(\Gamma(0, t) + e^{-t} \sum_{a=0}^{|\mu|-1} \frac{a!}{(-t)^{a+1}}\right), & \text{when } -\mu \in \mathbb{N}, \text{ and } t \neq 0, \end{cases}$$

$\Gamma(0, t)$ can also be written as $-\operatorname{Ei}(-t)$ using the *exponential integral* $\operatorname{Ei}(t) :=$ $-\int_{-t}^{\infty} e^{-w} \frac{dw}{w}$, and the equality

$$(3.3) \qquad \frac{d}{dt} \Gamma(\mu, t) = -e^{-t} t^{\mu-1}$$

holds whenever $\Gamma(\mu, t)$ is defined.

Modifying the antiderivative from equation (3.1), we now define

$$(3.4) \qquad \operatorname{e}(\xi) := -\frac{\operatorname{sgn}(\xi)}{\sqrt{2}} \Gamma\left(\tfrac{1}{2}, \tfrac{\xi^2}{2}\right) = -\operatorname{sgn}(\xi) \int_{|\xi|}^{\infty} e^{-w^2/2} dw \quad \text{for} \quad \xi \neq 0.$$

It decays rapidly as $|\xi| \to \infty$, but it is discontinuous at $\xi = 0$ with the jump

$$\lim_{\xi \to 0^+} \operatorname{e}(\xi) - \lim_{\xi \to 0^-} \operatorname{e}(\xi) = -\sqrt{2} \Gamma\left(\tfrac{1}{2}, 0\right) = -\sqrt{2\pi}.$$

We therefore have, as distributions on $\mathbb{R}$, the equality

$$(3.5) \qquad \frac{d}{d\xi} \operatorname{e}(\xi) = \operatorname{g}(\xi) - \sqrt{2\pi} \cdot \delta(\xi),$$

where $\delta(\xi)$ is the Dirac delta distribution. Our next goal is to find higher-order antiderivatives of $\operatorname{g}(\xi)$.

3.2 Two families of polynomials

First, we will consider two sequences of polynomials in $\mathbb{Q}[\xi]$, which we denote by P_ν and Q_ν with $\nu \in \mathbb{N}$ and defined recursively as follows. They will show up in equation (3.17) and Proposition 3.14, whence their importance. Set

$$(3.6) \qquad \begin{aligned} &P_0(\xi) = 1 \quad \text{and} \quad Q_0(\xi) = 0, \qquad \text{as well as} \\ &P_\nu'(\xi) = P_{\nu-1}(\xi) \quad \text{and} \quad P_\nu(\xi) + Q_\nu'(\xi) - \xi Q_\nu(\xi) = Q_{\nu-1}(\xi) \text{ for } \nu \geq 1. \end{aligned}$$

The fact that equation (3.6) defines unique sequences of polynomials, and their parity properties, are established via the following lemma.

Lemma 3.1 *Let p and q be two polynomials in $\mathbb{Q}[\xi]$. Then there is a unique pair (P, Q) of polynomials P and Q such that $P' = p$ and $P + Q' - \xi Q = q$. Moreover, if p and q have opposite parities, then so do P and Q, with that of P (resp. Q) coinciding with that of q (resp. p). In addition, if $p \neq 0$ has leading coefficient r and $\deg q \leq \deg p$, then $\deg P = \deg Q + 1 = \deg p + 1$ and the leading coefficients of both P and Q are $\frac{r}{\deg P}$.*

Proof Let $\operatorname{g}(\xi) := e^{-\xi^2/2}$ denote the usual Gaussian. Then, for each $k \geq 1$, the polynomial

$$p_k(\xi) := \operatorname{g}(\xi)^{-1} \frac{d}{d\xi} \left(\operatorname{g}(\xi) \xi^{k-1}\right) = (k-1) \xi^{k-2} - \xi^k \in \mathbb{Q}[\xi]$$

has degree k, parity $(-1)^k$, and leading coefficient -1. It is thus clear that $\{1\} \cup \{p_k\}_{k \geq 1}$ is a basis of $\mathbb{Q}[\xi]$ over $\mathbb{Q}$, that respects the parity decomposition. Therefore, there exist unique $\tilde{P} \in \mathbb{Q}[\xi]$, $c \in \mathbb{Q}$, and $Q \in \mathbb{Q}[\xi]$ such that

$$\tilde{P}' = p, \quad \tilde{P}(0) = 0, \quad \text{and} \quad g(\xi)\big(q(\xi) - \tilde{P}(\xi) + c\big) = \frac{d}{d\xi}\big(g(\xi)Q(\xi)\big)$$
$$= g(\xi)\big(Q'(\xi) - \xi Q(\xi)\big).$$

Setting $P := \tilde{P} - c$ proves the first assertion. For even q and odd p, $\tilde{P}$ and P are also even, and Q is thus odd. If q is odd and p is even, then $\tilde{P}$ is also odd, and then integrating from $-\infty$ to ∞ shows that $c = 0$ in this case, completing the proof of the parity assertion. The assertion about the degrees and the leading coefficients when $\deg q \leq \deg p$ are now easily checked from this construction. This proves the lemma. ∎

Corollary 3.2 *Equation* (3.6) *defines unique sequences* $\{P_v\}_{v=0}^{\infty}$ *and* $\{Q_v\}_{v=0}^{\infty}$ *of polynomials in* ξ. *Moreover,* P_v *is a polynomial of degree* v *with leading coefficient* $\frac{1}{v!}$ *and parity* $(-1)^v$ *for any* $v \geq 0$, *and* Q_v *is a polynomial of degree* $v - 1$ *with leading coefficient* $\frac{1}{v!}$ *and parity* $(-1)^{v-1}$ *for any* $v \geq 1$.

Proof All the statements hold for $v = 0$ by equation (3.6), and once they hold for $v - 1$, taking $p = P_{v-1}$ and $q = Q_{v-1}$ in Lemma 3.1 determines P_v and Q_v as P and Q, respectively, with the required properties. This proves the corollary. ∎

For convenience, we list the polynomials P_v and Q_v for $0 \leq v \leq 4$:

$$P_0(\xi) = 1, \; P_1(\xi) = \xi, \; P_2(\xi) = \frac{\xi^2 + 1}{2}, \; P_3(\xi) = \frac{\xi^3 + 3\xi}{6}, \; P_4(\xi) = \frac{\xi^4 + 6\xi^2 + 3}{24},$$
$$Q_0(\xi) = 0, \; Q_1(\xi) = 1, \; Q_2(\xi) = \frac{\xi}{2}, \; Q_3(\xi) = \frac{\xi^2 + 2}{6}, \; Q_4(\xi) = \frac{\xi^3 + 5\xi}{24}.$$

It will be useful for us to consider the (ordinary) generating series

$$(3.7) \qquad \Psi(\xi, t) := \sum_{v=0}^{\infty} P_v(\xi) t^v \qquad \text{and} \qquad \Upsilon(\xi, t) := \sum_{v=0}^{\infty} Q_v(\xi) t^v,$$

considered, at the moment, as formal power series in ξ and t, on which derivatives operate in the usual, formal way. They can be characterized in the following way.

Proposition 3.3 *The formal power series* $\Psi = \Psi(\xi, t)$ *(resp.* $\Upsilon = \Upsilon(\xi, t)$*) is characterized as the unique real-analytic function satisfying the following properties:*

(i) *For each power* t^v *of* t, *the power series in* ξ *that multiplies it is finite, i.e., a polynomial.*

(ii) *At* $t = 0$, *we have the equality* $\Psi(\xi, 0) = 1$ *(resp.* $\Upsilon(\xi, 0) = 0$*) as series in* ξ.

(iii) *The power series satisfies the differential equation*

$$\partial_\xi \Psi(\xi, t) = t \cdot \Psi(\xi, t) \quad \text{(resp.} \quad \Psi(\xi, t) + \partial_\xi \Upsilon(\xi, t) - \xi \cdot \Upsilon(\xi, t) = 1 + t \cdot \Upsilon(\xi, t)\text{).}$$

Proof Assume that Ψ and Υ satisfy these properties, and write their expansion in t as in equation (3.7), with polynomial coefficients $\tilde{P}_v(\xi)$ and $\tilde{Q}_v(\xi)$, respectively, by Property (i). Now, Property (ii) implies that $\tilde{P}_v$ and $\tilde{Q}_v$ satisfy the condition for $v = 0$ in equation (3.6), and also explain the existence of the term 1 in the differential equation for Υ in Property (iii) (set $t = 0$ there). Next, comparing the coefficient of t^v with $v \geq 1$ in the series resulting from substituting these expansions into the differential equations from Property (iii) yields the other part of equation (3.6).

Hence, $\{\tilde{P}_\nu\}_{\nu=0}^\infty$ and $\{\tilde{Q}_\nu\}_{\nu=0}^\infty$ are sequences satisfying that equation, so that $\tilde{P}_\nu = P_\nu$ and $\tilde{Q}_\nu = Q_\nu$ for every $\nu \in \mathbb{N}$ by Corollary 3.2. This proves the proposition. ∎

Proposition 3.3 allows us to determine the series Ψ and Υ explicitly as the expansions of real-analytic functions, which also shows that their Taylor expansion in (3.7) converges absolutely for all t and ξ.

Regarding Property (i) in that proposition, see Remark 3.5.

Theorem 3.4 *The power series Ψ and Υ from equation (3.7) describe the functions*

$$\Psi(\xi, t) = e^{\xi t + t^2/2} \quad \text{and} \quad \Upsilon(\xi, t) = e^{(\xi+t)^2/2} \int_\xi^{\xi+t} e^{-w^2/2} dw$$

$$= e^{\xi t + t^2/2} \int_0^t e^{-\xi w - w^2/2} dw.$$

These functions also satisfy the differential equations

$$(\partial_t - \xi)\Psi(\xi, t) = t\Psi(\xi, t) \quad \text{and} \quad (\partial_t - \xi)\Upsilon(\xi, t) = t\Upsilon(\xi, t) + 1.$$

Proof It suffices to show that the asserted series have the properties from Proposition 3.3. Property (ii) is immediate for both (by substituting $t = 0$), Property (i) for Ψ is easy to check by expanding it in t, and the two differential equations (that from Property (iii) and the one asserted here) are easily established by simple differentiation. Differentiation also shows that

$$\partial_\xi \Upsilon(\xi, t) = (\xi + t)\Upsilon(\xi, t) + e^{(\xi+t)^2/2}\big[e^{-(\xi+t)^2/2} - e^{-\xi^2/2}\big] = (\xi + t)\Upsilon(\xi, t) + 1 - \Psi(\xi, t),$$

from which Property (iii) for Υ quickly follows.

Next, note that Property (i) for Υ is equivalent to $\partial_t^\nu \Upsilon(\xi, t)\big|_{t=0}$ being a polynomial in ξ for every $\nu \in \mathbb{N}$. To see this, we first evaluate $\partial_t \Upsilon(\xi, t)$ as $1 + (\xi + t)\Upsilon(\xi, t)$, yielding the remaining differential equation as well. Simple induction now shows that, for every ν, the derivative $\partial_t^\nu \Upsilon(\xi, t)$ is a polynomial in $\xi + t$ plus another polynomial in $\xi + t$ times $\Upsilon(\xi, t)$. After substituting $t = 0$, the fact that $\Upsilon(\xi, 0) = 0$ yields the desired assertion. This proves the theorem. ∎

Remark 3.5 If one considers Property (iii) from Proposition 3.3 as differential equations for functions, then solving them implies that there exist functions $\psi(t)$ and $\phi(t)$ such that as functions, $\Psi(\xi, t)$ and $\Upsilon(\xi, t)$ are

$$\psi(t)e^{\xi t + t^2/2} \quad \text{and} \quad e^{(\xi+t)^2/2}\Bigg[\phi(t) + \int_\xi^{\xi+t} e^{-w^2/2} dw + \big(\psi(t) - 1\big)\int_\xi^\infty e^{-w^2/2} dw\Bigg],$$

respectively. The initial conditions from Property (ii) amount to the equalities $\psi(0) = 1$ and $\phi(0) = 0$. Showing that the polynomial property of the Taylor expansions from Property (i) in Proposition 3.3 is equivalent to ψ and ϕ being the constant functions with the respective values seems, however, not very straightforward.

Remark 3.6 The differential equations from Theorem 3.4 also imply that

$$\nu P_\nu(\xi) - \xi P_{\nu-1}(\xi) = P_{\nu-2}(\xi) \quad \text{and} \quad \nu Q_\nu(\xi) - \xi Q_{\nu-1}(\xi) = Q_{\nu-2}(\xi) \quad \text{for all} \quad \nu \geq 2.$$

For the explicit expressions for the polynomials from equation (3.6), recall the well-known formula for the generating function of the (probabilists') Hermite polynomials, stating that

$$(3.8) \qquad e^{\xi t - t^2/2} = \sum_{\nu=0}^{\infty} \mathrm{He}_\nu(\xi) \frac{t^\nu}{\nu!}.$$

It can be proved, for example, by comparing equation (2.22) with the Taylor series of $e^{-(\xi-t)^2/2}$ at $t = 0$. Theorem 3.4 then implies the following result.

Corollary 3.7 *For every $\nu \in \mathbb{N}$, we have*

$$P_\nu(\xi) = \frac{(-i)^\nu}{\nu!} \mathrm{He}_\nu(i\xi) := \frac{e^{-\xi^2/2}}{\nu!} \frac{d^\nu}{d\xi^\nu} e^{\xi^2/2} = \left(\xi + \frac{d}{d\xi}\right)^\nu \cdot \frac{1}{\nu!} = \sum_{a=0}^{\lfloor \nu/2 \rfloor} \frac{\xi^{\nu-2a}}{a!(\nu-2a)!2^a}.$$

In particular, P_ν is a polynomial of degree ν and parity $(-1)^\nu$ such that $\nu! P_\nu \in \mathbb{Z}[\xi]$ and $P_\nu(0)$ is $\frac{1}{2^{\nu/2}(\nu/2)!}$ for even ν and 0 for odd ν.

Note that the proof of Theorem 3.4 yields the equality $\partial_t \Upsilon(\xi, t) = \partial_\xi \Upsilon(\xi, t) + \Psi(\xi, t)$, from which we deduce that $Q_{\nu+1}(\xi) = \frac{Q_\nu'(\xi) + P_\nu(\xi)}{\nu+1}$, and using equation (3.6), we get

$$(3.9) \qquad Q_\nu(\xi) = \sum_{a=0}^{\nu-1} \frac{(\nu-1-a)!}{\nu!} \frac{d^a}{d\xi^a} P_{\nu-1-a}(\xi) = \sum_{a=0}^{\lfloor (\nu-1)/2 \rfloor} \frac{(\nu-1-a)!}{\nu!} P_{\nu-1-2a}(\xi),$$

with $P_\mu(\xi)$ given in Corollary 3.7. It is thus indeed a polynomial of degree $\nu - 1$ and parity $(-1)^{\nu-1}$ such that $\nu! Q_\nu \in \mathbb{Z}[\xi]$. It follows that $Q_\nu(0) = 0$ for even ν, whereas if ν is odd, then

$$(3.10) \qquad Q_\nu(0) = \frac{2^{\frac{\nu-1}{2}} \left(\frac{\nu-1}{2}\right)!}{\nu!} = \prod_{j=0}^{(\nu-1)/2} \frac{1}{2j+1}.$$

Remark 3.8 The recursion (3.6) extends naturally to $\nu \in \mathbb{Z}$, in which case $P_\nu(\xi) = 0$ and

$$Q_\nu(\xi) = (-1)^{1-\nu} \mathrm{He}_{-1-\nu}(\xi)$$

for $\nu \le -1$. We take these as the definitions of P_ν and Q_ν in these cases. Then Remark 3.6 extends to all $\nu \in \mathbb{Z}$.

The polynomials He_ν form an *Appell sequence* in the sense that $\mathrm{He}_\nu' = \nu \mathrm{He}_{\nu-1}$ (see, e.g., the exponential example in Section 5 of [Ze5] and some of the references therein), and Corollary 3.7 shows that the same applies also to the polynomials $\nu! P_\nu$. This means explicitly that the equalities

$$(3.11) \qquad \mathrm{He}_\nu(\xi + \eta) = \sum_{j=0}^{\nu} \binom{\nu}{j} \eta^j \mathrm{He}_{\nu-j}(\xi) \quad \text{and} \quad P_\nu(\xi + \eta) = \sum_{j=0}^{\nu} \frac{\eta^j P_{\nu-j}(\xi)}{j!}$$

hold. This either follows from the generating function $\Psi(\xi, t)$ (as well as the exponential generating function $e^{\xi t - t^2/2}$) being $e^{\xi t}$ times a function of ξ alone (see, e.g., [Ze5], even though this was known much earlier), or by simple computations using

the explicit formulae. A change of variable in equation (3.11) produces, for every $l \in \mathbb{N}$, the equality

$$(3.12) \qquad \sum_{v=0}^{l} \frac{(-1)^v}{(l-v)!}(\xi+\zeta)^{l-v}P_v(\xi) = P_l(\zeta) \in \mathbb{Q}[\xi,\zeta].$$

3.3 Auxiliary polynomials

We now define a few other families of polynomials, which will appear later in the Fourier expansion of the Shintani lift.

Lemma 3.9 *For any $l \in \mathbb{N}$, the polynomial*

$$\Pi_l(\xi,\zeta) := \sum_{v=0}^{l} \frac{(-1)^v}{(l-v)!}(\xi+\zeta)^{l-v}Q_v(\xi) \in \mathbb{Q}[\xi,\zeta]$$

has degree $l-1$ in ξ, and it satisfies the equality $\Pi_l(-\zeta,\zeta) = -Q_l(\zeta)$. We also have

$$\partial_\zeta \Pi_l(\xi,\zeta) = \Pi_{l-1}(\xi,\zeta) \qquad and \qquad \partial_\xi\big(\Pi_l(\xi,\zeta)g(\xi)\big) = \left(\frac{(\xi+\zeta)^l}{l!} - P_l(\zeta)\right)g(\xi)$$

for every $l \in \mathbb{N}$, and the generating series

$$\sum_{l=0}^{\infty} \Pi_l(\xi,\zeta)t^l = -e^{t^2/2+\zeta t}\int_0^t e^{\xi w - w^2/2}dw.$$

Proof The degree in ξ and the value of $\Pi_l(-\zeta,\zeta)$ are immediate from the definition. Substituting the definition of $\Pi_l(\xi,\zeta)$ inside the generating series produces

$$\sum_{l=0}^{\infty}\sum_{v=0}^{l} \frac{(-1)^v t^l}{(l-v)!}(\xi+\zeta)^{l-v}Q_v(\xi) = \sum_{v=0}^{\infty}\sum_{\mu=0}^{\infty} \frac{t^\mu}{\mu!}(\xi+\zeta)^\mu(-t)^v Q_v(\xi) = e^{t(\xi+\zeta)}\Upsilon(\xi,-t)$$

(with $\mu = l - v \geq 0$), where we have substituted equation (3.7). The value of the series, which we denote by $\Phi(\xi,\zeta,t)$, now follows from Theorem 3.4. One checks directly that this series satisfies the differential equations

$$\partial_\zeta \Phi(\xi,\zeta,t) = t\Phi(\xi,\zeta,t) \text{ and } (\partial_\xi - \xi)\Phi(\xi,\zeta,t) = e^{t^2/2+\zeta t}\int_0^t (\xi - w)e^{\xi w - w^2/2}dw$$

$$= e^{(\xi+\zeta)t} - \Psi(\zeta,t)$$

(using Theorem 3.4 again), from which the two required equalities follow for every l after expanding everything in t. This proves the lemma. ∎

Remark 3.10 Write the sum $\Pi_l(\xi,\zeta) + Q_l(\zeta)$ as $\tilde{\Pi}_l(\xi+\zeta,\zeta)$ for some polynomial $\tilde{\Pi}_l$. The equality $\Pi_l(-\zeta,\zeta) = -Q_l(\zeta)$ from Lemma 3.9 implies that $\tilde{\Pi}_l(\omega,\zeta) \in \omega\mathbb{Q}[\omega,\zeta]$.

Using the polynomial $\tilde{\Pi}_l$ from Remark 3.10, we define

$$(3.13) \qquad E_l(\zeta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{\tilde{\Pi}_l(\xi+\zeta,\zeta)}{\xi+\zeta}g(\xi)d\xi \in \mathbb{R}[\zeta].$$

We shall need their following properties.

Lemma 3.11   *We have $E_0 = E_1 = 0$, $E_l(-\zeta) = (-1)^l E_l(\zeta)$, and $E_l(\zeta) \in \mathbb{Q}[\zeta]$ for the polynomials from equation* (3.13). *Moreover, if $H_n := \sum_{a=1}^{n} \frac{1}{a}$ denotes the nth harmonic number, then we have*

$$E_l(0) = -P_l(0) \sum_{a=1,\, 2 \nmid a}^{l} \frac{1}{a} = -\frac{P_l(0)}{2}(2H_l - H_{\lfloor l/2 \rfloor}) \qquad \text{for} \qquad l \in \mathbb{N}.$$

Proof   Theorem 3.4 and Lemma 3.9 evaluate the generating series

$$\sum_{l=0}^{\infty} E_l(\zeta) t^l = \sum_{l=0}^{\infty} \int_{-\infty}^{\infty} g(\xi) \frac{\Pi_l(\xi, \zeta) + Q_l(\zeta)}{\xi + \zeta} t^l \, d\xi$$

$$= \int_0^t e^{\zeta(t-w) + (t^2 - w^2)/2} \int_{-\infty}^{\infty} g(\xi) \frac{1 - e^{(\xi + \zeta)w}}{\xi + \zeta} \, d\xi \, dw.$$

We now claim that for every real ζ, w, and h, we have

(3.14) $$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{(\xi + \zeta)w} - 1}{\xi + \zeta} g(\xi + h) \, d\xi = \int_0^w e^{(\zeta - h)s + s^2/2} \, ds.$$

To see this, we differentiate the left-hand side with respect to w and apply the usual trick of completion to the square in order to obtain $\frac{1}{\sqrt{2\pi}} e^{(\zeta - h)w + w^2/2}$ times $\int_{-\infty}^{\infty} g(\xi + h - w) \, d\xi$. As the latter integral equals $\sqrt{2\pi}$, we deduce that both sides have the same derivative with respect to w, and since they both vanish for $w = 0$, they are equal for every w. Substituting equation (3.14) into our generating series yields

$$\sum_{l=0}^{\infty} E_l(\zeta) t^l = -e^{\zeta t + t^2/2} \int_0^t e^{-\zeta w - w^2/2} \int_0^w e^{\zeta s + s^2/2} \, ds \, dw,$$

yielding the vanishing of E_0 and E_1. We apply the operator $\zeta + t - \partial_t$ to both sides, and using Theorem 3.4 again, with equation (3.7), we get

$$\sum_{l=1}^{\infty} \left(\zeta E_l(\zeta) + E_{l-1}(\zeta) - (l+1)E_{l+1}(\zeta) \right) t^l = \int_0^t e^{\zeta s + s^2/2} \, ds$$

$$= \int_0^t \Psi(\zeta, s) \, ds = \sum_{l=1}^{\infty} \frac{P_{l-1}(\zeta)}{l} t^l.$$

The resulting recurrence relation establishes the rationality and the parity, and as the asserted values for the constant terms satisfy the resulting relation for $\zeta = 0$, these values follow as well. This proves the lemma.   ∎

Given $l \in \mathbb{N}$, equation (3.12) and the definition of the polynomials Π_l in Lemma 3.9 show that $\frac{P_l(\xi) - (-1)^l P_l(\zeta)}{\xi + \zeta}$ and $\frac{Q_l(\xi) - (-1)^l \Pi_l(\xi, \zeta)}{\xi + \zeta}$ are polynomials in $\mathbb{Q}[\xi, \zeta]$. We can then define

(3.15)

$$\Omega_l(\zeta) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\left(P_l(\xi) - (-1)^l P_l(\zeta) \right) e(\xi) + \left(Q_l(\xi) - (-1)^l \Pi_l(\xi, \zeta) \right) g(\xi)}{\xi + \zeta} \, d\xi,$$

and deduce the following properties.

Lemma 3.12   *The generating series of the polynomials from equation* (3.15) *is*

$$\sum_{l=0}^{\infty} \Omega_l(\zeta) t^l = \int_0^t e^{-\zeta s} \frac{e^{t^2/2} - e^{s^2/2}}{t - s} ds.$$

Proof   By applying Theorem 3.4 and the generating series from Lemma 3.9, we can write

$$\sum_{l=0}^{\infty} \Omega_l(\zeta) t^l = \frac{e^{-\zeta t + t^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{(\zeta+\xi)t} - 1}{\zeta + \xi} \left(e(\xi) + g(\xi) \int_0^t e^{-\xi w - w^2/2} dw \right) d\xi,$$

where the second term in the parentheses is $\int_0^t g(\xi + w) dw$. We now claim that

$$(3.16) \qquad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{(\zeta+\xi)t} - 1}{\zeta + \xi} e(\xi) d\xi = \int_0^t e^{\zeta s} \frac{1 - e^{s^2/2}}{s} ds.$$

Indeed, as with equation (3.14), both sides vanish for $t = 0$, and for comparing their derivatives with respect to t, we employ integration by parts and use equation (3.5) and the equality $\int_{-\infty}^{\infty} g(\xi) d\xi = \sqrt{2\pi}$ again. Substituting equations (3.14) and (3.16) transforms our expression for the generating series into

$$e^{-\zeta t + t^2/2} \left(\int_0^t e^{\zeta s} \frac{1 - e^{s^2/2}}{s} ds + \int_0^t \int_0^t e^{(\zeta - w)s + s^2/2} ds \, dw \right)$$

$$= e^{-\zeta t + t^2/2} \int_0^t e^{\zeta s} \frac{1 - e^{s^2/2 - ts}}{s} ds,$$

from which the desired formula follows by taking $s \mapsto t - s$. This proves the lemma. ∎

Remark 3.13   Expanding all the exponents in the generating series from Lemma 3.12 and integrating yields the series $\sum_{m=0}^{\infty} \sum_{b=1}^{\infty} \sum_{r=0}^{2b-1} \frac{t^{2b+m}(-\zeta)^m}{2^b b! m! (2b+m-r)}$, where the internal sum over r can be written as the difference $H_{2b+m} - H_m$ between two harmonic numbers. Writing $\frac{1}{2^b b!}$ as $P_{2b}(0)$ as well as $P_{2b-1}(0) = 0$ using Corollary 3.7, we deduce that

$$\Omega_l(\zeta) = (-1)^l \sum_{\nu=1}^{l} P_\nu(0)(H_l - H_{l-\nu}) \frac{\zeta^{l-\nu}}{(l-\nu)!} = (-1)^l \left(l - \tfrac{1}{2} \right) \frac{\zeta^{l-2}}{l!} + O(\zeta^{l-4})$$

is a rational polynomial of degree $l - 2$ and parity $(-1)^l$ every $l \in \mathbb{N}$. We shall later also need, for $k \in \mathbb{N}$, the rational, $(-1)^k$-symmetric, degree $k - 2$ polynomial

$$\tilde{\Omega}_k(\eta) := (-i)^k \Omega_k(i\eta) = (-1)^k \sum_{\nu=1}^{k} \frac{\mathrm{He}_\nu(0)}{\nu!} (H_k - H_{k-\nu}) \frac{\eta^{k-\nu}}{(k-\nu)!}.$$

3.4   Singular Schwartz functions

For every $\nu \in \mathbb{Z}$ and $\xi \in \mathbb{R}$, we shall now define

$$(3.17) \qquad h_\nu(\xi) := P_\nu(\xi) e(\xi) + Q_\nu(\xi) g(\xi),$$

which has the following property generalizing equation (3.5).

Proposition 3.14 *For any $v \in \mathbb{Z}$, the function h_v has parity $(-1)^{v-1}$, and we have*

$$\frac{d}{d\xi}h_v(\xi) = h_{v-1}(\xi) - \sqrt{2\pi} \cdot P_v(0) \cdot \delta(\xi).$$

Proof The parity follows from that of the polynomials P_v and Q_v from Corollary 3.2. The second claim follows from equation (3.5), the equality $g'(\xi) = -\xi g(\xi)$, and the relation from equation (3.6). This proves the proposition. ∎

Remark 3.15 The decay of g and e from equations (3.1) and (3.4) imply that

$$|h_v(\xi)| = o_{\varepsilon,v}\left(e^{-(1-\varepsilon)\xi^2/2}\right) \quad \text{as} \quad |\xi| \to \infty \qquad \text{for} \quad v \in \mathbb{Z} \quad \text{and} \quad \varepsilon > 0.$$

Lemma 3.16 *For any $v \in \mathbb{Z}$ and $\xi > 0$, we have the equalities $vh_v(\xi) - \xi h_{v-1}(\xi) = h_{v-2}(\xi)$ and $\xi^3 \frac{d}{d\xi}\frac{h_v(\xi)}{\xi^v} = -\frac{h_{v-2}(\xi)}{\xi^{v-2}}$.*

Proof The first equality is a direct consequence of Remarks 3.6 and 3.8, and the second one follows from the first via Proposition 3.14. This proves the lemma. ∎

Since h_v are in L^1 with exponential decay, their Fourier transforms $\widehat{h_v}$ should be bounded and C^∞. However, for $v \geq 0$, h_v is not C^∞, hence not a Schwartz function, and thus $\widehat{h_v}$ need not be in L^1. The explicit formula is given in the following result.

Proposition 3.17 *For every $v \geq -1$ and $t \in \mathbb{R}$, we have the equality*

$$\widehat{h_v}(t) := \int_{-\infty}^{\infty} h_v(\xi)\mathbf{e}(-\xi t)d\xi = \sqrt{2\pi}\left(\frac{g(2\pi t)}{(2\pi i t)^{v+1}} - \sum_{r=0}^{v} \frac{P_r(0)}{(2\pi i t)^{v-r+1}}\right)$$

$$= \sqrt{2\pi} \sum_{r=v+1}^{\infty} \frac{P_r(0)}{(2\pi i t)^{v-r+1}}.$$

In particular, $\widehat{h_v}$ is bounded and C^∞.

Proof The case $v = -1$ is just the Fourier transform of the Gaussian, combined with equation (3.8) and Corollary 3.7. Now, applying the Fourier transform to Proposition 3.14 yields the equality $2\pi i t \cdot \widehat{h_v}(t) = \widehat{h_{v-1}}(t) - \sqrt{2\pi} \cdot P_v(0)$ for every $v \in \mathbb{Z}$. The general formula follows by induction on v (to both sides), and implies the boundedness and C^∞ properties. This proves the proposition. ∎

For two indices κ and v in $\mathbb{Z}$, we define the function

$$(3.18) \qquad \varphi_{\kappa,v}(\lambda, z) := \frac{(\lambda, Z(z))^\kappa}{(2\pi)^{(v+1)/2}} h_v\left(\sqrt{2\pi}(\lambda, Z^\perp(z))\right),$$

with $\lambda \in V_\mathbb{R}$ and $z \in \mathcal{H}$ such that $(\lambda, Z^\perp(z)) \neq 0$. For $\kappa < 0$, we also impose the condition $(\lambda, Z(z)) \neq 0$. These functions decay like Schwartz functions by Remark 3.15. But near points where $(\lambda, Z^\perp(z))$ or $(\lambda, Z(z))$ vanishes, they may become discontinuous. They are therefore "singular" Schwartz functions. The reason for introducing them will be clear in Proposition 3.18.

The parity from Proposition 3.14 implies that

$$(3.19) \qquad \varphi_{\kappa,v}(-\lambda, z) = (-1)^{\kappa+v+1}\varphi_{\kappa,v}(\lambda, z) \quad \text{for every} \quad \lambda \in V_\mathbb{R} \quad \text{and} \quad z \in \mathcal{H},$$

and the behavior of $Z(z)$ and $Z^\perp(z)$ under the action of $\mathrm{SL}_2(\mathbb{R})$ implies that the function $\varphi_{\kappa,\nu}(\lambda, z)$ from equation (3.18) has the modularity property

$$(3.20) \quad \varphi_{\kappa,\nu}(\gamma\lambda, \gamma z) = j(\gamma, z)^{-2\kappa}\varphi_{\kappa,\nu}(\lambda, z) \quad \text{for} \quad \lambda \in V_{\mathbb{R}},\ z \in \mathcal{H}, \text{ and } \gamma \in \mathrm{SL}_2(\mathbb{R}).$$

Note that for all $k \in \mathbb{N}$, the functions from equations (2.19) and (3.18), and the expansions from equations (2.20) and (2.21), are related by

$$(3.21)$$
$$\tilde{\varphi}_k(\lambda; \tau, z) = \nu^{-k/2}q_\tau^{Q(\lambda)}\varphi_{k,-1}\left(\sqrt{\nu}\lambda, z\right) \text{ and } \Theta_{k,L,h}(\tau, z) = \nu^{\frac{1-k}{2}}\sum_{m\in\mathbb{Z}+Q(h)}\sum_{\lambda\in L_{m,h}}\varphi_{k,-1}\left(\sqrt{\nu}\lambda, z\right)q_\tau^m,$$

since h_{-1} is just the Gaussian g. Moreover, since $Z(z) - yZ^\perp(z) = \frac{iy}{\sqrt{N}}\left(\begin{smallmatrix} 1 & -2z \\ 0 & 1 \end{smallmatrix}\right)$, we can write

$$(3.22)$$
$$\varphi_{\kappa,\nu}(\lambda, z) = \frac{y^\kappa(\xi + i\eta)^\kappa}{(2\pi)^{(\kappa+\nu+1)/2}}h_\nu(\xi) \quad \text{with} \quad \xi = \sqrt{2\pi}\left(\lambda, Z^\perp(z)\right) \quad \text{and} \quad \eta = \sqrt{\frac{2\pi}{N}}\left(\lambda, \left(\begin{smallmatrix} 1 & -2z \\ 0 & 1 \end{smallmatrix}\right)\right),$$

and prove the following result.

Proposition 3.18 *Take κ and ν in $\mathbb{Z}$ as well as an element $0 \neq \lambda \in V_{\mathbb{R}}$. Then we have*

$$-L_z\varphi_{\kappa,\nu}(\lambda, z) = \varphi_{\kappa+1,\nu-1}(\lambda, z)$$

at every point $z \in \mathcal{H}$ such that $\left(\lambda, Z^\perp(z)\right) \neq 0$, where if $\kappa < 0$, then we assume that z must also satisfy $\left(\lambda, Z(z)\right) \neq 0$.

Proof Write $\varphi_{\kappa,\nu}(\lambda, z)$ as in equation (3.22), and then simple calculations give that $L_z y = y^2$, $L_z\xi = -y(\xi + i\eta)$, and $L_z\eta = 0$ in these parameters. Combining this with Proposition 3.14 produces the desired result. This proves the proposition. ∎

Note that the assumption $\left(\lambda, Z(z)\right) \neq 0$ is always satisfied when $Q(\lambda) \geq 0$ (and $\lambda \neq 0$), and if $Q(\lambda) < 0$, then it holds for every $z \in \mathcal{H}$ except a single point, in which $Z^\perp(z)$ is a scalar multiple of λ.

3.5 Fourier transforms

We shall also need the Fourier expansion of the following generalization of the functions h_ν. Take $\kappa \in \mathbb{Z}$, and then for $\xi \in \mathbb{R}$ and $\eta \in \mathbb{R}$, we define

$$(3.23) \quad \begin{aligned} \mathrm{e}_\kappa(\xi; \eta) &:= (\xi + i\eta)^{\kappa-1}h_0(\xi) = (\xi + i\eta)^{\kappa-1}\mathrm{e}(\xi) \quad \text{and} \\ \mathrm{g}_\kappa(\xi; \eta) &:= (\xi + i\eta)^{\kappa-1}h_{-1}(\xi) = (\xi + i\eta)^{\kappa-1}\mathrm{g}(\xi). \end{aligned}$$

For any $l \in \mathbb{N}$, we use equation (3.12) and the polynomials from Lemma 3.9 and define

$$(3.24)$$
$$\mathrm{g}_{\kappa,l}(\xi; \eta) := P_l(i\eta)\mathrm{e}_\kappa(\xi; \eta) + \Pi_l(\xi, i\eta)\mathrm{g}_\kappa(\xi; \eta) = \sum_{v=0}^{l}\frac{(-1)^v}{(l-v)!}(\xi + i\eta)^{\kappa+l-v-1}h_v(\xi).$$

By applying equation (3.5) and the derivative formula from that lemma (or Proposition 3.14 with equation (3.12)), we get

$$(3.25) \qquad \partial_\xi \mathbf{g}_{\kappa+1,l}(\xi;\eta) = \kappa \mathbf{g}_{\kappa,l}(\xi;\eta) + \frac{g_{\kappa+l+1}(\xi;\eta)}{l!} - \sqrt{2\pi}(i\eta)^\kappa P_l(i\eta)\delta(\xi).$$

We assume that $\eta \neq 0$ in case $\kappa \leq 0$, and then the expression from equations (3.23) and (3.24) are L^1 as functions of ξ. Let thus $\widehat{e_\kappa}$, $\widehat{g_\kappa}$, and $\widehat{\mathbf{g}_{\kappa,l}}$ denote the Fourier transform of e_κ, g_κ, and $\mathbf{g}_{\kappa,l}$ in ξ, respectively. The results for $\eta < 0$ can be obtained from those with $\eta > 0$ because we have

$$\mathbf{g}_{\kappa,l}(\xi;-\eta) = (-1)^{\kappa+l}\mathbf{g}_{\kappa,l}(-\xi;\eta) \ \text{ and hence } \ \widehat{\mathbf{g}_{\kappa,l}}(t;-\eta) = (-1)^{\kappa+l}\overline{\widehat{\mathbf{g}_{\kappa,l}}}(-t;\eta).$$

For analyzing the behavior of $\widehat{\mathbf{g}_{\kappa,l}}(t;\eta)$, we shall need additional special functions. For every $v \in \mathbb{Z}$, $j \in \mathbb{N}$, $t \in \mathbb{R}$, and $\eta > 0$, we define

$$(3.26)$$
$$I_{v,j}(\eta,t) := \int_{-t}^\infty \frac{(w+t)^j}{j!} e^{-\eta w}\left(e^{-w^2/2} - \sum_{\mu=0}^v \frac{\mathrm{He}_\mu(0)}{\mu!}w^\mu\right)\frac{dw}{w^{v+1}}, \quad I_v(\eta) := I_{v,0}(\eta,0).$$

For $v \leq -1$, this is defined for all $\eta \in \mathbb{R}$. It is easy to check that

$$(3.27)$$
$$\partial_\eta I_{v,j}(\eta,t) = I_{v-1,j}(\eta,t) + \frac{\mathrm{He}_v(0)}{v!j!}\int_{-t}^\infty (w+t)^j e^{-\eta w}dw = \frac{\mathrm{He}_v(0)}{v!\eta^{j+1}}e^{\eta t} - I_{v-1,j}(\eta,t)$$

for all v and j (note the difference in the summation over μ in $I_{v,j}$ and in $I_{v-1,j}$) and thus in particular $I_v'(\eta) = \frac{\mathrm{He}_v(0)}{v!\eta} - I_{v-1}(\eta)$, while equation (3.1) and simple differentiation give

$$(3.28)$$
$$I_{-1}(\eta) = -e^{\eta^2/2}e(\eta) \qquad \text{and} \qquad \partial_t I_{v,j}(\eta,t) = \begin{cases} I_{v,j-1}(\eta,t), & \text{if } j \geq 1, \\ e^{\eta t}g(t), & \text{if } j = 0 \text{ and } v = -1. \end{cases}$$

Using the functions from equation (3.26), we can now evaluate the Fourier transforms $\widehat{e_\kappa}$ and $\widehat{g_\kappa}$ of the functions from equation (3.23) as follows.

Lemma 3.19 For $\kappa \geq 1$, we have the equalities

$$\widehat{e_\kappa}(t;\eta) = \sqrt{2\pi}\sum_{\mu=0}^{\kappa-1}\binom{\kappa-1}{\mu}\frac{(i\eta)^{\kappa-1-\mu}}{(-2\pi i)^\mu}\cdot\left(\frac{d}{dt}\right)^\mu\left(\frac{g(2\pi t)-1}{2\pi i t}\right) \qquad \text{and}$$

$$\widehat{g_\kappa}(t;\eta) = \sqrt{2\pi}(-i)^{\kappa-1}\mathrm{He}_{\kappa-1}(2\pi t - \eta)g(2\pi t).$$

On the other hand, when $\kappa \leq 0$ and $\eta > 0$, we get

$$\widehat{e_\kappa}(t;\eta) = \sqrt{2\pi}\cdot i^\kappa e^{-2\pi\eta t}I_{0,-\kappa}(\eta,2\pi t) \quad \text{and} \quad \widehat{g_\kappa}(t;\eta) = \sqrt{2\pi}\cdot i^{\kappa-1}e^{-2\pi\eta t}I_{-1,-\kappa}(\eta,2\pi t).$$

Proof The results for $\kappa \geq 1$ follow from applying the differential operator $\left(i\eta - \frac{\partial_t}{2\pi i}\right)^{\kappa-1}$ to the cases $v = 0$ or $v = -1$ of Proposition 3.17, respectively, combined

with equation (3.11) for the latter. For the case $\kappa \leq 0$, we recall that

$$\int_{-\infty}^{\infty} (\xi + i\eta)^{\kappa-1} \mathbf{e}(-\xi t)\, d\xi = \begin{cases} 2\pi \cdot i^{\kappa-1} \dfrac{(2\pi t)^{|\kappa|} e^{-2\pi\eta t}}{|\kappa|!}, & \text{if } t > 0, \\[2mm] 0, & \text{if } t < 0, \end{cases}$$

and we can express the Fourier transform of the product as the convolution of the Fourier transforms. This proves the lemma. ∎

Remark 3.20 Consider the function $\xi \mapsto (\xi + i\eta)^{\kappa-1} h_\nu(\xi)$, generalizing equation (3.23) to any ν. Its Fourier transform can be evaluated via the proof of Lemma 3.19, using the full Proposition 3.17 and Corollary 3.7. When $\kappa \leq 0$, this gives just $\sqrt{2\pi} i^{\kappa+\nu} I_{\nu,\kappa}(\eta, t)$. For $\kappa = 0$ and $t = 0$, this reduces to the equality $\int_{-\infty}^{\infty} \frac{h_\nu(\xi)}{\xi+i\eta}\, d\xi = \sqrt{2\pi} i^\nu I_\nu(\eta)$.

We can now deduce the following useful property.

Lemma 3.21 For every $\nu \in \mathbb{Z}$ and $\eta > 0$, we have the equality $\eta^3 \frac{d}{d\eta} \frac{I_\nu(\eta)}{\eta^\nu} = \frac{I_{\nu-2}(\eta)}{\eta^{\nu-2}}$.

Proof Using equation (3.27), Remark 3.20, and the first equality in Lemma 3.16, we get

$$\begin{aligned}
\eta^{1+\nu} \frac{d}{d\eta} \frac{I_\nu(\eta)}{\eta^\nu} &= \frac{\mathrm{He}_\nu(0)}{\nu!} - \big(\eta I_{\nu-1}(\eta) + \nu I_\nu(\eta)\big) \\
&= \frac{\mathrm{He}_\nu(0)}{\nu!} - \frac{(-i)^\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\nu h_\nu(\xi) + i\eta h_{\nu-1}(\xi)}{\xi + i\eta}\, d\xi \\
&= \frac{\mathrm{He}_\nu(0)}{\nu!} - \frac{(-i)^\nu}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{h_{\nu-2}(\xi)}{\xi + i\eta} + h_{\nu-1}(\xi) \right) d\xi.
\end{aligned}$$

The first integral produces the desired result $I_{\nu-2}(\eta)$ by Remark 3.20 again, and integrating Proposition 3.14 evaluates, via Remark 3.15, the second one as $-(-1)^\nu P_\nu(0)$. As this cancels with the first term by Corollary 3.7, this proves the lemma. ∎

One can also show, using the polynomials from equation (3.30), that

(3.29)
$$I_\nu = \widetilde{P}_\nu I_0 - \widetilde{Q}_\nu I_{-1} + \hat{\Omega}_\nu, \quad \text{with a polynomial } \hat{\Omega}_\nu \text{ of degree } |\nu| - 2 \text{ and parity } (-1)^\nu.$$

We can also evaluate the Fourier transform of $\mathbf{g}_{\kappa,l}$ at $t = 0$.

Proposition 3.22 Given any $\kappa \in \mathbb{Z}$, $l \in \mathbb{N}$, and $\eta > 0$, we have

$$\widehat{\mathbf{g}_{\kappa,l}}(0;\eta) = \begin{cases} -\dfrac{\sqrt{2\pi} i^{\kappa+l}}{\kappa \cdot l!} \Big(\mathrm{He}_{\kappa+l}(\eta) - \eta^\kappa \, \mathrm{He}_l(\eta) \Big), & \text{if } \kappa \neq 0 \text{ and } \kappa + l \geq 0, \\[3mm] \sqrt{2\pi} \cdot (-i)^l \big(I_l(\eta) - \tilde{\Omega}_l(\eta) \big), & \text{if } \kappa = 0, \end{cases}$$

where $I_{\nu,j}$ and $\tilde{\Omega}_k$ are defined in equation (3.26) and Remark 3.13, respectively.

Proof When $\kappa \neq 0$ and $\eta > 0$, we can integrate equation (3.25) (using the fact that Remark 3.15 extends to the functions $\mathbf{g}_{\kappa,l}$) to obtain

$$\widehat{\mathbf{g}_{\kappa,l}}(0;\eta) = \int_{-\infty}^{\infty} \mathbf{g}_{\kappa,l}(\xi;\eta)d\xi = \frac{1}{\kappa}\int_{-\infty}^{\infty}\left(-\frac{\mathbf{g}_{\kappa+l+1}(\xi;\eta)}{l!} + \sqrt{2\pi}(i\eta)^\kappa P_l(i\eta)\delta(\xi)\right)d\xi$$

$$= \frac{1}{\kappa}\left(-\frac{\widehat{\mathbf{g}_{\kappa+l+1}}(0;\eta)}{l!} + \sqrt{2\pi}P_l(i\eta)(i\eta)^\kappa\right),$$

which gives the desired value by Lemma 3.19 and Corollary 3.7.

When $\kappa = 0$, we evaluate directly, where the formula from Remark 3.20 gives

$$\widehat{\mathbf{g}_{0,l}}(0;\eta) = \sqrt{2\pi}(-i)^l I_l(\eta) - (-1)^l \int_{-\infty}^{\infty} \frac{h_l(\xi) - (-1)^l(P_l(i\eta)\mathsf{e}(\xi) + \Pi_l(\xi,i\eta)\mathsf{g}(\xi))}{\xi + i\eta}d\xi.$$

As equation (3.17) transforms the latter integral into $(-1)^l\sqrt{2\pi}\Omega_l(i\eta)$ via equation (3.15), we get the desired value by Remark 3.13. This proves the proposition. ∎

3.6 Asymptotic estimates

For determining the asymptotic behavior of the functions $I_{v,j}$ and the Fourier transforms $\widehat{\mathbf{g}_{\kappa,l}}$, we shall also need the following special functions. Like the definition of $\tilde{\Omega}_k$ in Remark 3.13, the polynomials P_v and Q_v for $v \in \mathbb{Z}$ given in equation (3.6) and Remark 3.8 have the modifications

$$(3.30) \qquad \widetilde{P}_v(\eta) := i^v P_v(i\eta) \qquad \text{and} \qquad \widetilde{Q}_v(\eta) := i^{v-1}Q_v(i\eta),$$

of the same parities as P_v and Q_v. Using these, we define the functions

$$(3.31)$$

$$J_{-1}(\eta) := \mathsf{g}(i\eta) = e^{\eta^2/2}, \quad J_0(\eta) := -\int_0^\eta e^{r^2/2}dr = \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\eta s} - e^{\eta s}}{s}\mathsf{g}(s)ds,$$

$$\text{and} \qquad J_v(\eta) := \tilde{P}_v(\eta)J_0(\eta) - \tilde{Q}_v(\eta)J_{-1}(\eta) \qquad \text{for} \qquad v \in \mathbb{Z}$$

(the two expressions for $J_0(\eta)$ are the same because they vanish at $\eta = 0$ and have the same derivative, and since $\widetilde{P}_0 = \widetilde{Q}_{-1} = 1$ and $\widetilde{P}_{-1} = \widetilde{Q}_0 = 0$, the two definitions for $v = 0$ and for $v = -1$ coincide). Since the polynomials from equation (3.30) satisfy identities analogous to those from equation (3.6) and Remark 3.6, the latter of which yields the equality $\eta J_v(\eta) + \eta J_{v-1}(\eta) = -J_{v-2}(\eta)$ for every η and v, evaluating the derivatives of J_0 and J_{-1} implies, for all $v \in \mathbb{Z}$, the relations

$$(3.32) \qquad J_v' = -J_{v-1}, \quad J_v(-\eta) = (-1)^{v-1}J_v(\eta), \quad \text{and} \quad \eta^3\frac{d}{d\eta}\frac{J_v(\eta)}{\eta^v} = \frac{J_{v-2}(\eta)}{\eta^{v-2}}.$$

The following estimates will be helpful when we evaluate sums of Fourier transforms later.

Lemma 3.23 *For any $\varepsilon > 0$ and fixed $\eta \in \mathbb{R}$, we have the asymptotic growth*

$$I_{-1,0}(\eta,t) = \begin{cases} \sqrt{2\pi}J_{-1}(\eta) + o_{\eta,\varepsilon}(e^{-(1-\varepsilon)t^2/2}), & t \to \infty, \\ o_{\eta,\varepsilon}(e^{-(1-\varepsilon)t^2/2}), & t \to -\infty, \end{cases}$$

with $J_{-1}(\eta)$ from equation (3.31). More generally, given $j \in \mathbb{N}$, we have

$$I_{-1,j}(\eta,t) = P_j(t-\eta)I_{-1,0}(\eta,t) + o_{j,\eta,\varepsilon}(e^{-(1-\varepsilon)t^2/2}) \qquad \text{as} \qquad t \to \pm\infty.$$

Moreover, given $\eta > 0$ and such ε, and with J_0 from equation (3.31), *we get*

$$\frac{j!}{t^j}I_{0,j}(\eta, t) = -(-1)^j j!\Gamma(-j, -\eta t) + \begin{cases} \sqrt{2\pi}J_0(\eta) + O_{j,\eta}\left(\frac{1}{t}\right), & t \to \infty, \\ o_{j,\eta,\varepsilon}\left(e^{-(1-\varepsilon)t^2/2}\right), & t \to -\infty. \end{cases}$$

Proof Using equation (3.4) and the value $\sqrt{2\pi}$ of $\int_{-\infty}^{\infty} g(\xi)d\xi$, the first equation follows from

$$I_{-1,0}(\eta, t) = \int_{-t}^{\infty} e^{-\eta w - w^2/2}dw = e^{\eta^2/2}\int_{\eta-t}^{\infty} g(s)ds = \begin{cases} e^{\eta^2/2}\left(\sqrt{2\pi} - e(t-\eta)\right), & t > \eta, \\ -e^{\eta^2/2}e(\eta - t), & t < \eta, \end{cases}$$

and Remark 3.15. Next, using the definition in equation (3.26) and simple algebra, we get, for every $\eta \in \mathbb{R}$ and $j \in \mathbb{N}$, the equality

$$\delta_{j,0}e^{\eta t - t^2/2} = -\int_{-t}^{\infty}\frac{d}{dw}\left(\frac{(w+t)^j}{j!}e^{-\eta w - w^2/2}\right)dw = (j+1)I_{-1,j+1} - (t-\eta)I_{-1,j} - (1-\delta_{j,0})I_{-1,j-1}.$$

Since the left-hand side times any polynomial is $o_\varepsilon\left(e^{-(1-\varepsilon)t^2/2}\right)$ as $t \to \pm\infty$, a simple induction on j combines with Remark 3.6 to prove the second relation.

Now, suppose $t \neq 0$ and $\eta > 0$. For $j = 0$, we evaluate

$$I_{0,0}(\eta, t) = \int_{-t}^{\infty} e^{-\eta w}\frac{g(w) - 1}{w}dw = \int_{-t}^{|t|} e^{-\eta w}\frac{g(w)}{w}dw + \int_{|t|}^{\infty} e^{-\eta w}\frac{g(w)}{w}dw - \Gamma(0, -\eta t).$$

As $t \to -\infty$, the first term above vanishes and the second term is $o_{\eta,\varepsilon}\left(e^{-(1-\varepsilon)t^2/2}\right)$ for any fixed $\eta \in \mathbb{R}$. As $t \to \infty$, the second term behaves the same, whereas using equation (3.31), we see that the first term contributes

$$\int_{-t}^{t} e^{-\eta w}\frac{g(w)}{w}dw = \int_{0}^{\infty}\frac{e^{-\eta w} - e^{\eta w}}{w}g(w)dw - \int_{t}^{\infty}\frac{e^{-\eta w} - e^{\eta w}}{w}g(w)dw$$

$$= \sqrt{2\pi}J_0(\eta) + o_{\eta,\varepsilon}\left(e^{-(1-\varepsilon)t^2/2}\right).$$

This proves the third equality for $j = 0$. When $j \geq 1$, we can write $\frac{j!}{t^j}I_{0,j}(\eta, t)$ as

$$\frac{1}{t^j}\int_{-t}^{\infty}\frac{(w+t)^j - t^j}{w}e^{-\eta w - w^2/2}dw - \frac{e^{\eta t}}{t^j}\int_{0}^{\infty}\frac{s^j - t^j}{s - t}e^{-\eta s}ds + \int_{-t}^{\infty} e^{-\eta w}\frac{g(w) - 1}{w}dw.$$

After expanding $(w + t)^j$ binomially, the first term is seen to be $o_{j,\eta,\varepsilon}\left(e^{-(1-\varepsilon)t^2/2}\right)$ as $t \to -\infty$ and $O_{j,\eta}\left(\frac{1}{t}\right)$ as $t \to \infty$. Expanding $\frac{s^j - s^j}{s - t}$ and applying equation (3.2), the second term becomes $-(-1)^j j!\Gamma(-j, -\eta t) + \Gamma(0, -\eta t)$, and the third term is just the expression for $j = 0$. Putting everything together proves the last desired equality. This proves the lemma. ∎

In fact, the first term in the last equation in the proof of Lemma 3.23 can easily be evaluated explicitly up to $o_{j,\eta,\varepsilon}\left(e^{-(1-\varepsilon)t^2/2}\right)$ also when $t \to \infty$, but the estimate $O_{j,\eta}\left(\frac{1}{t}\right)$ will be sufficient for our purposes.

Remark 3.24 The generating series $\sum_{j=0}^{\infty} I_{-1,j}(0,0)X^j$ of the constants $\{I_{-1,j}(0,0)\}_{j=0}^{\infty}$ is

$$\int_0^{\infty} e^{wX-w^2/2}dw = e^{X^2/2}\int_0^{\infty} e^{-(w-X)^2/2}dw = e^{X^2/2}\left(\int_0^{\infty} e^{-w^2/2}dw + \int_0^X e^{-w^2/2}dw\right)$$

(by symmetry), which equals $\Upsilon(0,X) + \sqrt{\pi/2}\Psi(0,X)$ by Theorem 3.4. It therefore follows from equations (3.7) and (3.17) and the parity from Proposition 3.14 that

$$I_{-1,j}(0,0) = Q_j(0) + \sqrt{\pi/2}P_j(0) = \lim_{\xi\to 0^-} h_j(\xi) = (-1)^{j-1}\lim_{\xi\to 0^+} h_j(\xi) \quad \text{for} \quad j \in \mathbb{N}.$$

We can now state the asymptotic expansion of $\widehat{\mathbf{g}_{\kappa,l}}(t;\eta)$.

Proposition 3.25 *Take $\kappa \in \mathbb{Z}$, $l \in \mathbb{N}$, and $\eta \in \mathbb{R}$, with $\eta > 0$ in case $\kappa \leq 0$. We then have*

$$\widehat{\mathbf{g}_{\kappa,l}}(t;\eta) = -\sqrt{2\pi}\frac{i^{\kappa+l}\operatorname{He}_l(\eta)}{l!(-2\pi t)^{\kappa}}e^{-2\pi\eta t}\Gamma(\kappa,-2\pi\eta t) + o_{\kappa,l,\eta,\varepsilon}(e^{-2\pi^2(1-\varepsilon)t^2}),$$

an expression holding as $t \to -\infty$ as well as in the limit $t \to \infty$ when $\kappa \geq 1$. On the other hand, if $\kappa \leq 0$ and $\eta > 0$, then the limit as $t \to \infty$ is given by the same formula plus the term

$$\frac{2\pi(-i)^{\kappa+l}}{|\kappa|!(-2\pi t)^{\kappa}}e^{-2\pi\eta t}\left[J_l(\eta) + O_{\kappa,l,\eta}\left(\tfrac{1}{t}\right)\right],$$

where $J_l(\eta)$ is the function defined in equation (3.31).

Proof Taking the Fourier transform of equation (3.24) allows us to write

$$\widehat{\mathbf{g}_{\kappa,l}}(t;\eta) = P_l(i\eta)\widehat{e_\kappa}(t;\eta) + \Pi_l\left(\tfrac{\partial_t}{-2\pi i}, i\eta\right)\widehat{g_\kappa}(t;\eta).$$

When $\kappa \geq 1$, we apply Lemma 3.19 and equation (3.2) to obtain that

$$\widehat{e_\kappa}(t;\eta) = -\sqrt{2\pi}\frac{e^{-2\pi\eta t}\Gamma(\kappa,-2\pi\eta t)}{(2\pi it)^{\kappa}} + o_{\kappa,l,\varepsilon}(\eta^{\kappa-1}e^{-(1-\varepsilon)t^2/2}) \qquad \text{as} \qquad t \to \pm\infty$$

and that $\Pi_l\left(\tfrac{\partial_t}{-2\pi i}, i\eta\right)\widehat{g_\kappa}(t;\eta)$ is bounded by the same error term. This establishes, via Corollary 3.7, the desired formula in this case. On the other hand, for $\kappa \leq 0$ and $\eta > 0$, we deduce from Lemma 3.19 that

$$\begin{aligned}
\widehat{\mathbf{g}_{\kappa,l}}(t;\eta) &= \sqrt{2\pi}\cdot i^{\kappa}e^{-2\pi\eta t}P_l(i\eta)I_{0,-\kappa}(\eta,2\pi t)\\
&\quad + \sqrt{2\pi}\cdot i^{\kappa-1}\Pi_l\left(\tfrac{\partial_t}{-2\pi i}, i\eta\right)e^{-2\pi\eta t}I_{-1,-\kappa}(\eta,2\pi t).
\end{aligned}$$

Lemma 3.23 now shows that, in the limit $t \to -\infty$, the first summand is

$$-\sqrt{2\pi}\cdot i^{\kappa}P_l(i\eta)\frac{e^{-2\pi\eta t}\Gamma(\kappa,-2\pi\eta t)}{(-2\pi t)^{\kappa}} + o_{\kappa,l,\varepsilon}(e^{-2\pi^2(1-\varepsilon)t^2})$$

and the second one goes into the error term, which yields the required formula also here by another application of Corollary 3.7. In the limit $t \to \infty$, we get the same contribution, but we need to consider the additional terms from Lemma 3.23 in that

limit. The term arising from the first summand is

$$(3.33) \qquad \frac{2\pi \cdot i^{\kappa} e^{-2\pi\eta t}}{|\kappa|!(2\pi t)^{\kappa}}\Big[P_l(i\eta)J_0(\eta) + O_{\kappa,l,\eta}\big(\tfrac{1}{t}\big)\Big].$$

For the one from the second summand, Remark 3.10 allows us to write

$$\Pi_l\big(\tfrac{\partial_t}{-2\pi i}, i\eta\big)e^{-2\pi\eta t}I_{-1,-\kappa}(\eta, 2\pi t)$$
$$= \Big[-Q_l(i\eta) + \tilde{\Pi}_l\big(\tfrac{\partial_t}{-2\pi i} + i\eta, i\eta\big)\Big]e^{-2\pi\eta t}I_{-1,-\kappa}(\eta, 2\pi t)$$
$$= e^{-2\pi\eta t}\Big[-Q_l(i\eta)I_{-1,-\kappa}(\eta, 2\pi t) + \tilde{\Pi}_l\big(\tfrac{\partial_t}{-2\pi i}, i\eta\big)I_{-1,-\kappa}(\eta, 2\pi t)\Big]$$

(by the action on that exponent). Lemma 3.23 and the fact that Corollary 3.2 gives the estimate $P_{|\kappa|}(2\pi t - \eta) = \frac{(2\pi t)^{|\kappa|}}{|\kappa|!}\big[1 + O_{\kappa,\eta}\big(\tfrac{1}{t}\big)\big]$ show that $\sqrt{2\pi} \cdot i^{\kappa-1}$ times the first summand here is

$$-\frac{2\pi \cdot i^{\kappa-1} e^{-2\pi\eta t}}{|\kappa|!(2\pi t)^{\kappa}}\Big[Q_l(i\eta)J_{-1}(\eta) + O_{\kappa,l,\eta}\big(\tfrac{1}{t}\big)\Big].$$

Equations (3.30) and (3.31) now show that this expression combines with the one from equation (3.33) to the asserted extra term, up to the required error term. Finally, equation (3.28) and the property of $\tilde{\Pi}_l$ from Remark 3.10 imply, via Lemma 3.23 again, that the expression involving that polynomial also goes into the error term. This proves the proposition. ∎

3.7 Lattice sums

To evaluate the constant term of the Shintani lift, we need to calculate certain lattice sums involving the function $\mathbf{g}_{\kappa,l}$ defined in equation (3.24). For $\kappa \in \mathbb{Z}$, $l \in \mathbb{N}$, $\eta \in \mathbb{R}$, a real number $v > 0$, and an element $\omega \in \mathbb{R}/\mathbb{Z}$, we consider the sum

$$(3.34) \qquad \mathbf{G}_{\kappa,l}(\omega; v, \eta) := \sum_{0 \neq \xi \in \mathbb{Z}+\omega} \mathbf{g}_{\kappa,l}(v\xi; \eta),$$

which converges absolutely by Remark 3.15 and defines a continuous function in η. We will be interested in its value at $\eta = 0$, denoted by

$$(3.35) \qquad \mathbf{G}_{\kappa,l}(\omega; v) := \mathbf{G}_{\kappa,l}(\omega; v, 0) = \lim_{\eta \to 0^+} \mathbf{G}_{\kappa,l}(\omega; v, \eta),$$

and its asymptotic expansion as $v \to 0^+$.

Remark 3.26 The function $\mathbf{G}_{\kappa,l}$ from equation (3.34), or its variant from equation (3.35), is defined as a lattice sum, and would naturally be expressible also via its Fourier series, which amounts to the Poisson Summation Formula. Indeed, the summed function $\mathbf{g}_{\kappa,l}$ is differentiable away from 0 with a strongly decaying derivative, making $\mathbf{G}_{\kappa,l}$ differentiable with respect to ω when $\omega \neq 0$. Therefore, the Fourier expansion of the lattice sum $\mathbf{G}_{\kappa,l}$ converges pointwise to it for every $\omega \neq 0$. Since the mth Fourier coefficient of $\mathbf{G}_{\kappa,l}$ (for $\eta \neq 0$) is, by simple integration, $\frac{1}{v}\widehat{\mathbf{g}_{\kappa,l}}\big(\tfrac{m}{v}; \eta\big)$, we obtain that the Poisson Summation Formula for $\mathbf{G}_{\kappa,l}$ is valid for $\omega \neq 0$. In fact, since the one-sided derivatives of $\mathbf{g}_{\kappa,l}$ also exist (provided that $\mathbf{g}_{\kappa,l}(0)$ is modified to have the limit value from the appropriate side), the Fourier series converges also at $\omega = 0$ to a value

that can be described (both Fourier series convergence statements follow, for example, from the Dirichlet–Dini Criterion). However, it is easier to consider the value at $\omega = 0$ in terms of equation (3.44).

For analyzing it, we first need to recall a few familiar functions. Let $\{B_\mu(\omega)\}_{\mu=0}^\infty$ (for $\omega \in \mathbb{R}$) be the *Bernoulli polynomials* defined as

$$\frac{te^{\omega t}}{e^t - 1} = \sum_{\mu=0}^\infty B_\mu(\omega) \frac{t^\mu}{\mu!}, \quad \text{so that in particular} \quad B_1(\omega) = \omega - \tfrac{1}{2},$$

and then $B_\mu := B_\mu(0)$ are the *Bernoulli numbers*. Moreover, one defines $\mathbb{B}_\mu$ to be the 1-periodic function that coincides with B_μ on the interval $(0,1)$, and whose value on the integers is 0 in case $\mu = 1$ and B_μ otherwise. Then $\mathbb{B}_\mu$ with $\mu \geq 2$ is continuous on $\mathbb{R}$ (and $\mathbb{B}_0$ is the constant function 1), and we have

(3.36)
$$\mathbb{B}_1(0) = 0 = \lim_{\omega \to 0^+} \mathbb{B}_1(\omega) + \tfrac{1}{2} = \lim_{\omega \to 0^-} \mathbb{B}_1(\omega) - \tfrac{1}{2} \quad \text{and} \quad \mathbb{B}_\mu(\omega) = -\sum_{0 \neq m \in \mathbb{Z}} \frac{\mu! \mathbf{e}(m\omega)}{(2\pi i m)^\mu},$$

the latter Fourier expansion being valid for every $\omega \in \mathbb{R}/\mathbb{Z}$ and $0 < \mu \in \mathbb{N}$ (this is essentially equations (13)–(15) in Section 1.13 of [EMOT]).

We also recall from Section 1.11 of [EMOT] the function

(3.37)
$$F(q, s) = \sum_{m=1}^\infty \frac{q^m}{m^s} \quad \text{for} \quad s \in \mathbb{C} \quad \text{and} \quad q \in \mathbb{C} \quad \text{with} \quad |q| < 1.$$

Since we shall use this function only when $s = -j$ for $j \in \mathbb{N}$, where $F(q, -j)$ is a polynomial in q divided by $(1 - q)^{j+1}$, the analytic continuation to any $q \in \mathbb{C} \backslash [1, \infty)$, and even to any $1 \neq q \in \mathbb{C}$, is immediate. Writing $q = \mathbf{e}(\omega)$ for $\omega \in \mathbb{R}$, the fact that $F(q, 0) = \frac{q}{1-q}$ combines with equation (15) of Section 1.11 of [EMOT] (for $j \geq 1$) to give, for all $j \in \mathbb{N}$, the expansion

(3.38)
$$F\big(\mathbf{e}(\omega), -j\big) = \frac{j!}{(-2\pi i \cdot \omega)^{j+1}} - \frac{B_{j+1} + \delta_{j,0}}{j+1} + O(\omega).$$

Another function to recall is the *polygamma function*, defined for $m \in \mathbb{N}$ and $z \in \mathbb{C}$ as

(3.39)
$$\psi^{(m)}(z) := \frac{d^{m+1}}{dz^{m+1}} \log \Gamma(z) = -\gamma \delta_{m,0} + (-1)^m m! \sum_{a=0}^\infty \left(\frac{\delta_{m,0}}{(a+1)^{m+1}} - \frac{1}{(a+z)^{m+1}} \right),$$

where $\delta_{m,0}$ is the Kronecker δ-symbol again.

Let $\tilde{\psi}^{(m)}$ be the 1-periodic function that coincides with $\psi^{(m)}$ from equation (3.39) on $(0, 1]$. Then, for $\kappa \in \mathbb{Z}$ and $\omega \in \mathbb{R}/\mathbb{Z}$, we define

(3.40)
$$\Phi_\kappa(\omega) := \begin{cases} -\mathbb{B}_\kappa(\omega)/\kappa, & \kappa \geq 1, \\ -\big[\tilde{\psi}^{(|\kappa|)}(-\omega) + (-1)^\kappa \tilde{\psi}^{(|\kappa|)}(\omega)\big]/2|\kappa|!, & \kappa \leq 0, \end{cases}$$

as well as

$$
(3.41) \qquad \Xi_\kappa(\omega) := \frac{(-2\pi i)^{1-\kappa}}{\sqrt{2\pi}} \begin{cases} F(\mathbf{e}(\omega),\kappa)/|\kappa|! + \delta_{\kappa,0}/2, & \kappa \le 0 \text{ and } \omega \notin \mathbb{Z}, \\ -\mathbb{B}_{1-\kappa}(0)/(1-\kappa)!, & \kappa \le 1 \text{ and } \omega \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}
$$

Note that, for all $\kappa \in 2\mathbb{Z}$, we have the equality

$$
(3.42) \qquad \Phi_\kappa(1) = \mathrm{CT}_{s=1-\kappa}\, \zeta(s).
$$

Recalling the notation H_n for the nth harmonic number, we shall also need the constant

$$
(3.43) \qquad C_l := \frac{\gamma + \log 2 - 2H_l + H_{\lfloor l/2\rfloor}}{2} = \frac{\gamma + \log 2}{2} - \sum_{a=1,\, 2\nmid a}^{l} \frac{1}{a}.
$$

We remark that it is easy to see, via the asymptotic $H_n = \log n + \gamma + o(1)$ as $n \to \infty$, that C_l from equation (3.43) grows as $-\frac{\log l}{2} + o(1)$ as $l \to \infty$.

The evaluation of the expression that we need is now carried out as follows.

Proposition 3.27　*Take $\kappa \in \mathbb{Z}$, $l \in \mathbb{N}$, $\eta \in \mathbb{R}$, $v > 0$, and $\omega \in \mathbb{R}/\mathbb{Z}$. Then the value of the expression $\mathbf{G}_{\kappa,l}(\omega;v)$ from equation (3.35) is*

$$
-\sqrt{2\pi}\, v^{\kappa-1}\big[P_l(0)\Phi_\kappa(\omega) + Q_l(0)\Xi_\kappa(\omega)\big]
$$
$$
+\begin{cases} -\dfrac{\sqrt{2\pi}\cdot i^{\kappa+l}\,\mathrm{He}_{\kappa+l}(0)}{v\kappa\cdot l!} + o_{\kappa,l,\varepsilon}\big(e^{-2\pi^2(1-\varepsilon)/v^2}\big), & \kappa \ge 1, \\[2mm] \dfrac{\sqrt{2\pi}}{v}\delta_{\kappa,0}P_l(0)(\log v + C_l) + O_{\kappa,l,\omega}(v^\kappa), & \kappa \le 0, \end{cases}
$$

with the O-notation concerning the behavior as $v \to 0^+$.

Proof　Consider first the case where $\kappa \ge 1$. For $\omega \ne 0$, we use the Poisson Summation Formula (justified by Remark 3.26), and applying Propositions 3.22 and 3.25 gives us

$$
\mathbf{G}_{\kappa,l}(\omega;v)
$$
$$
= \frac{1}{v}\Bigg(\widehat{\mathbf{g}_{\kappa,l}}(0;0) + \lim_{\eta\to0}\sum_{0\ne m\in\mathbb{Z}} \mathbf{e}(m\omega)\widehat{\mathbf{g}_{\kappa,l}}\Big(\tfrac{m}{v};\eta\Big)\Bigg)
$$
$$
= -\frac{\sqrt{2\pi}\,i^{\kappa+l}}{v\kappa}\Bigg(\frac{\mathrm{He}_{\kappa+l}(0)}{l!} + \frac{\kappa\,\mathrm{He}_l(0)}{l!}\lim_{\eta\to0}\sum_{0\ne m\in\mathbb{Z}} \frac{\mathbf{e}(m\omega)v^\kappa}{(-2\pi m)^\kappa}e^{-2\pi\eta m/v}\Gamma\Big(\kappa,-\tfrac{2\pi\eta m}{v}\Big)\Bigg)
$$
$$
= \frac{\sqrt{2\pi}}{v\kappa}\Bigg(\frac{-i^{\kappa+l}\,\mathrm{He}_{\kappa+l}(0)}{l!} - \kappa!P_l(0)\sum_{0\ne m\in\mathbb{Z}} \frac{\mathbf{e}(m\omega)v^\kappa}{(2\pi i m)^\kappa}\Bigg)
$$
$$
= \frac{\sqrt{2\pi}}{v\kappa}\Bigg(\frac{-i^{\kappa+l}\,\mathrm{He}_{\kappa+l}(0)}{l!} + P_l(0)v^\kappa\mathbb{B}_\kappa(\omega)\Bigg),
$$

via Corollary 3.7 and equation (3.36), up to an error term of $o_{\kappa,l,\varepsilon}\big(e^{-2\pi^2(1-\varepsilon)/v^2}\big)$. From the definition, it is also clear that

$$
(3.44) \qquad \mathbf{G}_{\kappa,l}(0;v) = \lim_{\omega\to0^+}\big(\mathbf{G}_{\kappa,l}(\omega+\mathbb{Z};v) - \mathbf{g}_{\kappa,l}(v\omega;0)\big).
$$

Equation (3.24) implies that $-\lim_{\xi\to0}\mathbf{g}_{\kappa,l}(\xi;0)$ vanishes for $\kappa>1$, and we have

$$-\lim_{\xi\to0^+}\mathbf{g}_{1,l}(\xi;0)=\lim_{\xi\to0^+}(-1)^{l+1}h_l(\xi)=I_{-1,l}(0,0)=P_l(0)\sqrt{\pi/2}+Q_l(0)$$

by Remark 3.24. Substituting these into equation (3.44), and applying equation (3.36) for $\kappa=1$, completes the proof for $\kappa\geq1$.

For $\kappa\leq0$, we will first consider the case where $l=1$, in which

$$\mathbf{g}_{\kappa,1}(\xi;0)=\xi^\kappa h_0(\xi)-\xi^{\kappa-1}h_1(\xi)=-\xi^{\kappa-1}g(\xi);\quad\text{hence}\quad\mathbf{G}_{\kappa,1}(\omega;v)=-\lim_{\eta\to0^+}\sum_{0\neq\xi\in\mathbb{Z}+\omega}g_\kappa(v\xi;\eta).$$

Assuming that $\omega\neq0$, we apply equation (3.35), the Poisson summation formula again, Lemmas 3.19 and 3.23, and equations (3.11) and equation (3.37) (with its analytic continuation), which compares $\mathbf{G}_{\kappa,1}(\omega;v)$ with

$$-\frac{\sqrt{2\pi}i^{\kappa-1}}{v}\left(\lim_{\eta\to0^+}I_{-1,-\kappa}(\eta,0)+\lim_{\eta\to0^+}\sum_{0\neq m\in\mathbb{Z}}e^{-2\pi\eta m/v}I_{-1,-\kappa}\left(\eta,\frac{2\pi m}{v}\right)\mathbf{e}(m\omega)\right)$$

$$=-\frac{\sqrt{2\pi}i^{\kappa-1}}{v}\left(I_{-1,-\kappa}(0,0)+\sum_{m=1}^\infty\left(P_{|\kappa|}\left(\frac{2\pi m}{v}\right)+o_{\kappa,\varepsilon}\left(e^{-2\pi^2(1-\varepsilon)m^2/v^2}\right)\right)\sqrt{2\pi}\mathbf{e}(m\omega)\right)$$

$$=-\frac{\sqrt{2\pi}i^{\kappa-1}}{v}\left(I_{-1,-\kappa}(0,0)+\sqrt{2\pi}\sum_{j=0}^{|\kappa|}\left(\frac{2\pi}{v}\right)^j\frac{P_{|\kappa|-j}(0)}{j!}F(\mathbf{e}(\omega),-j)\right)+o_{\kappa,\varepsilon}\left(e^{-2\pi^2(1-\varepsilon)/v^2}\right).$$

(3.45)

Now, the summands with $j<|\kappa|$ give $O_{\kappa,\omega}(v^\kappa)$, and the same applies to the first term when $\kappa\leq-1$. Since $I_{-1,0}(0,0)=\sqrt{\frac{\pi}{2}}$ by Remark 3.24, this is indeed the desired value, since $P_1(0)=0$. For $\omega=0$, we apply equation (3.44), where we have seen that the second term there is now $+\frac{g(v\omega)}{(v\omega)^{|\kappa|+1}}$. We expand the term $F(\mathbf{e}(\omega),\kappa)$ from equation (3.45) as in equation (3.38), and observe that the singularities in ω cancel with those of the Laurent expansion of $\frac{g(v\omega)}{(v\omega)^{|\kappa|+1}}$, which is $\sum_{\nu=0}^\infty i^\nu P_\nu(0)(v\omega)^{\nu+\kappa-1}$ by equation (3.8) and Corollary 3.7. Substituting into the limit from equation (3.44) yields

$$-\frac{\sqrt{2\pi}i^{\kappa-1}}{v}\left(I_{-1,-\kappa}(0,0)-\sqrt{2\pi}\sum_{j=0}^{|\kappa|}\left(\frac{2\pi}{v}\right)^j\frac{P_{|\kappa|-j}(0)}{(j+1)!}(B_{j+1}+\delta_{j,0})\right)+1+o_{\kappa,\varepsilon}\left(e^{-2\pi^2(1-\varepsilon)/v^2}\right),$$

where again the same terms (and the 1) go into the error term. Since for $\kappa=0$ the two terms cancel, the result follows also in this case.

We now consider the case $l=0$, where equations (3.24), (3.34), and (3.35) and the trick from the proof of Lemma 8.5 of [BFI] evaluate $\mathbf{G}_{\kappa,0}(\omega;v)$ as

(3.46)

$$\sum_{0\neq\xi\in\mathbb{Z}+\omega}\frac{\mathbf{e}(v\xi)}{(v\xi)^{|\kappa|+1}}=-\sum_{0\neq\xi\in\mathbb{Z}+\omega}\frac{\mathrm{sgn}(\xi)}{(v\xi)^{|\kappa|+1}}\mathrm{CT}_{s=0}(v|\xi|)^{-s}\left(\int_0^\infty e^{-w^2/2}w^s\,dw-\int_0^{v|\xi|}e^{-w^2/2}w^s\,dw\right).$$

Recalling the Hurwitz zeta function $\zeta(s,z):=\sum_{n=1}^\infty\frac{1}{(n+z)^s}$, the first term in equation (3.46) is the constant term at $s=0$ of $-\frac{2^{(s-1)/2}}{v^{|\kappa|+1+s}}\Gamma\left(\frac{s+1}{2}\right)$ times

$$\sum_{0 < \xi \in \mathbb{Z}+\omega} \frac{1}{\xi^{|\kappa|+1+s}} + \sum_{0 < \xi \in \mathbb{Z}-\omega} \frac{(-1)^{\kappa}}{\xi^{|\kappa|+1+s}} = \zeta(|\kappa|+1+s,\omega) + (-1)^{\kappa}\zeta(|\kappa|+1+s,-\omega),$$

where $\pm\omega$ here means the corresponding representatives in $(0,1]$. Since $\zeta(m+1+s,z)$ with $m \in \mathbb{N}$ expands as $\frac{\delta_{m,0}}{s} - \frac{(-1)^m \psi^{(m)}(z)}{m!} + O(s)$ (see equation (9) on Section 1.10 of [EMOT] for $m=0$ and just equation (3.39) for $m>0$), the Taylor expansion of the remaining functions and the value $-\gamma - 2\log 2$ of $\psi\left(\frac{1}{2}\right)$ produce the constant term

$$-\sqrt{2\pi}v^{\kappa-1}\left(-\frac{\tilde{\psi}^{(|\kappa|)}(-\omega) + (-1)^{\kappa}\tilde{\psi}^{(|\kappa|)}(\omega)}{2|\kappa|!} - \delta_{\kappa,0}\frac{\gamma + \log 2 + 2\log v}{2}\right),$$

which is the desired expression since $C_0 = \frac{\gamma+\log 2}{2}$ by equation (3.43). The second term in equation (3.46) becomes, after a simple substitution,

$$\mathrm{CT}_{s=0}\sum_{0 \neq \xi \in \mathbb{Z}+\omega}\int_0^1 \frac{g(v\rho\xi)}{(v\xi)^{|\kappa|}}\rho^s d\rho = -\mathrm{CT}_{s=0}\int_0^1 \mathbf{G}_{\kappa+1,1}(\omega;v\rho)\rho^{s-\kappa}d\rho.$$

For $\kappa \leq -1$, our expression for $\mathbf{G}_{\kappa+1,1}(\omega;v\rho)$ is $O_\kappa((v\rho)^\kappa)$; hence the integral converges at $s=0$ and is $O_\kappa(v^\kappa)$. When $\kappa=0$, we have $\mathbf{G}_{1,1}(\omega;v\rho) = -\frac{\sqrt{2\pi}}{v\rho} - \sqrt{2\pi}\Xi_1(\omega)$ up to rapidly decreasing functions, so that the integral is $-\frac{\sqrt{2\pi}}{vs}$ (with no constant term), again plus $O(1) = O_\kappa(v^\kappa)$. This proves the result for $l=0$ as well.

For general $l \in \mathbb{N}$, equation (3.24) and Remark 3.10 allow us to write

$$\mathbf{G}_{\kappa,l}(\omega;v) = \sum_{0 \neq \xi \in \mathbb{Z}+\omega}\mathbf{g}_{\kappa,l}(v\xi;0) = P_l(0)\sum_{0 \neq \xi \in \mathbb{Z}+\omega}\frac{\mathrm{e}(v\xi)}{(v\xi)^{|\kappa|+1}} + \sum_{0 \neq \xi \in \mathbb{Z}+\omega}\Pi_l(v\xi,0)\frac{g(v\xi)}{(v\xi)^{|\kappa|+1}}$$

$$= P_l(0)\mathbf{G}_{\kappa,0}(\omega;v) + Q_l(0)\mathbf{G}_{\kappa,1}(\omega;v) + \sum_{0 \neq \xi \in \mathbb{Z}+\omega}\frac{\tilde{\Pi}_l(v\xi,0)}{v\xi}\frac{g(v\xi)}{(v\xi)^{|\kappa|}},$$

where $\frac{\tilde{\Pi}_l(v\xi)}{v\xi}$ is a polynomial in $v\xi$. The first two terms now give the desired formula, up to $\frac{\sqrt{2\pi}}{v}P_l(0)(C_l - C_0)$ in case $\kappa=0$. When $\kappa \leq -1$, it suffices to view the third term as a linear combination of $\mathbf{G}_{j,1}(\omega;v)$ with $j \geq \kappa+1$, all of which are $O(v^{j-1})$ when $j \leq 0$ and $O\left(\frac{1}{v}\right)$ in case $j > 0$, since these are all $O(v^\kappa)$. For $\kappa=0$, we evaluate $\sum_{\xi \in \mathbb{Z}+\omega}\frac{\tilde{\Pi}_l(v\xi,0)}{v\xi}g(v\xi)$ using the Poisson Summation Formula, where all the Fourier terms with $m \neq 0$ give $o_{l,\varepsilon}(e^{-2\pi^2(1-\varepsilon)/v^2})$ once again. Finally, $\frac{1}{v}$ times the zeroth Fourier term is $\frac{\sqrt{2\pi}}{v}E_l(0)$ by equation (3.13), which is precisely the required expression by Lemma 3.11 and equation (3.43). This completes the proof of the proposition. ∎

4 Nearly holomorphic modular forms

In this section, we shall prove our result in the most general case, evaluating the Shintani lift $\mathcal{I}_{k,L}(\tau,f)$ from equation (2.24) for a nearly holomorphic modular form $f \in \widetilde{M}_{2k}^!(\Gamma)$. Each of the Fourier coefficients of the lift is evaluated separately, with the partition into types as is already seen in [ANS, BFI, DIT] and others, into negative indices, non-square positive indices, square positive indices, and vanishing indices. This requires the definition of several types of traces that show up in the calculations.

The reader who wishes to compare our formulae with those from [ANS] should consult Remark 2.8, due to the difference in conventions that is explained there.

4.1 Traces and regularizations

Recall that if $\lambda \in L^*$ satisfies $Q(\lambda) < 0$, then the stabilizer Γ_λ of λ in Γ is finite, and λ is a multiple of $Z^\perp(z_\lambda)$ for a unique $z_\lambda \in \mathcal{H}$. We then define, for every $k \in \mathbb{Z}$ and $f \in \mathcal{A}_0^!(\Gamma)$, the *trace*

$$(4.1) \qquad \mathrm{Tr}_\lambda^{(k)}(f) := \frac{\left[-\mathrm{sgn}\left(\lambda, Z^\perp(z_\lambda)\right)\right]^k}{|\Gamma_\lambda|} f(z_\lambda).$$

If $Q(\lambda) > 0$, then we recall the geodesic $c_\lambda \subseteq \mathcal{H}$ and its image $c(\lambda) \subseteq Y$ from equation (2.15), and that when λ is not split-hyperbolic, i.e., when $\iota(\lambda) = 0$ in the notation of equation (2.16), the latter is a closed geodesic inside the open modular curve Y. We can then define, for every $g \in \mathcal{A}_{2k}^!(\Gamma)$, the *trace*

$$(4.2) \qquad \mathrm{Tr}_\lambda(g) := \oint_{c(\lambda)} g(z)\left(\lambda, Z(z)\right)^{k-1} dz.$$

On the other hand, when λ is split-hyperbolic, i.e., when $\iota(\lambda) = 1$, the image $c(\lambda)$ of c_λ in Y is not compact, and if g grows toward the cusps, then the integral corresponding to that from equation (4.2) does not converge. We shall regularize it only for nearly holomorphic modular forms, i.e., for $g \in \widetilde{M}_{2k}^!(\Gamma)$. Then its Fourier expansion near the cusp associated with some $\ell \in \mathrm{Iso}(V)$ is given, in the coordinates from equation (2.12), by

$$(4.3)$$
$$g_\ell(z_\ell) := (g \mid_{2k} \sigma_\ell)(z_\ell) = \sum_{l=0}^p \frac{g_{\ell,l}(z_\ell)}{y_\ell^l} = \sum_{l=0}^p \sum_{n \in \mathbb{Z}} \frac{c_\ell(n,l) q_\ell^n}{y_\ell^l} = \sum_{l=0}^p \sum_{n \le 0} \frac{c_\ell(n,l) q_\ell^n}{y_\ell^l} + g_\ell^0(z_\ell),$$

where p is the depth of g, $c_\ell(n,l) = 0$ for all $0 \le l \le p$ when $n \ll 0$, and the latter decomposition is into the (finite) principal part and the cuspidal part g_ℓ^0. Recall our extension of the incomplete Gamma function in (3.2), and the truncated modular curve Y_T from equation (2.13) for any $T > 1$. We also set for n and κ in $\mathbb{Z}$, positive reals c and T, split-hyperbolic $\lambda \in V$, and $g \in \widetilde{M}_{2k}^!(\Gamma)$ with expansion as in equation (4.3) for $\ell = \ell_\lambda$, the quantities

$$(4.4) \qquad \phi_n(\kappa, T; r) := \begin{cases} \Gamma(\kappa, rnT)/(rn)^\kappa, & n \ne 0, \\ -T^\kappa/\kappa, & n = 0 \text{ and } \kappa \ne 0, \\ -\log T, & n = 0 \text{ and } \kappa = 0 \end{cases} \qquad \text{and}$$

$$\mathrm{Sing}_\lambda(g, T) := i^k (2\sqrt{Q(\lambda)})^{k-1} \sum_{l=0}^p \sum_{n \in \mathbb{Z}} c_{\ell_\lambda}(n, l) \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) \phi_n\left(k - l, T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right).$$

Note that when $n = 0$, ϕ_0 is independent of r, and we can then omit it from the notation. In addition, assuming that $f \in \widetilde{M}_{2k}^!(\Gamma)$ expands as in equation (4.3), the weight lowering property of the operator L_z implies that for every $v \in \mathbb{N}$ we can write $(L_z^v f)_\ell(z_\ell)$ as

(4.5)

$$(L_z^\nu f \mid_{2k-2\nu} \sigma_\ell)(z_\ell) = L_{z_\ell}^\nu f_\ell(z_\ell) = (-1)^\nu \sum_{l=\nu}^{p} \frac{l! f_{\ell,l}(z_\ell)}{(l-\nu)! y_\ell^l} = (-1)^\nu \sum_{n \in \mathbb{Z}} \sum_{l=\nu}^{p} \frac{l! c_\ell(n,l) q_\ell^n}{(l-\nu)! y_\ell^{l-\nu}}.$$

We can now define the *trace* to be

(4.6)

$$\mathrm{Tr}_\lambda(g) := \lim_{T \to \infty} \left(\int_{c(\lambda) \cap Y_T} g(z) \big(\lambda, Z(z)\big)^{k-1} dz + \mathrm{Sing}_\lambda(g, T) + (-1)^k \, \mathrm{Sing}_{-\lambda}(g, T) \right).$$

We now prove that this is a regularization of the required trace.

Proposition 4.1 *The quantity in the limit defining* $\mathrm{Tr}_\lambda(g)$ *in equation* (4.6) *exists and is independent of the choice of T when it is sufficiently large.*

Proof Since c_λ only intersects the cusps $\ell_{\pm\lambda}$, there exists some $R > 1$ such that for all $T > R$, we find that $c(\lambda) \cap Y_T \cong c_\lambda \cap \mathcal{H}_T$ is contained in $\mathcal{H}_R \cup \sigma_{\ell_\lambda} \mathcal{F}_T^{\alpha_{\ell_\lambda}} \cup \sigma_{\ell_{-\lambda}} \mathcal{F}_T^{\alpha_{\ell_{-\lambda}}}$. We thus obtain

$$\int_{c(\lambda) \cap Y_T} g(z)\big(\lambda, Z(z)\big)^{k-1} dz = \int_{c_\lambda \cap \mathcal{H}_R} \tilde{g}(z) dz + \int_{c_\lambda \cap \sigma_{\ell_\lambda} \mathcal{F}_T^{\alpha_{\ell_\lambda}} \setminus \mathcal{H}_R} \tilde{g}(z) dz$$

$$+ \int_{c_\lambda \cap \sigma_{\ell_{-\lambda}} \mathcal{F}_T^{\alpha_{\ell_{-\lambda}}} \setminus \mathcal{H}_R} \tilde{g}(z) dz$$

for every $T > R$, where we wrote $\tilde{g}(z)$ for $g(z)\big(\lambda, Z(z)\big)^{k-1}$. The first term is independent of T, and if we change, in the integral corresponding to $\pm\lambda$, the variable to z_ℓ for $\ell = \ell_{\pm\lambda}$ from equation (2.12), then it becomes

$$(\pm 1)^k \int_{r_{\pm\lambda}+iR}^{r_{\pm\lambda}+iT} g_\ell(z_\ell) \big(\sigma_\ell^{-1}(\pm\lambda), Z(z_\ell)\big)^{k-1} dz_\ell = (\pm i)^k \big(2\sqrt{Q(\lambda)}\big)^{k-1}$$

$$\times \int_R^T g_\ell(r_{\pm\lambda} + iy_\ell) y_\ell^{k-1} dy_\ell$$

via equation (2.17). This expression is a differentiable function of T, and equations (3.3) and (4.3) show that its derivative is minus that of $\mathrm{Sing}_\lambda(g, T)$ from equation (4.4). Hence, the expression from equation (4.6) is independent of T as long as $T > R$, and in particular $\mathrm{Tr}_\lambda(g)$ exists. This proves the proposition. ∎

Note that $\lim_{T \to \infty} \phi_n(\kappa, T; r) = 0$ for any $r > 0$ when $n > 0$, and the integral of the part $g_{\ell_{\pm\lambda}}^0$ from equation (4.3) converges as $T \to \infty$. Hence, the regularization from Proposition 4.1 is essentially only of the integral of the principal part. This regularized integral can also be viewed as the special value at $s = k$ of an appropriately regularized L-function of g, as in, e.g., [BFK].

Following [ANS, BFI], we now give an equivalent expression for the regularized theta lift $\mathfrak{I}_{k,L}(\tau, f)$ of f from equation (2.24).

Proposition 4.2 *For* $f \in \widetilde{M}_{2k}^!(\Gamma)$ *with asymptotic expansion at the cusp ℓ as in equation* (4.3)*, the regularized theta lift* $\mathfrak{I}_{k,L}(\tau, f)$ *of f from equation* (2.24) *can be written as*

$$\lim_{T\to\infty}\left(\int_{Y_T}f(z)\Theta_{k,L}(\tau,z)d\mu(z)+\sum_{\ell\in\Gamma\setminus\mathrm{Iso}(V)}i^k\frac{\varepsilon_\ell}{\sqrt{N}}\Theta_{k,\ell}(\tau)\sum_{l=0}^{p}c_\ell(0,l)\phi_0(k-l,T)\right),$$

where ϕ_0 is defined in equation (4.4).[5]

Proof We argue as in Proposition 5.2 of [ANS], focusing on the modifications required for our statement, and leaving many details out as they are identical. For fixed T, the constant term in the integral over Y_T is the substitution $s = 0$, and near a cusp ℓ, Lemma 2.9 expresses $\Theta_{k,L}(\tau,z)$ as a multiple of $\Theta_{k,\ell}(\tau)$ plus decreasing terms, whose integral vanishes as $T \to \infty$. Here, instead of the first equation on page 2319 of [ANS], the integral of our nearly holomorphic modular form yields

$$\int_1^\infty\int_0^{\alpha_\ell}(f|_{2k}\sigma_\ell)(z_\ell)y_\ell^{k-1-s}dx_\ell dy_\ell=\alpha_\ell\sum_{l=0}^{p}c_\ell(0,l)\int_1^\infty y_\ell^{k-s-1-l}dy_\ell=\alpha_\ell\sum_{l=0}^{p}\frac{c_\ell(0,l)}{k-l-s}.$$

The same identifications of these constant terms with our functions $\phi_0(k-l,T)$ yield the desired result. This proves the proposition. ∎

For k, $f \in \mathcal{A}_0^!(\Gamma)$, and $g \in \mathcal{A}_{2k}^!(\Gamma)$ as above and an index $m \neq 0$, we can define the combinations

(4.7)
$$\mathrm{Tr}_{m,h}^{(k)}(f):=\sum_{\lambda\in\Gamma\setminus L_{m,h}}\mathrm{Tr}_\lambda^{(k)}(f)\ \text{for}\ m<0,\quad\mathrm{Tr}_{m,h}(g):=\sum_{\lambda\in\Gamma\setminus L_{m,h}}\mathrm{Tr}_\lambda(g)\ \text{for}\ m>0.$$

Note that $\Gamma\setminus L_{m,h}$ is finite when $m \neq 0$, so that there is no question of convergence in equation (4.7). On the other hand, when $g \in \widetilde{M}_{2k}^!(\Gamma)$ has the usual expansion and $Q(\lambda) = m = 0$, we will define the trace to be

(4.8)
$$\mathrm{Tr}_{0,h}(g):=\sum_{\ell\in\Gamma\setminus\mathrm{Iso}(V)}\frac{\varepsilon_\ell}{\sqrt{N}}\iota_\ell(0,h)c_\ell(0,0)(\sqrt{N}\beta_\ell)^k\Phi_k(\omega_{\ell,h}),$$

where $\iota_\ell(m,h)$, $\omega_{\ell,h}$, and Φ_κ are defined in equations (2.18), (2.11), and (3.40), respectively.

The main term of the Shintani lift from Theorem 4.3 will have the traces from equations (4.7) and (4.8) as coefficients. However, for the terms with $\iota(m) = 1$, we need to define some corrections. Recall that when $m > 0$ and $\iota_\ell(m,h) = 1$, Remark 2.7 implies that the numbers r_λ for oriented $\lambda \in L_{m,h} \cap \ell^\perp$ are all the same modulo $\frac{\beta_\ell}{2}\sqrt{\frac{N}{m}}\mathbb{Z}$. We can thus define, for our element $f \in \widetilde{M}_{2k}^!(\Gamma)$ of depth p expanded as in equation (4.3) near each cusp $\ell \in \Gamma\setminus\mathrm{Iso}(V)$, the *complementary trace*

(4.9)
$$\mathrm{Tr}_{m,h}^c(f,v):=-(-2i\sqrt{m})^k\cdot\sqrt{2\pi}\sum_{\ell\in\Gamma\setminus\mathrm{Iso}(V)}\left(\iota_\ell(m,h)+(-1)^k\iota_\ell(m,-h)\right)\frac{\varepsilon_\ell}{\sqrt{N}}$$

$$\times\sum_{\substack{0>n\in\mathbb{Z}\\n\equiv0\ \mathrm{mod}\ 2\varepsilon_\ell\sqrt{m/N}}}\mathbf{e}\left(\frac{nr_\lambda}{\alpha_\ell}\right)\sum_{l=k}^{p}\left(\frac{2\pi n}{\alpha_\ell}\right)^{l-k}\frac{l!c_\ell(n,l)}{(l-k)!}\cdot\frac{J_l\left(2\sqrt{2\pi mv}\right)}{\left(2\sqrt{2\pi mv}\right)^l},$$

---

[5]Note that $c_\ell(0,l)$ is well defined for $\ell \in \Gamma\setminus\mathrm{Iso}(V)$ by the modularity of f.

with $J_l(\eta)$ from equation (3.31). Note that the sum over n in equation (4.9) is finite, and is empty for all but finitely many values of m.

We also define the *complementary trace from constants*

$$
\text{(4.10)}\quad
\begin{aligned}
\text{Tr}^{\text{cc}}_{m,h}(f,v) &:= -(-2i\sqrt{m})^k \sum_{\ell \in \Gamma \backslash \text{Iso}(V)} \left(\iota_\ell(m,h) + (-1)^k \iota_\ell(m,-h)\right) \frac{\varepsilon_\ell}{\sqrt{N}} \\
&\times k! c_\ell(0,k) \frac{I_k(2\sqrt{2\pi m v}) - \tilde{\Omega}_k(2\sqrt{2\pi m v})}{(2\sqrt{2\pi m v})^k},
\end{aligned}
$$

where I_k and $\tilde{\Omega}_k$ are defined in equation (3.26) and Remark 3.13, respectively. The name represents the fact that only the constant terms $c_\ell(0,k)$ contribute to it.

When $m = 0$, there is only a complementary trace from constants, which is defined as

$$
\text{(4.11)}
$$

$$
\text{Tr}^{\text{cc}}_{0,h}(f,v) := \delta_{k,0}\delta_{h,0}\sqrt{v} \int_Y^{\text{reg}} f(z)\,d\mu(z) + \sum_{\ell \in \Gamma \backslash \text{Iso}(V)} \iota_\ell(0,h) \frac{\varepsilon_\ell}{\sqrt{N}} \times
$$

$$
\left(-k! c_\ell(0,k) P_k(0) \frac{\log\left(\sqrt{2\pi N v}\beta_\ell\right) + C_k}{(2\pi v)^{k/2}} + \sum_{l=0}^p l! c_\ell(0,l) Q_l(0) \frac{(\sqrt{N}\beta_\ell)^{k-l} \Xi_{k-l}(\omega_{\ell,h})}{(2\pi v)^{l/2}} \right),
$$

where C_k and Ξ_κ are defined in equations (3.43) and (3.41) respectively, and for $f \in \widetilde{M}^!_{2k}(\Gamma)$, the regularized integral $\int_Y^{\text{reg}} f(z)\,d\mu(z)$ is the (convergent) limit $\lim_{T\to\infty} \int_{Y_T} f(z)\,d\mu(z)$.

4.2 Main theorem and proof

We can now state and prove our main theorem. Given $k \in \mathbb{N}$ and an element $f \in \widetilde{M}^!_{2k}(\Gamma)$, we gather the traces from equations (4.7) and (4.8) and define

$$
\text{(4.12)}\quad \mathcal{J}^{\text{nh}}_{k,L,h}(\tau,f) := \sum_{b=0}^{\lfloor p/2 \rfloor} \sum_{0 \le m \in \mathbb{Z}+Q(h)} \frac{\text{Tr}_{m,h}(L_z^{2b} f)}{(4\pi v)^b b!} q_\tau^m,
$$

which is a nearly holomorphic function of depth $\lfloor \frac{p}{2} \rfloor$ on $\mathcal{H}$ that is bounded at ∞. Using the negative index case of equation (4.7), we also define

$$
\text{(4.13)}
$$

$$
\mathcal{J}^{\text{neg}}_{k,L,h}(\tau,f) := \sum_{0 > m \in \mathbb{Z}+Q(h)} \sum_{l=k}^p \frac{4^k \sqrt{\pi} |m|^{\frac{k-1}{2}} h_l(2\sqrt{2\pi|m|v})}{\sqrt{2}(4\sqrt{2\pi|m|v})^l (l-k)!} \text{Tr}^{(k)}_{m,h}(R^{l-k}_{2k-2l} L_z^l f) q_\tau^m,
$$

which resembles the non-holomorphic part of a harmonic weak Maass form with cuspidal ξ-image (see also the proof of Proposition 4.5). We also gather the traces from equations 4.9–4.11, and set

$$
\text{(4.14)}
$$

$$
\mathcal{J}^{\text{c}}_{k,L,h}(\tau,f) := \sum_{\substack{0 < m \in \mathbb{Z}+Q(h) \\ \iota(m)=1}} \text{Tr}^{\text{c}}_{m,h}(f,v) q_\tau^m \quad \text{and} \quad \mathcal{J}^{\text{cc}}_{k,L,h}(\tau,f) := \sum_{\substack{0 \le m \in \mathbb{Z}+Q(h) \\ \iota(m)=1}} \text{Tr}^{\text{cc}}_{m,h}(f,v) q_\tau^m,
$$

where the former is a finite sum of increasing terms, and the second one is infinite but converges.

The main result, which evaluates the regularized Shintani lift of f, now reads as follows.

Theorem 4.3 *Write the vector-valued Shintani lift $\mathcal{I}_{k,L}(\tau, f)$ of $f \in \widetilde{M}^!_{2k}(\Gamma)$, which is defined in equation* (2.24) *and lies in $\mathcal{A}_{k+\frac{1}{2}}\big(\mathrm{Mp}_2(\mathbb{Z}), \rho_L\big)$, as $\sum_{h \in D_L} \mathcal{I}_{k,L,h}(\tau, f)\mathfrak{e}_h$. Then the scalar-valued coefficient associated with $h \in D_L$ is given by*

$$\mathcal{I}_{k,L,h}(\tau, f) = \mathcal{I}^{\mathrm{nh}}_{k,L,h}(\tau, f) + \mathcal{I}^{\mathrm{neg}}_{k,L,h}(\tau, f) + \mathcal{I}^{\mathrm{c}}_{k,L,h}(\tau, f) + \mathcal{I}^{\mathrm{cc}}_{k,L,h}(\tau, f),$$

where the terms are defined in equations (4.12)*–*(4.14).

Proof We apply Proposition 4.2, and expand $\Theta_{k,L}(\tau, z)$ and $\Theta_{k,\ell}(\tau)$ using equations (3.21) and (2.23) respectively, which gives

(4.15)

$$\mathcal{I}_{k,L,h}(\tau, f)$$

$$= \lim_{T \to \infty} \begin{pmatrix} v^{\frac{1-k}{2}} \displaystyle\int_{Y_T} f(z) \sum_{m \in \mathbb{Z}+Q(h)} q^m_\tau \sum_{\lambda \in L_{m,h}} \varphi_{k,-1}\big(\sqrt{v}\lambda, z\big)d\mu(z) + \\ \displaystyle\sum_{\substack{0 \le m \in \mathbb{Z}+Q(h) \\ \iota(m)=1}} q^m_\tau \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} i^k \frac{\varepsilon_\ell}{\sqrt{N}} a(\Theta_{k,\ell}, m, h, v) \sum_{l=0}^{p} c_\ell(0, l)\phi_0(k - l, T) \end{pmatrix}.$$

We may interchange the order of integration and summation as both are absolutely convergent for fixed T. Propositions 4.10–4.13 now evaluate the coefficient of q^m_τ to be the one implied by the asserted sum. This proves the theorem. ∎

A much simpler but interesting special case is the one where the depth $p < k$.

Corollary 4.4 *Assume that $f \in \widetilde{M}^{!,\le p}_{2k}(\Gamma)$, and that $p < k$. Then*

$$\mathcal{I}_{k,L}(\tau, f) = \sum_{h \in D_L} \mathcal{I}^{\mathrm{nh}}_{k,L,h}(\tau, f)\mathfrak{e}_h \in \widetilde{M}^{\le \lfloor p/2 \rfloor}_{k+\frac{1}{2}}\big(\mathrm{Mp}_2(\mathbb{Z}), \rho_L\big).$$

For $p = 0$, we recover the part of Theorem 6.1 of [ANS] *with weakly holomorphic input.*

Proof If $p < k$, then $\mathcal{I}^{\mathrm{neg}}_{k,L,h}$ and $\mathcal{I}^{\mathrm{c}}_{k,L,h}$ vanish identically (because both involve sums over $k \le l \le p$), and equations (4.10) and (4.11) imply that $\mathcal{I}^{\mathrm{cc}}_{k,L,h}$ consists only of the constant term

$$\mathrm{Tr}^{\mathrm{cc}}_{0,h}(f, v) = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \iota_\ell(0, h) \frac{\varepsilon_\ell}{\sqrt{N}} \sum_{l=0}^{k-1} l! c_\ell(0, l) Q_l(0) \frac{(\sqrt{N}\beta_\ell)^{k-l} \Xi_{k-l}(\omega_{\ell,h})}{(2\pi v)^{l/2}}$$

(because $c_\ell(0, k) = 0$ when $p < k$). Since $k - l > 0$, the only term $\Xi_{k-l}(\omega_{\ell,h})$ from equation (3.41) that may not vanish is when $l = k - 1$ and $\omega_{\ell,h} = 0$, where it equals the constant $-\frac{1}{\sqrt{2\pi}}$. However, equation (2.11) shows that the latter equality only holds when $h = 0$, and thus equation (2.18) yields $\iota_\ell(0, h) = 1$ for all ℓ, and the terms involving ε_ℓ, β_ℓ, and $\sqrt{N}$ reduce to α_ℓ when $l = k - 1$ by equation (2.10). This reduces our expression for $\mathcal{I}^{\mathrm{cc}}_{k,L,h}$ to $-\frac{Q_{k-1}(0)}{(2\pi)^{k/2}v^{(k-1)/2}}$ times $\sum_\ell \alpha_\ell c_\ell(0, k - 1)$. However,

the $c_\ell(0, k - 1)$'s are, up to constant multiple that is independent of ℓ, the constant terms of the weakly holomorphic modular form $L_z^{k-1} f \in M_2^!(\Gamma)$, and by equation (2.12), multiplying each by α_ℓ yields the residue of the corresponding meromorphic differential $L_z^{k-1} f dz$ at the cusp ℓ (again, up to a constant multiple that is independent of ℓ). Thus, $\mathcal{I}_{k,L,h}^{\mathrm{cc}}$ is a multiple of the sum of residues of a meromorphic differential on the compact curve associated with Γ at all of its poles, which therefore vanishes (this also explains Remark 1.4). This proves the corollary. ∎

When $p \geq k$, Corollary 4.4 no longer holds, but one can obtain a nearly holomorphic modular form after applying an appropriate weight-changing operator.

Proposition 4.5 *For any $f \in \widetilde{M}_{2k}^{1, \leq p}(\Gamma)$ for which the constant terms $c_\ell(0, k)$ vanish for every ℓ, we have*

$$\xi_{k-2\lfloor p/2\rfloor+\frac{1}{2}} L_\tau^{\lfloor p/2\rfloor} \mathcal{I}_{k,L}(\tau, f) \in \widetilde{M}_{2\lfloor p/2\rfloor-k+\frac{3}{2}}^{\leq p-k} \big(\mathrm{Mp}_2(\mathbb{Z}), \overline{\rho}_L \big).$$

Note that, for $p < k$, the depth is negative, so that the space $\widetilde{M}_{2\lfloor p/2\rfloor-k+\frac{3}{2}}^{\leq p-k}(\overline{\rho}_L)$ is trivial, and indeed the modular forms that are annihilated by the operator in question are those lying in $\widetilde{M}_{k+\frac{1}{2}}^{\leq \lfloor p/2\rfloor}\big(\mathrm{Mp}_2(\mathbb{Z}), \rho_L \big)$.

Proof The fact that $\xi_{k-2\lfloor p/2\rfloor+\frac{1}{2}} L_\tau^{\lfloor p/2\rfloor} \mathcal{I}_{k,L}(\tau, f)$ lies in $\mathcal{A}_{2\lfloor p/2\rfloor-k+\frac{3}{2}}\big(\mathrm{Mp}_2(\mathbb{Z}), \overline{\rho}_L \big)$ follows from the weight changing properties of L_τ and $\xi_{k-2\lfloor p/2\rfloor+\frac{1}{2}}$, including the conjugation of the representation in equation (3.41). For investigating its analytic properties, we need to evaluate the operation of L_τ on each of the summands in Theorem 4.3.

The first summand, from equation (4.12), is nearly holomorphic of depth $\lfloor \frac{p}{2} \rfloor$. Every application of L_τ reduces the depth by 1, and then the application of the ξ-operator, which is again L_τ composed with a modified complex conjugation, annihilates the weakly holomorphic expression $L_\tau^{\lfloor p/2\rfloor} \mathcal{I}_{k,L,h}^{\mathrm{nh}}(\tau, f)$. Now, the terms from equation (4.13), as well as those from equation (4.14) defined in equation (4.9), involve the expressions $\frac{h_l(\eta)}{\eta^l}$ and $\frac{I_l(\eta)}{\eta^l}$ respectively, for $\eta = 2\sqrt{2\pi|m|v}$ and $k \leq l \leq p$, times constants times q_τ^m. As applying $L_\tau = -2iv^2 \partial_{\overline{\tau}}$ to a function of η is the same as $\frac{\eta^3}{16\pi|m|} \frac{d}{d\eta}$, Lemma 3.16 and equation (3.32) imply that this effect amounts, up to scalar multiples, to subtracting 2 from the index l. After doing so $\lfloor \frac{p}{2} \rfloor + 1$ times (including the action of the ξ-operator), we remain only with indices $k - 2\lfloor \frac{p}{2} \rfloor - 2 \leq l \leq p - 2\lfloor \frac{p}{2} \rfloor - 2$, that are thus negative.

However, equations (3.17) and (3.31), the vanishing of P_v and $\widetilde{P}_v$ via Remark 3.8 and equation (3.30), and the parity of Q_v and $\widetilde{Q}_v$ imply that for negative l, such expressions are odd polynomials in η, of degree at most $4\lfloor \frac{p}{2} \rfloor + 3 - 2k$ multiplied by $e^{\pm \eta^2/2}$ respectively. Recalling that for the effect of $\xi_{k-2\lfloor p/2\rfloor+\frac{1}{2}}$ we must conjugate and multiply by v to the negative half-integral power $k - 2\lfloor \frac{p}{2} \rfloor - \frac{3}{2}$, and substituting the value $2\sqrt{2\pi|m|v}$ of η, we indeed obtain a decaying nearly holomorphic expression from the image of $\mathcal{I}_{k,L,h}^{\mathrm{neg}}(\tau, f)$ for each $h \in D_L$, and a finite increasing nearly holomorphic expression from the image of $\mathcal{I}_{k,L,h}^{\mathrm{c}}(\tau, f)$ for every h. For the depth bound, we need the lowest power of η in the expressions $\eta^{|l|} Q_l(\eta)$ or $\eta^{|l|} \widetilde{Q}_l(\eta)$ for negative l in our

range, which arises from the value $p - 2\left\lfloor \frac{p}{2} \right\rfloor - 2$ of l that is minimal in absolute value, divided by $\eta^{4\lfloor p/2 \rfloor + 3 - 2k}$. For odd p, this is $l = -1$ with η multiplied by a constant term, yielding a denominator of $v^{2\lfloor p/2 \rfloor + 1 - k} = v^{p-k}$, and if p is even then $l = -2$ is multiplied by a linear polynomial in η, and the denominator is $v^{2\lfloor p/2 \rfloor - k} = v^{p-k}$ as well.

It remains to consider the constant terms in equation (4.14), defined in equation (4.11). The first term, appearing only when $k = 0$, is a multiple of $\sqrt{v}$, which after $\left\lfloor \frac{p}{2} \right\rfloor + 1$ operations of L_τ becomes a multiple of $v^{\lfloor p/2 \rfloor + 3/2}$, and after conjugating and multiplying by $v^{-2\lfloor p/2 \rfloor - \frac{3}{2}}$ (with $k = 0$) yields $v^{-\lfloor p/2 \rfloor}$, indeed nearly holomorphic with the required degree bound. Similarly, $\left\lfloor \frac{p}{2} \right\rfloor + 1$ actions of L_τ and multiplication by $v^{k - 2\lfloor p/2 \rfloor - \frac{3}{2}}$ take the lth term in the last in equation (4.11) to a multiple of $v^{k - \lfloor p/2 \rfloor - \frac{l+1}{2}}$. Since the multiplier $Q_l(0)$ restricts l to be odd, and with $l \leq p$ the maximal value of $\frac{l+1}{2}$ is $\left\lceil \frac{p}{2} \right\rceil$, the result here is nearly holomorphic of degree at most $p - k$. As this covers all the expressions that are not multiples of the coefficients $c_\ell(0, k)$, this proves the proposition. ∎

Remark 4.6 A natural question to ask, in view of Proposition 4.5, is what can be said about the $\xi_{k - 2\lfloor p/2 \rfloor + \frac{1}{2}} L_\tau^{\lfloor p/2 \rfloor} \mathcal{I}_{k, L}(\tau, f)$ in case the coefficients $c_\ell(0, k)$ do not vanish. Following the proof of that proposition, the terms involving $\frac{I_k(\eta)}{\eta^k}$ from equation (4.10) inside equation (4.14) behave like those with $\frac{h_l(\eta)}{\eta^l}$ or $\frac{J_l(\eta)}{\eta^l}$ (see Lemma 3.21 and equation (3.29)), and as the polynomial $\tilde{\Omega}_k$ from Remark 3.13 has degree $k - 2$ and parity $(-1)^k$, the corresponding expression from equation (4.10) is nearly holomorphic of degree at most $\left\lfloor \frac{p}{2} \right\rfloor$ and is thus annihilated by $L_\tau^{\lfloor p/2 \rfloor}$. Now, we can express I_v for $v := k - 2\left\lfloor \frac{p}{2} \right\rfloor - 2 < 0$ via equation (3.29), where using equation (3.8) the multiplier of I_{-1} becomes a multiple of P_{-1-v}. As one can verify by induction that the polynomial $\hat{\Omega}_v$ from equation (3.29) equals minus the same multiple of Q_{-1-v} for $v < 0$, this determines $I_v(\eta)$ as a multiple of $e^{\eta^2/2} h_{-1-v}(\eta)$, and with the powers of m and v, this evaluates the image of the expression from equation (4.10) under $\xi_{k - 2\lfloor p/2 \rfloor + \frac{1}{2}} L_\tau^{\lfloor p/2 \rfloor}$ as a multiple of $\frac{h_{|v|-1}(\eta)}{\eta^{|v|-1}} q_\tau^{-m}$ with our η. The term with the scalar and the logarithm in the constant term from equation (4.11), which shows up only for even k because of the multiplier $P_k(0)$, becomes, after this operator, a constant divided by $v^{\frac{|v|-1}{2}}$. Recall that the nearly holomorphic part from Proposition 4.5 is annihilated after applying $\xi_{k - \lceil p/2 \rceil + \frac{3}{2}} L_\tau^{k-p}$, so one can apply this operator to the remaining expression. Using similar considerations once again, and verifying that all the multipliers match (including the constant terms) shows, via equation (2.23), that the resulting expression is a constant multiple of the sum $\sum_{\ell \in \Gamma \setminus \text{Iso}(V)} c_\ell(0, k) \Theta_{2\lceil p/2 \rceil - k, \ell}(\tau)$. In total, for $p < k$, the Shintani lift is annihilated by one application of $\xi_* L_\tau^*$ (by Corollary 4.4), when $p \geq k$ and $c_\ell(0, k) = 0$ for all ℓ, it is killed after two applications of such operators, resembling harmonicity (this is Proposition 4.5), and with the extra term, a third application is required for sending the lift to 0, which is a type of a sesqui-harmonic property (because the theta function from equation (2.23) is nearly holomorphic itself).

We can now deduce the theorems mentioned in the Introduction.

Proof of Theorem 1.1   We apply Theorem 4.3 to the scaled lattice $L_\Delta := \Delta L$ for L from equation (2.8), with $Q = -\frac{\det}{|\Delta|}$. Then $L_\Delta^* = L^*$ and it is well known (see [GKZ] or [AE]) that if $g \in \mathcal{A}_{\frac{3}{2}}^!\big(\mathrm{Mp}_2(\mathbb{Z}), \rho_{L_\Delta}\big)$ and $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$, then

$$\Upsilon_\Delta(g)(z) := \frac{1}{[\Gamma : \Gamma_{L_\Delta}]} \sum_{\delta \in D_{L_\Delta}} \chi_\Delta(\delta) g_\delta(4z) \in \mathcal{A}_{\frac{3}{2}}^!\big(\tilde{\Gamma}_0(4)\big),$$

where χ_Δ is the character from equation (1.1) and $\tilde{\Gamma}_0(4) \subseteq \mathrm{Mp}_2(\mathbb{Z})$ is the metaplectic cover of $\Gamma_0(4)$. Moreover, it is in the Kohnen plus space. We denote $\mathcal{I}_\Delta^*(z, f) := \Upsilon_\Delta\big(\mathcal{I}_{1,L_\Delta}^*(z, f)\big)$ for $* \in \{\mathrm{nh}, \mathrm{neg}, \mathrm{c}, \mathrm{cc}\}$. When $f = J \cdot E_2^*$, we have $k = p = 1$, and only nonzero coefficients of the principal part are $c_\ell(\Delta, 0) = 1$, $c_\ell(\Delta, 1) = -\frac{3}{\pi}$, and $c_\ell(0, 0) = -24$ for any $\ell \in \mathrm{Iso}(V)$. The fact that $c_\ell(0, 1) = 0$ and $Q_0 = 0$ implies that $\mathcal{I}_\Delta^{\mathrm{cc}}(z, f) = 0$, and we have

(4.16)
$$\Gamma \backslash \mathcal{Q}_{\Delta^2} = \Gamma\big\{[0, |\Delta|, C] \big| C \in \mathbb{Z}/\Delta\mathbb{Z}\big\}, \quad \Gamma_{L_\Delta, \infty} \backslash \ell_\infty = \Gamma_{L_\Delta, \infty}\big\{[0, 0, C] \big| C \in \mathbb{Z}/\Delta\mathbb{Z}\big\},$$

with the value $\chi_\Delta\big([0, |\Delta|, C]\big) = \chi_\Delta\big([0, 0, C]\big) = \left(\frac{C}{\Delta}\right)$, the real part $r_{[0,|\Delta|,C]} = -\frac{C}{|\Delta|}$, and $\mathbb{B}_1\big(\omega_{\ell_\infty, [0,0,C]}\big) = \mathbb{B}_1\big(\frac{C}{|\Delta|}\big) = \frac{C}{|\Delta|} - \frac{1}{2}$ for $0 < C < |\Delta|$. Since $L_z f = \frac{3}{\pi} J$, χ_Δ is antisymmetric, and the sign from equation (4.1) is that of $-A$ from λ as in equation (2.8), we have

$$\mathcal{I}_\Delta^{\mathrm{neg}}(z, f) = \frac{-12}{\sqrt{2\pi}} \sum_{0 < D \in \mathbb{Z}} \left(\sum_{0 \ll \lambda \in \Gamma \backslash \mathcal{Q}_{\Delta D}} \frac{\chi_\Delta(\lambda)}{|\Gamma_\lambda|} J(z_\lambda) \right) \frac{h_1\big(2\sqrt{2\pi D y}\big)}{2\sqrt{2\pi D y}} q^{-D}$$

as well as

$$\mathcal{I}_\Delta^{\mathrm{nh}}(z, f) = 48|\Delta| H(-\Delta) + \sum_{0 > D \in \mathbb{Z}} \left(\sum_{\lambda \in \Gamma \backslash \mathcal{Q}_{\Delta D}} \chi_\Delta(\lambda) \mathrm{Tr}_\lambda(f) \right) q^{-D},$$

where for the latter constant we use equation (4.16) and the equality

$$\sum_{C=1}^{|\Delta|-1} \left(\frac{C}{\Delta}\right)\left(\frac{C}{|\Delta|} - \frac{1}{2}\right) = \frac{1}{|\Delta|} \sum_{C=1}^{|\Delta|-1} C\left(\frac{C}{\Delta}\right) = \frac{\sqrt{|\Delta|}}{\pi} L\big(\big(\frac{\cdot}{\Delta}\big), 1\big) = 2H(-\Delta).$$

Moreover, the congruence from equation (4.9) gives $a|\Delta$, the fundamentality of Δ leaves only $a = |\Delta|$ after Υ_Δ, and then equation (4.16) and a standard Gauss sum evaluation yield $\mathcal{I}_\Delta^{\mathrm{c}}(z, f) = \frac{12}{\sqrt{2\pi}} \sqrt{|\Delta|} \cdot \frac{J_1\big(2\sqrt{2\pi|\Delta|y}\big)}{2\sqrt{2\pi|\Delta|y}} q^{|\Delta|}$.

Now, let $\tilde{f}_{-\Delta}(z)$ denote the holomorphic part $\mathcal{I}_\Delta^{\mathrm{nh}}(z, f)$, which has the required expansion. Since Lemma 3.16 and equation (3.32) imply that $\xi_{\frac{3}{2}}$ takes the functions $\frac{h_1(2\sqrt{2\pi D y})}{(2\sqrt{2\pi D y})} q^{-D}$ and $\frac{J_1\big(2\sqrt{2\pi|\Delta|y}\big)}{2\sqrt{2\pi|\Delta|y}} q^{|\Delta|}$ to $-\frac{q^D}{4\sqrt{2\pi D}}$ and $+\frac{q^\Delta}{4\sqrt{2\pi|\Delta|}}$ respectively, the complement $\mathcal{I}_\Delta^{\mathrm{neg}}(z, f) + \mathcal{I}_\Delta^{\mathrm{c}}(z, f)$ of $\tilde{f}_{-\Delta}(z)$ is indeed harmonic, with the asserted $\xi_{\frac{3}{2}}$-image $\frac{3}{2\pi} f_{-\Delta}$. This completes the proof of the theorem. ∎

Proof of Theorem 1.2 We apply Theorem 4.3 to the lattice L in equation (2.8), where for which the parameters are $N = \alpha_\ell = \beta_\ell = \varepsilon_\ell = 1$, combined with the isomorphism from [K] to scalar-valued modular forms. The vanishing of $c(0, k)$ implies the vanishing of the main terms of $\mathcal{J}_{k,L}^{cc}$. As equations (3.10) and (3.41) evaluate the remaining term from equation (4.11) to be the asserted one, and the first term in equation (4.11) produces the one mentioned in Remark 1.3, this proves the first claim of the (extended) theorem. The remaining two claims follow directly from Corollary 4.4 (there exists only one cusp, so $c(0, k - 1) = 0$ in this case) and Proposition 4.5. This proves the theorem. ∎

Remark 4.7 Using [Ze3, Ze7], the sum $\sum_{a=0}^{\lfloor p/2 \rfloor} \frac{1}{v^a a!} L_\tau^a \mathcal{J}_{k,L}(\tau, f)$ is a (vector-valued) quasi-modular form of weight $k + \frac{1}{2}$ and depth $\lfloor \frac{p}{2} \rfloor$, and one checks that the contribution of $\mathcal{J}_{k,L,h}^{nh}(\tau, f)$ is just $\sum_{0 \le m \in \mathbb{Z} + Q(h)} \mathrm{Tr}_{m,h}(f) q_\tau^m$ for every $h \in D_L$. Moreover, Lemmas 3.16 and 3.21 and equation (3.32) show that applying this combination to $\mathcal{J}_{k,L,h}^{neg}(\tau, f)$, $\mathcal{J}_{k,L,h}^{c}(\tau, f)$, and $\mathcal{J}_{k,L,h}^{cc}(\tau, f)$ amounts to replacing h_l from equation (4.13), J_l in equation (4.9), and I_k appearing in equation (4.10) by $\sum_a h_{l-2a}(-2)^a a!$, $\sum_a J_{l-2a} 2^a a!$, and $\sum_a I_{k-2a} 2^a a!$, respectively. Moreover, after substituting equations (3.17), (3.31), and (3.29), the respective coefficient of h_0, J_0, and I_0 in these combinations is just the corresponding denominator η^l or η^k. It would be interesting to investigate these functions further. We also note that as long as $2a \le k$, the term $L_\tau^a \mathcal{J}_{k,L}(\tau, f)$ is easily verified to be $\frac{1}{(-4\pi)^a}$ times the Shintani lift $\mathcal{J}_{k-2a,L}(\tau, L_z^{2a} f)$.

4.3 Orbital integrals

In order to prove Theorem 4.3, we need to evaluate equation (4.15), which we can do for every $m \in \mathbb{Z} + Q(h)$ separately. Moreover, the integral over Y_T can be replaced by an integral over $\mathcal{F}_T$ from equation (2.14), and integrals over (nice) regions in $\mathcal{H}$ can be expressed using the following lemma.

Lemma 4.8 *Let $\lambda \in V_\mathbb{R}$ and $f \in \widetilde{M}_{2k}^!(\Gamma)$ of some depth p be given, and take a connected domain $\mathcal{R} \subseteq \mathcal{H}$ with a piecewise smooth positively oriented boundary $\partial\mathcal{R}$. Then we have*

$$\int_\mathcal{R} f(z) \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z) = \sum_{v=0}^p \oint_{\partial\mathcal{R}} (L_z^v f)(z) \varphi_{k-v-1,v}(\sqrt{v}\lambda, z) dz.$$

Proof We apply Lemma 2.3 repeatedly, where in the vth step f is replaced by $L_z^v f$ and $g(z) = \varphi_{k-v,v-1}(\sqrt{v}\lambda, z)$. Then Proposition 3.18 allows us to take $G(z) = \varphi_{k-v-1,v}(\sqrt{v}\lambda, z)$, and the sum ends after $v = p$ since $L_z^{p+1} f = 0$ by assumption. This proves the lemma. ∎

In view of equation (4.12), we will need the following lemma later.

Lemma 4.9 *Let $\lambda \in V$ with $\iota(\lambda) = 1$, $v > 0$, and $f \in \widetilde{M}_{2k}^!(\Gamma)$ of some depth p be given, and denote $\eta = 2\sqrt{2\pi Q(\lambda)v}$. Then we have the equality*

$$\sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\mathrm{Sing}_\lambda(L_z^{2b} f)}{(4\pi v)^b b!} = i^k \left(2\sqrt{Q(\lambda)}\right)^{k-1} \sum_{n \in \mathbb{Z}} \sum_{l=0}^p c_{\ell_\lambda}(n, l) \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) \phi_n\left(k - l, T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right) \frac{\mathrm{He}_l(\eta)}{\eta^l}.$$

Proof As Corollary 3.7 presents the coefficient in front of $\mathrm{Sing}_\lambda(L_z^\mu f)$ on the left-hand side of the first equality as $\frac{P_\mu(0)}{(2\pi v)^{\mu/2}}$, substituting equation (4.5), with the summation index l replacing $l - \mu$, into equation (4.4), expresses the left-hand side of the first equality as

$$\sum_{\mu=0}^{p} \frac{P_\mu(0)}{(2\pi v)^{\mu/2}} i^{k-\mu} (2\sqrt{Q(\lambda)})^{k-\mu-1} \sum_{l=\mu}^{p} \sum_{n\in\mathbb{Z}} \frac{l! c_{\ell_\lambda}(n,l)}{(l-\mu)!} \mathbf{e}\!\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) \phi_n\!\left((k-\mu)-(l-\mu), T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right)$$

$$= i^k (2\sqrt{Q(\lambda)})^{k-1} \sum_{j=0}^{p} \sum_{n\in\mathbb{Z}} c_{\ell_\lambda}(n,l) \mathbf{e}\!\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) \phi_n\!\left(k-l, T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right) \sum_{\mu=0}^{l} \frac{l! P_\mu(0)}{(l-\mu)!(2i\sqrt{2\pi Q(\lambda)v})^\mu}.$$

Equation (3.11) and Corollary 3.7 now express the sum over μ as $\frac{l! P_l(i\eta)}{(i\eta)^l} = \frac{\mathrm{He}_l(\eta)}{\eta^l}$, as desired. This proves the lemma. ∎

Now, the coefficient of q^m in equation (4.15) is evaluated for $m < 0$, $m > 0$ with $\iota(m) = 0$, $m > 0$ with $\iota(m) = 1$, and $m = 0$ respectively, in the following four propositions.

Proposition 4.10 *For every $h \in D_L$ and $0 > m \in \mathbb{Z} + Q(h)$, we have the equality*

$$\lim_{T\to\infty} v^{\frac{1-k}{2}} \int_{Y_T} f(z) \sum_{\lambda\in L_{m,h}} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z)$$

$$= \sum_{l=k}^{p} \frac{4^k \sqrt{\pi} |m|^{\frac{k-1}{2}} h_l(2\sqrt{2\pi|m|v}) \operatorname{Tr}_{m,h}^{(k)}(R_{2k-2l}^{l-k} L_z^l f)}{\sqrt{2}(4\sqrt{2\pi|m|v})^l (l-k)!}.$$

Proof The proof is similar to that of Proposition 3.9 of [BFIL]. Remark 3.15 gives us the strong decay of $\varphi_{\kappa,v}(\sqrt{v}\lambda, z)$, so that we can take the integral over $z \in Y$. Unfolding, cutting out a small neighborhood $B_\varepsilon(z_\lambda)$ for each $\lambda \in \Gamma\backslash L_{m,h}$, and applying Lemma 4.8 yields

$$\int_Y f(z) \sum_{\lambda\in L_{m,h}} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z)$$

$$= \lim_{\varepsilon\to 0} \sum_{\lambda\in\Gamma\backslash L_{m,h}} \frac{-1}{|\Gamma_\lambda|} \sum_{l=0}^{p} \oint_{\partial B_\varepsilon(z_\lambda)} (L_z^l f)(z) \varphi_{k-1-l,l}(\sqrt{v}\lambda, z) dz.$$

Substituting equations (2.6) and (3.18) (with $\zeta = \sigma\sqrt{|m|}$ for $\sigma := -\operatorname{sgn}(\lambda, Z^\perp(z_\lambda))$), and multiplying by $v^{\frac{1-k}{2}}$, shows that the desired left-hand side is the sum over λ and l of

$$\sum_{\lambda\in\Gamma\backslash L_{m,h}} \sum_{l=0}^{p} \frac{-(4y_\lambda\sigma\sqrt{|m|})^{k-1-l}}{(2\pi)^{(l+1)/2} v^{l/2} |\Gamma_\lambda|} \lim_{\varepsilon\to 0} \oint_{\partial B_\varepsilon(z_\lambda)} \frac{(L_z^l f)(z) A_{z_\lambda}(z)^{k-1-l} dz}{(1-A_{z_\lambda}(z))^{2k-2l-2}} h_l\!\left(2\sigma\sqrt{2\pi m v}\frac{1+\varepsilon^2}{1-\varepsilon^2}\right).$$

Evaluating this limit via Corollary 2.2, using the parity from Proposition 3.14, and applying equations (4.1) and (4.7) yields the desired right-hand side. This proves the proposition. ∎

Proposition 4.11 *If $h \in D_L$ and $0 < m \in \mathbb{Z} + Q(h)$ with $\iota(m) = 0$, then we have*

$$v^{\frac{1-k}{2}} \lim_{T \to \infty} \int_{Y_T} f(z) \sum_{\lambda \in L_{m,h}} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z) = \sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\mathrm{Tr}_{m,h}(L_z^{2b} f)}{(4\pi v)^b b!}.$$

Proof An element $\lambda \in L_{m,h}$ is not perpendicular to ℓ for any $\ell \in \mathrm{Iso}(V)$, and its stabilizer Γ_λ is infinite cyclic by Lemma 2.5. Therefore, the functions $\varphi_{\kappa,\nu}(\sqrt{v}\lambda, z)$ again decay strongly, via Remark 3.15, toward any cusp ℓ. We combine this with the usual unfolding argument to express our left-hand side as

$$v^{\frac{1-k}{2}} \lim_{T \to \infty} \sum_{\lambda \in \Gamma \backslash L_{m,h}} \int_{\mathcal{F}_{\lambda,T}} f(z) \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z)$$

$$= v^{\frac{1-k}{2}} \sum_{\lambda \in \Gamma \backslash L_{m,h}} \int_{\mathcal{F}_\lambda} f(z) \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z),$$

where $\mathcal{F}_\lambda$ is a fundamental domain for the action of Γ_λ on $\mathcal{H}$ and $\mathcal{F}_{\lambda,T} := \mathcal{F}_\lambda \cap \mathcal{H}_T$ (this is well defined modulo Γ by equation (3.20) and the modularity of f). We now remove an ε-neighborhood of the geodesic c_λ from $\mathcal{F}_\lambda$ for applying Lemma 4.8, substitute the value of each $\varphi_{k-\nu-1,\nu}$ from equation (3.18), and gather powers of v to write our expression as

$$-\sum_{\lambda \in \Gamma \backslash L_{m,h}} \sum_{\nu=0}^{p} \frac{\lim_{\varepsilon \to 0^+} \left(h_\nu(\varepsilon) - h_\nu(-\varepsilon) \right)}{(2\pi)^{(\nu+1)/2} v^{\nu/2}} \int_{c_\lambda \cap \mathcal{F}_\lambda} (L_z^\nu f)(z)(\lambda, Z(z))^{k-\nu-1} dz.$$

However, the integral is $\mathrm{Tr}_\lambda(L_z^\nu f)$ from equation (4.2), summing over λ replaces it by $\mathrm{Tr}_{m,h}(L_z^\nu f)$ from equation (4.7), and we have $\lim_{\varepsilon \to 0^+} \left(h_\nu(\varepsilon) - h_\nu(-\varepsilon) \right) = -\sqrt{2\pi} P_\nu(0)$ by Proposition 3.14. The desired formula now follows from Corollary 3.7. This proves the proposition. ∎

When $\iota(m) = 1$, the coefficient $a(\Theta_{k,\ell}, m, h, \nu)$ from (2.23), as well as the traces from equations (4.9) and (4.10), may be nonzero. Recalling the functions ϕ_n from equation (4.4), the limit of the corresponding coefficient from equation (4.15) is evaluated as follows.

Proposition 4.12 *Let $h \in D_L$ and $0 < m \in \mathbb{Z} + Q(h)$ with $\iota(m) = 1$ be given. Then, for large $T > 1$, we have*

$$v^{\frac{1-k}{2}} \int_{Y_T} f(z) \sum_{\lambda \in L_{m,h}} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z) = \sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\mathrm{Tr}_{m,h}(L_z^{2b} f)}{(4\pi v)^b b!} + \mathrm{Tr}_{m,h}^c(f, \nu)$$

$$+ \mathrm{Tr}_{m,h}^{cc}(f, \nu) - \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \frac{\varepsilon_\ell i^k}{\sqrt{N}} a(\Theta_{k,\ell}, m, h, \nu) \sum_{l=0}^{p} c_\ell(0, l) \phi_0(k-l, T) + O\left(\frac{1}{T}\right).$$

Proof For $\lambda \in L_{m,h}$, we set $\mathcal{H}_{\lambda,T} := \mathcal{H} \backslash \left(B_\varepsilon(\ell_\lambda) \cup B_\varepsilon(\ell_{-\lambda}) \right)$ for $\varepsilon = e^{-2\pi T}$. The usual unfolding argument, the fact that the stabilizer of $\lambda \in L_{m,h}$ is trivial by Lemma 2.5,

and the decay from Remark 3.15 allow us to replace, as in Lemma 5.2 of [BF2], the left-hand side by

$$v^{\frac{1-k}{2}} \sum_{\lambda \in \Gamma \backslash L_{m,h}} \int_{H_T} f(z) \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z)$$

$$= v^{\frac{1-k}{2}} \sum_{\lambda \in \Gamma \backslash L_{m,h}} \int_{H_{\lambda,T}} f(z) \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z) + O\left(\frac{1}{T}\right)$$

(the error term here is, in fact, much better). The argument from the proof of Proposition 4.11 expresses the term associated with $\lambda \in \Gamma \backslash L_{m,h}$ as

$$(4.17) \qquad \sum_{b=0}^{\lfloor p/2 \rfloor} \frac{1}{(4\pi v)^b b!} \int_{c(\lambda) \cap Y_T} (L_z^{2b} f)(z) (\lambda, Z(z))^{k-1-2b} dz$$

$$+ \sum_{v=0}^{p} v^{\frac{1-k}{2}} \int_{\partial \mathcal{H}_{\lambda,T}} (L_z^v f)(z) \varphi_{k-v-1,v}(\sqrt{v}\lambda, z) dz.$$

Next, $\partial \mathcal{H}_{\lambda,T} = \partial B_\varepsilon(\ell_\lambda) \cup \partial B_\varepsilon(\ell_{-\lambda})$ for this ε, both with the opposite orientation. In the integral along the first part, as σ_ℓ^{-1} takes $\partial B_\varepsilon(\ell_\lambda)$ to $\mathbb{R} + iT$, equation (2.17) expresses the corresponding term in equation (4.17) as

$$-\sum_{v=0}^{p} \int_{-\infty}^{\infty} v^{\frac{1-k}{2}} (L_z^v f|_{2k-2v} \sigma_{\ell_\lambda})(x_{\ell_\lambda} + iT) \varphi_{k-v-1,v}\left(\sqrt{\frac{mv}{N}}\begin{pmatrix} 1 & -2r_\lambda \\ 0 & -1 \end{pmatrix}, x_{\ell_\lambda} + iT\right) dx_{\ell_\lambda}.$$

Equation (3.19) thus implies that the integral along $\partial B_\varepsilon(\ell_{-\lambda})$ is evaluated in the same manner, with λ replaced by $-\lambda$, and multiplied by $(-1)^k$.

Applying equation (3.18), noting that the pairings with $Z(x_{\ell_\lambda} + iT)$ and $Z^\perp(x_{\ell_\lambda} + iT)$ are $2\sqrt{mv}(x_{\ell_\lambda} + iT - r_\lambda)$ and $\frac{2\sqrt{mv}}{T}(x_{\ell_\lambda} - r_\lambda)$ respectively, we substitute $\eta := 2\sqrt{2\pi mv}$ and $\xi := \frac{\eta}{T}(x_{\ell_\lambda} - r_\lambda)$ and use equations (4.5) and (3.24) and the definition of the Fourier transform to present the latter expression as

$$-v^{\frac{1-k}{2}} \sum_{v=0}^{p} \int_{-\infty}^{\infty} \sum_{n \in \mathbb{Z}} \sum_{l=v}^{p} \frac{(-1)^v l! c_{\ell_\lambda}(n,l)}{(l-v)! T^{l-v}} e^{-2\pi n T/\alpha_{\ell_\lambda}} \mathbf{e}\left(\frac{nx_{\ell_\lambda}}{\alpha_{\ell_\lambda}}\right) \frac{T^{k-v-1}}{(2\pi)^{k/2}} (\xi + i\eta)^{k-v-1} h_v(\xi) dx_{\ell_\lambda}$$

$$= -v^{\frac{1-k}{2}} \sum_{n \in \mathbb{Z}} \sum_{l=0}^{p} \frac{l! c_{\ell_\lambda}(n,l) T^{k-l-1}}{(2\pi)^{k/2}} e^{-2\pi n T/\alpha_{\ell_\lambda}} \int_{-\infty}^{\infty} \mathbf{e}\left(\frac{nx_{\ell_\lambda}}{\alpha_{\ell_\lambda}}\right) \mathbf{g}_{k-l,l}(\xi; \eta) dx_{\ell_\lambda}$$

$$= -v^{\frac{1-k}{2}} \sum_{n \in \mathbb{Z}} \sum_{l=0}^{p} \frac{l! c_{\ell_\lambda}(n,l) T^{k-l}}{(2\pi)^{k/2}\eta} e^{-2\pi n T/\alpha_{\ell_\lambda}} \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) \widehat{\mathbf{g}_{k-l,l}}\left(-\frac{nT}{\alpha_{\ell_\lambda}\eta}; \eta\right).$$

Propositions 3.22 and 3.25 show, via equation (4.4), that the contribution of fixed n and l to the latter expression is $\frac{i^k \eta^{k-1}}{(2\pi v)^{(k-1)/2}} c_{\ell_\lambda}(n,l) \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right) = i^k (2\sqrt{m})^{k-1} c_{\ell_\lambda}(n,l) \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right)$ times

$$\begin{cases} \frac{\mathrm{He}_l(\eta)}{\eta^l} \phi_n\left(k-l, T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right), & n > 0 \text{ or } n < 0 \text{ and } l > k, \\ \frac{\mathrm{He}_l(\eta)}{\eta^l} \phi_n\left(k-l, T; \frac{2\pi}{\alpha_{\ell_\lambda}}\right) - (-1)^k \sqrt{2\pi} \left(\frac{2\pi n}{\alpha_{\ell_\lambda}}\right)^{l-k} \frac{l!}{(l-k)!} \frac{J_l(\eta)}{\eta^l}, & n < 0 \text{ and } l \geq k, \\ \left(\frac{\mathrm{He}_l(\eta)}{\eta^l} - \frac{\mathrm{He}_k(\eta)}{\eta^k}\right) \phi_0(k-l, T), & n = 0 \text{ and } l \neq k, \\ \left(\frac{\mathrm{He}_l(\eta)}{\eta^l} - \frac{\mathrm{He}_k(\eta)}{\eta^k}\right) \phi_0(k-l, T) - \frac{(-1)^k k!}{\eta^k} \left(I_k(\eta) - \tilde{\Omega}_k(\eta)\right), & n = 0 \text{ and } l = k, \end{cases}$$

where the error terms from the former proposition go into $O\left(\frac{1}{T}\right)$ and the first term with $l = k$ trivially vanishes.

Now, since $Q(\lambda) = m$ for our λ, Lemma 4.9 shows that the sum over n and l of the first terms gives $\sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\operatorname{Sing}_\lambda(L_z^{2b} f)}{(4\pi v)^b b!}$. Hence, these terms, the corresponding ones from the integral along $\partial B_\varepsilon(\ell_{-\lambda})$, and the first term from equation (4.17) combine, via equation (4.6), to $\sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\operatorname{Tr}_\lambda(L_z^{2b} f)}{(4\pi v)^b b!}$, and after summing over all $\lambda \in \Gamma \backslash L_{m,h}$, we get the first asserted term by equation (4.7). Now, fix $\ell \in \Gamma \backslash \operatorname{Iso}(V)$, and equation (2.18) implies that in the sum of the remaining terms over $\lambda \in \Gamma \backslash L_{m,h}$, we only get contributions to the integral along $\partial B_\varepsilon(\ell_\lambda)$ when $\iota_\ell(m, h) = 1$, and to the one over $\partial B_\varepsilon(\ell_{-\lambda})$ if $\iota_\ell(m, -h) = 1$, with the sign $(-1)^k$. Lemma 2.6 and Remark 2.7 now imply that, for such ℓ, the sum of $\mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right)$ over λ with $\ell_\lambda = \ell$ gives $2\varepsilon_\ell \sqrt{\frac{m}{N}} \mathbf{e}\left(\frac{nr_\lambda}{\alpha_{\ell_\lambda}}\right)$ in case $2\varepsilon_\ell \sqrt{\frac{m}{N}}$ divides n and 0 otherwise. The resulting sums thus produce the remaining required terms by equations (4.9), (4.10), and (2.23) and the value of η. This completes the proof of the proposition. ∎

The proof of Proposition 4.12 also shows that the regularized traces that show up in that proposition, interpreted as regularized L-values, are not those of $L_z^{2b} f$ but rather of its image under the Hecke U-operator of index $2\varepsilon_\ell \sqrt{\frac{m}{N}}$.

We can now state and prove the analog of Propositions 4.10–4.12 for $m = 0$, again using the expressions from equations (2.23) and (4.4), and with the trace from equation (4.11).

Proposition 4.13 *For an isotropic element $h \in D_L$ and large $T > 1$, we have*

$$v^{\frac{1-k}{2}} \int_{Y_T} f(z) \sum_{\lambda \in L_{0,h}} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z) = \sum_{b=0}^{\lfloor p/2 \rfloor} \frac{\operatorname{Tr}_{0,h}(L_z^{2b} f)}{(4\pi v)^b b!} + \operatorname{Tr}_{0,h}^{\mathrm{cc}}(f, v)$$

$$- \sum_{\ell \in \Gamma \backslash \operatorname{Iso}(V)} \frac{i^k \varepsilon_\ell}{\sqrt{N}} a(\Theta_{k,\ell}, 0, h, v) \sum_{l=0}^{p} c_\ell(0, l) \phi_0(k - l, T) + O\left(\frac{1}{T}\right).$$

Proof It is easy to see that $L_{0,h} \backslash \{0\} = \bigcup_{\{\ell \in \operatorname{Iso}(V) \,|\, \iota_\ell(0,h)=1\}} [(L + h) \cap \ell]$, a union that respects the Γ-action. The same unfolding argument from the proofs of Propositions 4.11 and 4.12 allows us to write this part of the left-hand side as

$$\sum_{\{\ell \in \Gamma \backslash \operatorname{Iso}(V) \,|\, \iota_\ell(0,h)=1\}} v^{\frac{1-k}{2}} \int_{\Gamma_\ell \backslash \mathcal{H}_T} f(z) \sum_{\lambda \in (L+h) \cap \ell} \varphi_{k,-1}(\sqrt{v}\lambda, z) d\mu(z),$$

and Remark 3.15 implies that replacing $\Gamma_\ell \backslash \mathcal{H}_T$ by $\Gamma_\ell \backslash \mathcal{H}_{\ell,T}$, for $\mathcal{H}_{\ell,T} := \mathcal{H}_T \backslash B_\varepsilon(\ell)$, produces an error term that is much smaller that $O\left(\frac{1}{T}\right)$[6]. As ℓ is perpendicular to neither $Z(z)$ nor $Z^\perp(z)$ for any $z \in \mathcal{H}$, when we invoke Lemma 4.8 as the proof of Proposition 4.12, only the (negatively oriented) integral along the boundary $\Gamma_\ell \backslash \partial B_\varepsilon(\ell)$ remains. Now, since equation (2.9) shows that the latter maps under σ_ℓ^{-1} onto $(\mathbb{R}/\alpha_\ell \mathbb{Z}) + iT$,

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[6] Also here, we actually work with representatives for $\Gamma \backslash \operatorname{Iso}(V)$, but we again allow this abuse of notation.

equation (2.11) expresses the summand associated with ℓ as

$$-\sum_{v=0}^{p} v^{\frac{1-k}{2}} \int_{\mathbb{R}/\alpha_\ell \mathbb{Z}} (L_z^v f \mid_{2k-2v} \sigma_\ell)(x_\ell + iT) \sum_{0 \neq \xi \in \mathbb{Z} + \omega_{\ell,h}} \varphi_{k-1-v,v}\left(\sqrt{v} \begin{pmatrix} 0 & \beta_\ell \xi \\ 0 & 0 \end{pmatrix}, x_\ell + iT\right) dx_\ell + O\left(\tfrac{1}{T}\right).$$

Here, after applying equation (3.18), the pairings with $Z(x_{\ell_\lambda} + iT)$ and $Z^\perp(x_{\ell_\lambda} + iT)$ are just $\sqrt{N}v\beta_\ell \xi$ and $\frac{\sqrt{N}v\beta_\ell}{T}$ respectively. Setting $v := \frac{\sqrt{2\pi N}v\beta_\ell}{T}$, equations (3.24), (3.34), (3.35), and (4.5) and simple Fourier integration evaluate the main term as

$$-\sum_{v=0}^{p} v^{\frac{1-k}{2}} \int_{\mathbb{R}/\alpha_\ell \mathbb{Z}} \sum_{n \in \mathbb{Z}} \sum_{l=v}^{p} \frac{(-1)^v l! c_\ell(n,l)}{(l-v)! T^{l-v}} e^{-2\pi n T/\alpha_\ell} \mathbf{e}\left(\frac{nx_\ell}{\alpha_\ell}\right) \sum_{0 \neq \xi \in \mathbb{Z} + \omega_{\ell,h}} \frac{(T v \xi)^{k-1-v}}{(2\pi)^{k/2}} h_v(v\xi) dx_\ell$$

$$= -\sqrt{v} \sum_{l=0}^{p} \frac{l! c_\ell(0,l)}{(2\pi v)^{k/2}} \alpha_\ell T^{k-1-l} \sum_{0 \neq \xi \in \mathbb{Z} + \omega_{\ell,h}} \mathbf{g}_{k-l,l}(v\xi;0) = -\sqrt{v} \sum_{l=0}^{p} \frac{l! c_\ell(0,l)}{(2\pi v)^{k/2}} \alpha_\ell T^{k-1-l} \mathbf{G}_{k-l,l}(\omega_{\ell,h};v).$$

We apply Proposition 3.27 and substitute the value of v, equation (2.10), and Corollary 3.7 for $P_k(0)$, and after summing over ℓ, our part of the left-hand side takes the form

$$\sum_{\substack{\ell \in \Gamma \backslash \mathrm{Iso}(V) \\ \iota_\ell(0,h)=1}} \sum_{l=0}^{p} \frac{l! c_\ell(0,l)}{(2\pi v)^{(k-1)/2}} \beta_\ell \varepsilon_\ell \left(\sqrt{2\pi N}v\beta_\ell\right)^{k-1-l} \left[P_l(0)\Phi_{k-l}(\omega_{\ell,h}) + Q_l(0)\Xi_{k-l}(\omega_{\ell,h})\right]$$

$$+ \frac{i^k \mathrm{He}_k(0)}{(2\pi v)^{(k-1)/2}} \sum_{\substack{\ell \in \Gamma \backslash \mathrm{Iso}(V) \\ \iota_\ell(0,h)=1}} \frac{\beta_\ell \varepsilon_\ell}{\sqrt{2\pi N}v\beta_\ell} \left(\sum_{\substack{0 \leq l \leq p \\ l \neq k}} c_\ell(0,l) \frac{T^{k-l}}{k-l} - c_\ell(0,k)\left(\log \frac{\sqrt{2\pi N}v\beta_\ell}{T} + C_k\right)\right) + O\left(\tfrac{1}{T}\right).$$

Note that the latter terms with $l > k$ do not arise through this argument, but as they decay like our error term, the expression is unaffected by adding them.

After canceling, writing $\iota_\ell(0,h)$ as a multiplier, and evaluating $P_l(0)$ using Corollary 3.7, equation (4.5) shows that the first terms combine to the desired combination of the traces from equation (4.8). Equation (4.4) then expresses the remaining expression as the main term from equation (4.11) plus

$$-\frac{i^k \mathrm{He}_k(0)}{(2\pi v)^{k/2}} \sum_{\ell \in \mathrm{Iso}(V)} \iota_\ell(0,h) \frac{\varepsilon_\ell}{\sqrt{N}} \sum_{l=0}^{p} c_\ell(0,l) \phi_n(k-l,T),$$

which is the last asserted term by equation (2.23). This gives the desired right-hand side when $h \neq 0$, and for $h = 0$, there is also the integral involving $\varphi_{k,-1}(0,z)$ from $0 \in L_{0,0}$. However, equation (3.18) evaluates it as $0^k h_{-1}(0) = \delta_{k,0}$, and since in weight 0 we have, by equation (4.3), the bound

$$\int_Y^{\mathrm{reg}} f(z) d\mu(z) - \int_{Y_T} f(z) d\mu(z) = \sum_{\ell \in \Gamma \backslash \mathrm{Iso}(V)} \int_T^\infty \alpha_\ell \sum_{l=0}^{p} \frac{c_\ell(0,l)}{y_\ell^{l+2}} = O\left(\tfrac{1}{T}\right),$$

we indeed get the remaining term from equation (4.11). This proves the proposition. ∎

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