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GROUPS OF SMALL PERIOD GROWTH

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Abstract We construct finitely generated groups of small period growth, i.e. groups where the maximum order of an element of word length n grows very slowly in n. This answers a question of Bradford related to the lawlessness growth of groups and is connected to an approximative version of the restricted Burnside problem.

Keywords: periodic groups; period growth; Burnside problems; groups acting on rooted trees; residually finite groups

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1. Introduction

In this paper we provide an affirmative answer to the following question posed by H. Bradford at the 'New Trends around Profinite Groups' conference in Levico Terme, 2021.

Q1 Is there a lawless finitely generated p-group of sublinear period growth?

A group is called *lawless* if it does not satisfy any non-trivial identity, i.e. if every word-map has a non-trivial image. Let G be a group generated by a finite set S. For any $n \in \mathbb{N}$, write $B_G^S(n)$ for the set of elements in G of word length at most n (with respect to S). The *period growth function* $\pi_G^S : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ *of* G *with respect to* S, first considered by Grigorchuk [5], is defined by

$$\pi_G^S(n) = \max\{\mathrm{ord}(g) \mid g \in B_G^S(n)\}.$$

Grigorchuk proved that the growth type of π_G^S is independent of the choice of S. Consequently, **Q1** is well-posed and we drop the superscript S in statements regarding the growth type of the period growth function of a group.

Bradford's question was motivated by an application to *lawlessness growth*, cf. [4, Example 2.7 and Question 10.2]. The lawlessness growth of a lawless group measures the minimal word length of witnesses to the non-triviality of the verbal subgroup $w(G)$ for group words w of increasing length. Since elements of order m do not satisfy any power words of length smaller than m, there is a close connection between the period and the lawlessness growth of p-groups G: any upper bound on the growth of $\pi_G^S(n)$ yields a lower bound for the lawlessness growth. Concretely, an example of a lawless p-group, p being some prime, with the properties required by **Q1** has superlinear lawlessness growth, see Proposition 5.1. For a detailed study on lawlessness growth, we refer to [4].

Clearly, a group with the properties demanded in **Q1** is infinite since it is lawless, and it is periodic since otherwise there exists some $n_0 \in \mathbb{N}$ such that $\pi_G^S(n) = \infty$ for all $n \geq n_0$. Little is known regarding the period growth of finitely generated infinite periodic groups. Grigorchuk proved that the (first) Grigorchuk group $\mathcal{G}$ fulfils $\pi_{\mathcal{G}} \precsim n^9$, where given two non-decreasing functions $f, g : \mathbb{N} \to \mathbb{R}_{>0}$, we write $f \precsim g$ if

$$\limsup_{n\to\infty} f(n)/g(n) < \infty.$$

This bound was improved by Bartholdi and Šunić [2] to $n^{3/2}$, also extending the result to certain generalizations of $\mathcal{G}$. In [4, Remark 5.7], Bradford constructs a Golod–Shafarevich p-group of at most linear period growth. We remark that the standard proof that the Gupta–Sidki 3-group Γ_3 is periodic yields $\pi_{\Gamma_3} \precsim n^{1/\log_3(4/3)}$.

To state our main result, we need to define some functions that grow very slowly. The *tetration function* $\mathrm{tetr}_k : \mathbb{N} \to \mathbb{N}$ *with base* k is defined recursively by $\mathrm{tetr}_k(0) = 1$ and $\mathrm{tetr}_k(n+1) = k^{\mathrm{tetr}_k(n)}$ for $n \in \mathbb{N}$. We define a left-inverse non-decreasing function by $\mathrm{slog}_k(n) = \max\{l \in \mathbb{N} \mid \mathrm{tetr}_k(l) \leq n\}$. Furthermore, given $r \in \mathbb{R}$, we write $\exp_r$ for the exponentiation function $\exp_r(k) = r^k$.

Now we may state our main result.

Theorem 1.1. *There exists a 4-generated infinite residually finite periodic 2-group G such that*

$$\pi_G \precsim \exp_8 \circ \mathrm{slog}_2 .$$

In particular, the function π_G grows slower than any iterated logarithm. Furthermore, this group is lawless, which is the content of Proposition 5.6. Thus, Theorem 1.1 gives an affirmative answer to **Q1**.

The group we construct to prove Theorem 1.1 is realized as a group of automorphisms of a spherically homogeneous locally finite rooted tree, whose valency is unbounded. In the theory of automorphisms of rooted trees, it is often interesting to obtain examples acting on regular trees, i.e. locally finite trees where all vertices (except the root vertex) have the same valency. On our way to prove Theorem 1.1, we obtain a family of groups of slow (albeit far faster than the growth described in Theorem 1.1) period growth that act on regular rooted trees without additional work.

Theorem 1.2. *Let $\epsilon > 0$. There exists a finitely generated infinite residually finite periodic 2-group G_ϵ acting on a regular rooted tree (depending on ϵ) such that*

$$\pi_{G_\epsilon} \precsim n^\epsilon.$$

We stress the fact that the groups we construct are residually finite. This is important in the context of the following approximative variant of the restricted Burnside problem. The restricted Burnside problem may be formulated as follows: Are residually finite groups with bounded period growth function finite? Thus, considering groups with slow but not bounded period growth as the next best thing to groups of finite exponent, we ask:

Q2 Among all m-generated residually finite, infinite p-groups G, what are the minimal growth types of π_G? What growth types are possible in general?

Since $\precsim$ is not a linear order, this question is likely very hard to answer in full generality. However, there is a universal lower bound: By Zel'manovs [11, 12] solution to the restricted Burnside problem, the finite residual res $\mathrm{B}(m, n)$ of the free Burnside group of rank m and exponent n is a finite group for all values of m and n. Define

$$\mathrm{zel}_m(n) = \max\{k \in \mathbb{N} \mid |\mathrm{res}\ \mathrm{B}(m,k)| \leq n\}.$$

Since **Q2** excludes finite groups, this function yields a lower bound for the period growth function of any m-generated residually finite infinite p-group. The best known lower bound for $\mathrm{zel}_m(n)$ is due to Groves and Vaughan-Lee [6], who prove that

$$\mathrm{zel}_m(n^{(4^n)}) \geq \mathrm{slog}_m(n).$$

Theorem 1.1 provides a group whose period growth comes close to the best known upper bound for zel_m,

$$\mathrm{zel}_m(2^{2^{\cdot^{\cdot^{\cdot^{2^m}}}}}) \leq 2^k,$$

with k appearances of the number 2 in the tower on the left side, which is due to Newman, whose argument is given in [10].

Organization

After some preliminary definitions, we first prove Theorem 1.2 and then use the groups constructed for this purpose as a model for the more involved construction of the group we use to prove Theorem 1.1. We then establish that all the groups constructed are lawless and thus constitute examples of groups with fast lawlessness growth. We end with some open questions related to the subject.

2. Groups of automorphisms of rooted trees

Let G be a group. For $x, y \in G$, we write $x^y = y^{-1}xy$ and $[x, y] = x^{-1}x^y$. Let S be a generating set for G. We write $\ell_S : G \to \mathbb{N}$ for the word length function of G with respect to S and $B_G^S(n)$ for the set of elements of G of length n with respect to S. For

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two integers $l, u \in \mathbb{Z}$, we denote by $[l, u]$ and $[l, u)$ the set of integer numbers within the corresponding intervals.

Let $(X_n)_{n\in\mathbb{N}_+}$ be a sequence of finite non-empty sets. The *(spherically homogeneous) rooted tree of type* $(X_n)_{n\in\mathbb{N}_+}$ is the tree T with finite strings $x_1 \dots x_k$, $x_i \in X_i$ for $i \in [1, k]$, as vertices and edges between strings that only differ by one letter. The empty string is called the root of the tree. Every vertex of distance k for some fixed $k \in \mathbb{N}$ from the root is a string of length k, which has valency $|X_{k+1}| + 1$. The set $\mathcal{L}_T(k)$ of vertices of distance k to the root is called the kth layer of the tree. We identify the first layer with the set X_1. Every vertex $u \in \mathcal{L}_T(k)$ is the root of a rooted subtree T_u of type $(X_n)_{n\geq k}$. We may compose strings in the following way: if $v \in \mathcal{L}_T(k)$ and $u \in T_v$, then the concatenation vu is a vertex of T.

If the sequence $(X_n)_{n\in\mathbb{N}_+}$ is constant, we call the corresponding tree *regular*. In this case, all subtrees T_u for $u \in T$ are isomorphic.

A *(tree) automorphism* of T is a (graph) automorphism of T fixing the root. Such a map must also leave the layers of T invariant. Let $v \in T$ and $u \in T_v$ be two vertices, and $a \in \mathrm{Aut}(T)$ an automorphism of T. Then the equation

$$(vu).a = (v.a)(u.(a|_v))$$

defines a unique automorphism $a|_v$ of T_v called the *section of a at v*.

Any automorphism a can be decomposed into its sections prescribing the action at the subtrees of the first layer and $a|^\epsilon$, the action of a on the first layer $\mathcal{L}_T(1) = X_1$. We adopt the convention that an X_1-indexed family $(x : a_x)_{x\in X_1}$ of automorphisms $a_x \in \mathrm{Aut}(T_x)$ is identified with the automorphism having section a_x at x, which stabilizes the first layer. Hence, for any $a \in \mathrm{Aut}(T)$, we write

$$a = (x : a|_x)_{x\in X_1} a|^\epsilon.$$

We record some important equalities for sections. Let $a \in \mathrm{Aut}(T), u \in T$ and $v \in T_u$. Then

$$(a|_u)|_v = a|_{uv}, \quad (ab)|_u = a|_u b|_{u.a}, \quad a^{-1}|_u = (a|_{u.a^{-1}})^{-1}.$$

We call an automorphism *rooted* if all its first layer sections are trivial, i.e. if it permutes the set of subtrees $\{T_x \mid x \in X_1\}$. The subgroup of rooted automorphisms is isomorphic to $\mathrm{Sym}(X_1)$.

Let $G \leq \mathrm{Aut}(T)$ be a group of automorphisms. The (pointwise) stabilizer of the kth layer of T in G is denoted $\mathrm{St}_G(k)$ and called the *kth layer stabilizer*. All layer stabilizers are normal subgroups of finite index in G. Their intersection is trivial; hence, the group G is residually finite. The group G is called *spherically transitive* if it acts transitively on every layer $\mathcal{L}_T(k)$.

The *kth rigid layer stabilizer* $\mathrm{Rist}_G(k)$ of a spherically transitive group G for some $k \in \mathbb{N}$ is the product of all (equivalently, the normal closure of a) *rigid vertex stabilizer* $\mathrm{rist}_G(u) = \{g \in G \mid g|_v = \mathrm{id}\,\text{for}\ v \in T \setminus T_u\}$, where $u \in \mathcal{L}_T(k)$. A spherically transitive group G is *weakly branch* if $\mathrm{Rist}_G(k)$ is non-trivial for all $k \in \mathbb{N}$. Every weakly branch group is lawless (cf. [1]).

If T is regular, a group $G \leq \mathrm{Aut}(T)$ is called *self-similar* if for all $u \in T$ the image of the section map $G|_u$ is contained in G. It is called *fractal* if $\mathrm{st}_G(x)|_x = G$ for all $x \in \mathcal{L}_T(1)$. The group G is called *weakly regular branch* if it contains a non-trivial subgroup $H \leq G$ such that $\mathrm{rist}_H(x)|_x \geq H$ for all $x \in \mathcal{L}_T(1)$. Every weakly regular branch group is weakly branch.

Since we aim to provide examples of periodic groups, we need the following criterion for periodicity, which is adopted from the methods developed by Grigorchuk, Gupta and Sidki (cf. [5, 7]). Since our criterion is adapted to a more general situation, we give a short proof.

Proposition 2.1. *Let $G \leq \mathrm{Aut}(T)$ be a group, let π be a set of primes and let $n \in \mathbb{N}$ be a positive integer, such that $G|_u/\,\mathrm{St}_{G|_u}(n)$ is a π-group for every $u \in T$. For every vertex $u \in T$, let $\ell_u : G|_u \to \mathbb{N}$ be a length function such that $\ell_u(g) \leq 1$ implies that g is a π-element.*

If for all vertices $u, v \in T$ such that $v = uw$ for some string w of length n, and all $g \in G|_u$, we have

$$\ell_v(g|_w) < \ell_u(g)/\exp(G|_u/\mathrm{St}_{G|_u}(n)), \tag{*}$$

then G is a π-group.

Proof. Let $g \in G|_u$ for some $u \in \mathcal{L}_T(k)$ and $k \in \mathbb{N}$. We prove that the order of g is finite and divisible by primes in π only. The statement then is obtained by considering $u = \epsilon$. We use induction on $\ell = \ell_u(g)$. If $\ell \leq 1$, the element is a π-element by assumption. If $\ell > 1$, write $q = \exp(G|_u/\,\mathrm{St}_{G|_u}(n))$. By assumption, q is only divisible by primes in π. Now g^q stabilizes the nth layer; hence, $g^q = (x : g^q|_x)_{x \in \mathcal{L}_{T_u}(n)}$ and $\mathrm{ord}(g)$ divides $q \cdot \mathrm{lcm}\{\mathrm{ord}(g^q|_x) \mid x \in \mathcal{L}_{T_u}(n)\}$. Using Equation (*), we obtain

$$\ell_{ux}(g^q|_x) < \ell_u(g^q)/q \leq \ell_u(g) = \ell$$

for all $x \in \mathcal{L}_{T_u}(n)$. Thus, by induction, $\mathrm{ord}(g^q|_x)$ is finite and divisible by primes in π only, and consequently, the same holds for g. □

3. Layerwise length reduction and the proof of Theorem 1.2

We construct a family of groups K_r, indexed by all integers $r \geq 2$, acting on regular rooted trees $T^{(r)}$ whose type depends on r. Fix an integer $r \geq 2$, and write $A_r = C_2^r$ for the elementary abelian 2-group of rank r. Also fix a (minimal) generating set $E_r = \{e_i \mid i \in [0, r)\}$. Let $T^{(r)}$ be the regular rooted tree of type $(A_r)_{n \in \mathbb{N}_+}$. We now construct K_r as a group of automorphisms of $T^{(r)}$, using a construction much in spirit of the Gupta–Sidki p-groups or the second Grigorchuk group. In fact, K_r is a (constant) spinal group in the terminology of [3, 9].

View the group A_r as rooted automorphisms of $T^{(r)}$ by embedding A_r into $\mathrm{Sym}(A_r)$ via its right multiplication action. Notice that we may see an element $a \in A_r$ both as a vertex of $T^{(r)}$ and as an automorphism acting on $T^{(r)}$. We fix a translation map of A_r, given by $a \mapsto \overline{a} := \prod_{i=0}^{r-1} e_i a$. Therefore, $\ell_{E_r}(\overline{e_i}) = r - 1$ for all $i \in [0, r)$.

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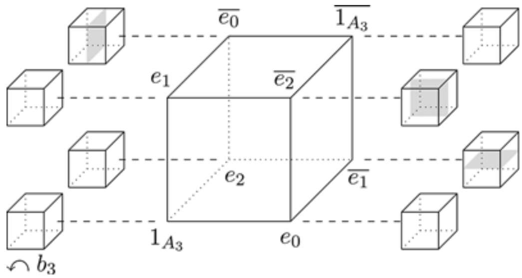

Figure 1. The action of the generator b_3 of K_3 on the first two layers of $T^{(3)}$.

Define $b_r \in \mathrm{Aut}(T^{(r)})$ by

$$b_r = (1_{A_r} : b_r;\ \overline{e_i} : e_i \text{ for } i \in [0, r);\ * : \mathrm{id}),$$

where $*$ stands for every element of A_r not referred to elsewhere in the tuple. Figure 1 depicts the case $r = 3$ as an example. Notice that $b_r \in \mathrm{St}(1)$ is an involution. We define

$$K_r = \langle A_r \cup \{b_r\}\rangle.$$

This is a group generated by $r + 1$ involutions. For $r = 2$, this group contains elements of infinite order, but for $r > 2$, the groups K_r are periodic by [9, Theorem A]. We do not need to rely on this result since the bounds establishing slow period growth also show that K_r is periodic for $r > 4$. Since we are mostly interested in K_r for big r, this suffices for our purposes.

We fix two generating sets for K_r,

$$\mathbb{E}_r = E_r \cup \{b_r\} \quad \text{and} \quad \mathbb{S}_r = A_r \cup b_r^{A_r}$$

and establish some basic properties of the groups K_r.

Lemma 3.1. *Let $r \in \mathbb{N}_+$ be a positive integer. The group K_r is self-similar, fractal and spherically transitive. In particular, it is infinite.*

Proof. The rooted group A_r acts transitively on the first layer. Since rooted elements have trivial sections, self-similarity follows from the fact that all sections of b_r are in $\mathbb{E}_r \subset K_r$. In fact, all elements of $\mathbb{E}_r$ appear as sections of $b_r \in \mathrm{St}_{K_r}(1)$. Conjugating by rooted elements, we may achieve any section of b_r at any first layer vertex; thus, K_r is fractal. By the transitivity of A_r, the group K_r acts transitively on the second layer, and inductively, K_r is spherically transitive. □

Now we come to the core of our argument for establishing slow period growth. We prove an inequality between the length of an element and its sections at vertices of the second layer, using that the automorphism b_r has short sections with respect to $\mathbb{E}_r$, but the only conjugates in b_r^A aside from b_r that have non-trivial section at the vertex 1_{A_r}

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are big with respect to $\mathbb{E}_r$. In preparation for the proof of Theorem 1.1, we prove this inequality for a more general class of groups than just those of the form K_r. Therefore, we need the following technical definition. Let $r \in \mathbb{N}_+$, and let $\tilde{T}$ be a rooted tree of type $(X_n)_{n\in\mathbb{N}_+}$ such that $X_1 = X_2 = A_r$. An element $b \in \mathrm{St}_{\mathrm{Aut}(\tilde{T})}(1)$ is said to be *two-layer resemble* b_r if the following three conditions hold:

(1) $b|_x = b_r|_x$ for $x \in A_r \setminus \{1_{A_r}\}$,
(2) $b|_{1_{A_r}} \in \mathrm{St}(1)$,
(3) $b|_{1_{A_r}x} = b_r|_{1_{A_r}x}$ for $x \in A_r \setminus \{1_{A_r}\}$.

A group $\mathcal{G} \leq \mathrm{Aut}(\tilde{T})$ is said to be *two-layer resemble* K_r *witnessed by* b if it is generated by a set $\mathcal{E} = E_r \cup \langle b\rangle$, where b is an automorphism that two-layer resembles b_r.

Clearly, b_r two-layer resembles itself. Notice that the coset $b_r\,\mathrm{St}(2)$ contains many elements that do not two-layer resemble b_r since the first (and second) layer sections of an element in $\mathrm{St}(2)$ do not need to be rooted. In fact, if the trees $\tilde{T}$ and $T^{(r)}$ coincide, the set of elements that two-layer resembles b_r is equal to the coset $b_r \cdot \mathrm{rist}_{\mathrm{Aut}(T)}(1_{A_r}1_{A_r})$.

Lemma 3.2. *Let $\mathcal{G} \leq \mathrm{Aut}(\tilde{T})$ be a group that two-layer resembles K_r witnessed by $b \in \mathrm{Aut}(\tilde{T})$. Write $\mathcal{S} = A_r \cup \langle b\rangle^{A_r}$ and $\mathcal{S}'' = A_r \cup \langle b|_{1_{A_r}1_{A_r}}\rangle^{A_r}$. Then for all $g \in \mathcal{G}$ and $u \in \mathcal{L}_{\tilde{T}}(2)$, we have*

$$\ell_{\mathcal{S}''}(g|_u) \leq \lceil \ell_{\mathcal{S}}(g)/r \rceil .$$

Proof. The reader less interested in the technicalities may consider this proof in its application to the example $b = b_r$, reading $\mathcal{G} = K_r$, $\mathcal{E} = \mathcal{E}' = \mathcal{E}'' = \mathbb{E}_r$ and $\mathcal{S} = \mathcal{S}'' = \mathbb{S}_r$, avoiding some of the cumbersome notation necessary to deal with the more delicate construction that is necessary for proving Theorem 1.1.

The main idea is the following. An $\mathcal{S}$-word representing an element g gives rise to an $\mathcal{E}'$-word of the same length representing a first layer section $g|_x$. Taking sections again, one finds that the cost of every letter of the form e_i in $g|_{xy}$ is a letter of the form $b^{\overline{e_i}}$ in $g|_x$. But rewriting the $\mathcal{E}'$-word to an $\mathcal{S}'$-word that contains elements of this form must give a far shorter expression. Consequently, the second layer sections are of shorter word length.

It is sufficient to prove $\ell_{\mathcal{S}''}(g|_u) \leq 1$ for all $g \in B_{\mathcal{G}}^{\mathcal{S}}(r)$. From this one derives the desired inequality by cutting a minimal $\mathcal{S}$-word representing g into pieces of length at most r. Thus, let $g \in B_{\mathcal{G}}^{\mathcal{S}}(r)$.

For convenience, we write $\underline{b} = b|_{1_{A_r}}$ and $\underline{\underline{b}} = b|_{1_{A_r}1_{A_r}}$. Furthermore, let $\mathcal{E} = E_r \cup \langle b\rangle$, $\mathcal{E}' = E_r \cup \langle \underline{b}\rangle$ and $\mathcal{E}'' = E_r \cup \langle \underline{\underline{b}}\rangle$.

Notice that for all $x \in A_r$, we have $\mathcal{S}|_x = \mathcal{E}'$, since $a|_x = \mathrm{id}$ for all $a \in A_r$ and, using the first property of automorphisms two-layer resembling b,

$$(b^n)^a|_x = b^n|_{xa^{-1}} = b^n|_{xa} = (b|_{xa})^n$$
$$= \begin{cases} \underline{b}^n & \text{if } x = a, \\ e_j^n & \text{if } xa = \overline{e_j} \text{ for some } j \in [0, r), \\ \mathrm{id} & \text{else.} \end{cases}$$

Thus, $g|_x \in B_{\mathcal{G}}^{\mathcal{E}'}(r)$.

Now minimally represent $g|_x$ as a word in $\mathcal{S}'$ and collect the A_r-type generators to the right, i.e. write

$$g|_x = (\underline{b}^{n_1})^{a_1} \cdots (\underline{b}^{n_{k-1}})^{a_{k-1}} a_k \tag{*}$$

for $k \in \mathbb{N}$, $a_i \in A_r$ and $n_i \in \mathbb{Z}$ for all $i \in [1, k]$. Arguing as above, but now using the second and third properties of automorphisms two-layer resembling b, one finds that the first layer sections of $(\underline{b}^n)^a$ (for $n \in \mathbb{Z}, a \in A_r$) are letters in $\mathcal{E}''$. Thus, given $y \in A_r$, the $\mathcal{S}'$-word representing $g|_x$ yields a $\mathcal{E}''$-word representing $g|_{xy}$. Without loss of generality, we may assume that all but the last letter of (*) contribute a non-trivial $\mathcal{E}''$-letter to $g|_{xy}$. This amounts to

$$a_i \in \{y\} \cup \{y\overline{e_j} \mid j \in [0, r)\}$$

for all $i \in [1, k)$. In case that all $a_i \neq y$ for all $i \in [1, k)$, all $\mathcal{E}''$-letters representing $g|_{xy}$ are in E_r; hence, $g|_{xy}$ is of $\mathcal{S}''$-length 1. Otherwise, there is some $a_i = y$. Without loss of generality, we may assume $a_1 = y$. If $k = 2$, we again find $\ell_{\mathcal{S}''}(g|_{xy}) = 1$. If $k > 2$, the word (*) has a prefix

$$(\underline{b}_1^n)^y (\underline{b}_2^n)^{y\overline{e_j}}$$

for some $j \in [0, r)$. Rewriting (*) as a $\mathcal{E}'$-word, it must have a prefix

$$y\underline{b}^{n_1}\overline{e_j}\underline{b}^{n_2}.$$

(The last part of the $\mathcal{S}'$-prefix may cancel.) But this prefix is of $\mathcal{E}'$-length

$$\ell_{A_r}(y) + 1 + \ell_{A_r}(\overline{e_j}) + 1 \geq r + 1.$$

This is impossible since we have established $g|_x \in B_{\mathcal{G}}^{\mathcal{E}'}(r)$. Consequently, if $a_i = y$, we must have $k = 2$, and $\ell_{\mathcal{S}''}(g|_{xy}) \leq 1$. □

Applying the lemma to $b = b_r$ and $G = K_r$ (and using the self-similarity of K_r), we obtain the following inequality.

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Lemma 3.3. *Let $g \in K_r$ be an element and let $u \in \mathcal{L}_{T(r)}(2)$. Then*

$$\ell_{\mathbb{S}_r}(g|_u) \leq \lceil \ell_{\mathbb{S}_r}(g)/r \rceil .$$

Proof of Theorem 1.2. Notice that Proposition 2.1, Lemma 3.1, Lemma 3.3 and the fact that $K_r/\operatorname{St}_{K_r}(2)$ is a subgroup of the permutational wreath product of elementary abelian 2-groups, hence itself a 2-group, show that K_r is an infinite 2-group in case $r > 4$.

We prove $\pi_{K_r}^{\mathbb{S}_r}(n) \leq n^{1/(\log_4(r)-1)}$ for every $n \in \mathbb{N}$ and $r > 4$. Clearly, choosing some big integer r, this proves the theorem.

Let $g \in K_r$ be an element. Write $n = \ell_{\mathbb{S}_r}(g)$. Since A_r is a group of exponent two, $g^2 \in \operatorname{St}_{K_r}(1)$ and $g^4 \in \operatorname{St}_{K_r}(2)$. Consequently, the order of g^4 is the least common multiple of the orders of $g^4|_u$ for $u \in \mathcal{L}_{T(r)}(2)$, which equals, since K_r is a 2-group, the maximum of their orders, i.e.

$$\operatorname{ord}(g) \leq 4 \cdot \max\{\operatorname{ord}(g^4|_u) \mid u \in \mathcal{L}_{T(r)}(2)\}.$$

In view of Lemma 3.3, we see $\ell_{\mathbb{S}_r}(g^4|_u) \leq \lceil \frac{4n}{r} \rceil$, so for $n \geq r$

$$\pi_{K_r}^{\mathbb{S}_r}(n) \leq 4 \cdot \pi_{K_r}^{\mathbb{S}_r}(\lceil 4n/r \rceil);$$

hence, using that K_r is generated by involutions,

$$\pi_{K_r}^{\mathbb{S}_r}\left(\left(\tfrac{r}{4}\right)^k\right) \leq 4^k \pi_{K_r}^{\mathbb{S}_r}(1) = 2 \cdot 4^k.$$

This implies

$$\pi_{K_r}^{\mathbb{S}_r} \precsim \exp_4 \circ \log_{\frac{r}{4}} \sim n^{1/(\log_4(r)-1)}. \qquad \square$$

4. Growing valency and the proof of Theorem 1.1

We now construct a group G with the properties described in Theorem 1.1. To achieve this, we take the generators b_r of the groups K_r constructed in the previous section and build a single automorphism d acting on a rooted tree with unbounded valency that resembles some b_{r_0} for two layers (where the valency is $2^{r_0} + 1$), then use one layer to increase the valency to $2^{r_1} + 1$ for some $r_1 > r_0$ that resembles b_{r_1} for two layers &c. This will allow us to use the reduction formulas for the b_r but with (rapidly) increasing r.

The slowest period growth (using this construction) will be achieved if one arranges the sequence $(r_n)_{n\in\mathbb{N}}$ to grow as fast as possible. For this, there is a natural upper bound. We want the sections of d at a given layer of valency $r_{n+1} + 1$ to generate an elementary abelian 2-group acting on the layer below but can use no more than $2^{r_n} - 1$ sections as generators. Hence, the maximum possible increase of valency is given by the following function $f : \mathbb{N} \to \mathbb{N}$. Let $f(0) = 3$ and $f(k + 1) = 2^{f(k)} - 1$ for $k \in \mathbb{N}$. Since we aim to increase the valency of our tree on every third layer, we also introduce $f_3(k) = f(\lfloor k/3 \rfloor)$, a function that takes every value of f thrice. These functions grow very quickly.

Lemma 4.1. *For all $k \in \mathbb{N}$, we have $f(k) \geq \mathrm{tetr}_2(k)$.*

Proof. We use induction on k for the statement $f(k)-1 \geq \mathrm{tetr}_2(k)$. Clearly, $f(0)-1 = 2 \geq 1 = \mathrm{tetr}_2(0)$. Now for all $k > 0$

$$f(k+1) - 1 = 2^{f(k)} - 2 \geq 2^{f(k)-1} \geq 2^{\mathrm{tetr}_2(k)} \geq \mathrm{tetr}_2(k+1). \qquad \square$$

Recall from the previous section that A_r denotes a copy of the elementary abelian 2-group with an (ordered) basis $E_r = \{e_0, \ldots, e_{r-1}\}$. We now fix some enumeration (which may depend on r) $\{a_i \mid i \in [0, 2^r)\} = A_r$ for these groups, such that a_0 is the trivial element. Also recall the translation map $a \mapsto \overline{a}^{(r)} = a \prod_{i=0}^{r-1} e_i$ defined in the previous section. We introduce the superscript to make precise within which group we are translating.

Now we define T as the rooted tree of type $(A_{f_3(k)})_{k \in \mathbb{N}}$. For any $k \equiv_3 0$ excluding $k = 0$, the kth, $(k+1)$st and $(k+2)$nd layers of T have valency $2^{f_3(k)} + 1$. Write T_k for the (isomorphism class) of any subtree of T_u for some $u \in \mathcal{L}_T(k)$, i.e. $T_0 = T$ and T_k of type $(A_{f_3(l)})_{l \geq k}$.

Again we view the group $A_{f_3(k)}$ as rooted automorphisms by their right multiplication action. Define a sequence of automorphisms $d_n \in \mathrm{Aut}(T_k)$ for $k \in \mathbb{N}$ by

$$d_k = (1_{A_{f_3(k)}} : d_{k+1};\ \overline{e_i}^{(f_3(k))} : e_i;\ * : \mathrm{id}) \qquad \text{for } k \equiv_3 0, 1 \text{ and}$$
$$d_k = (1_{A_{f_3(k)}} : d_{k+1};\ a_i : e_i \in A_{f_3(k+1)} \text{ for } i \in [1, 2^{f_3(k)})) \qquad \text{for } k \equiv_3 2.$$

Finally, we define $G_k = \langle A_{f_3(k)} \cup \{d_k\} \rangle \leq \mathrm{Aut}(T_k)$ and write G for G_0.

Note that among the sections of d_k are all the elements of E_{k+1}. Using this, we see that, for every $v \in T$ of length k, we have $G|_v = G_k$ and G acts spherically transitively on T.

For $k \in \mathbb{N}$, define $\mathrm{S}_k = A_{f_3(k)} \cup \{d_k\}^{A_{f_3(k)}}$ and $\mathrm{E}_k = E_{f_3(k)} \cup \{d_k\}$, filling the rôles of $\mathbb{S}_r$ and $\mathbb{E}_r$ of § 3. Both are generating sets for G_k. Note that $d_k^2 = 1$; hence, both sets consist of involutions.

Lemma 4.2. *Let $k \in \mathbb{N}$ be a positive integer such that $k \equiv_3 0$ and $g \in G_k$ an element. Then for all $v \in \mathcal{L}_{T_k}(2)$, we have*

$$\ell_{\mathrm{S}_{k+2}}(g|_v) \leq \left\lceil \frac{\ell_{\mathrm{S}_k}(g)}{f(k/3)} \right\rceil.$$

Proof. We apply Lemma 3.2. This is possible since by definition, d_k two-layer resembles $b_{f(k/3)}$. Notice that $\mathcal{S} = \mathrm{S}_k$ and $\mathcal{S}'' = \mathrm{S}_{k+2}$. $\square$

Lemma 4.3. *Let $k \in \mathbb{N}$ and let $g \in G_k$. Then for all $x \in \mathcal{L}_{T_k}(1)$*

$$\ell_{\mathrm{S}_{k+1}}(g^2|_x) \leq \ell_{\mathrm{S}_k}(g) + 1.$$

Proof. Since $\langle d_k \rangle^{A_{f_3(k)}}$ is closed under conjugation with $A_{f_3(k)}$, we may write $g = d_k^{a_1} \cdots d_k^{a_{\ell-1}} c$ for $\ell = \ell_{\mathrm{S}_n}(g)$, for some $a_i \in A_{f_3(k)}$ for $i \in [1, \ell)$ and $c \in \mathrm{S}_r \setminus \{\mathrm{id}\}$.

Then $g|_x = d_k^{a_1}|_x \cdots d_k^{a_{\ell-1}}|_x c|_x$. Now at most every second expression, $d_k^{a_i}|_x$ (including $c|_x$ if it is of this form) can evaluate to d_k. Otherwise, there is some i such that $a_i = a_{i+1} = x$, respectively, $a_{\ell-1} = x$ and $c = d_k^x$, which implies

$$g = d_k^{a_1} \cdots d_k^{a_{i-1}} d_k^x d_k^x d_k^{a_{i+2}} \cdots d_k^{a_{\ell-1}} c = d_k^{a_1} \cdots d_k^{a_{i-1}} d_k^{a_{i+2}} \cdots d_k^{a_{\ell-1}} c, \qquad (**)$$

and $g = d_k^{a_1} \cdots d_k^{a_{\ell-2}}$, respectively. But then $\ell_{\mathrm{S}_n}(g) \leq \ell - 2$, a contradiction. Hence, there are at most $\lceil \ell/2 \rceil$ symbols d_k in the product $d_k^{a_1}|_x \cdots d_k^{a_{\ell-1}}|_x c|_x$. Thus, we have

$$g|_x = d_k^{a'_1} \cdots d_k^{a'_{n-1}} a'_n$$

for some $n \leq \lceil \ell/2 \rceil + 1$ and $a'_i \in A_r$ for $i \in [1, n]$.

Now consider $g|_{x.g}$. If $g \in \mathrm{St}(1)$, we have $g|_{x.g} = g|_x$ and hence

$$g^2|_x = (g|_x)^2 = d_k^{a'_1} \cdots d_k^{a'_{n-1}} a'_n d_k^{a'_1} \cdots d_k^{a'_{n-1}} a'_n = d_k^{a'_1} \cdots d_k^{a'_{n-1}} d_k^{a'_1{}^{a'_n}} \cdots d_k^{a'_{n-1}{}^{a'_n}},$$

thus, $\ell_{\mathrm{S}_{k+1}}(g^2|_x) = 2(n-1) \leq \ell + 1$. It remains to consider the case $g \notin \mathrm{St}(1)$. Notice that every expression $d_k^{a_i}$ in (**) can only contribute one d_k-letter to all first-layer sections. Thus, in $g|_x$ and $g|_{x.g}$, cumulatively, there are at most ℓ such letters. Collecting the $A_{f_3(k)}$-letters to the right, the product $g|_x g|_{x.g}$ is at most of length $\ell + 1$. □

Lemma 4.4. *Let $k \equiv_3 0$ and let $g \in G_k$. Then for all $u \in \mathcal{L}_{T_k}(3)$,*

$$\ell_{\mathrm{S}_{k+3}}(g^8|_u) \leq \left\lceil \frac{4 \cdot \ell_{\mathrm{S}_k}(g)}{f(k/3)} \right\rceil + 1.$$

Proof. Since $A_{f_3(k)}$ and $A_{f_3(k+1)}$ are of exponent two, we have $g^4 \in \mathrm{St}_{G_k}(2)$. Hence, $g^8|_u = (g^4|_{u_1 u_2})^2|_{u_3}$, where $u = u_1 u_2 u_3$. Now

$$\begin{aligned} \ell_{\mathrm{S}_{k+3}}(g^8|_u) &= \ell_{\mathrm{S}_{k+3}}((g^4|_{u_1 u_2})^2|_{u_3}) \\ &\leq \ell_{\mathrm{S}_{k+2}}(g^4|_{u_1 u_2}) + 1 && \text{(by Lemma 4.3)} \\ &\leq \left\lceil \frac{\ell_{\mathrm{S}_k}(g^4)}{f(k/3)} \right\rceil + 1 && \text{(by Lemma 4.2)} \\ &\leq \left\lceil \frac{4 \cdot \ell_{\mathrm{S}_k}(g)}{f(k/3)} \right\rceil + 1. \end{aligned}$$

□

Lemma 4.5. *The group G is a 2-group.*

Proof. This follows from Proposition 2.1 and Lemma 4.2. Using the notation of Proposition 2.1, let $n = 10$. Since $G|_u / \mathrm{St}_{G|_u}(1)$ is an elementary abelian 2-group for all $u \in T_0$, we see that $\exp(G|_u / \mathrm{St}_{G|_u}(n)) \leq 2^n$. Now, regardless of the value of k modulo 3, taking the 10th section of some $g \in G_k$ allows us to invoke Lemma 4.2 at least

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three times. Hence, for all $w \in \mathcal{L}_{T_k}(10)$,

$$\ell_{\mathrm{S}_{k+10}}(g|_w) \le \left\lceil \frac{\ell_{\mathrm{S}_k}(g)}{f(0)f(1)f(2)} \right\rceil = \left\lceil \frac{\ell_{\mathrm{S}_k}(g)}{3 \cdot 7 \cdot 127} \right\rceil < \frac{\ell_{\mathrm{S}_k}(g)}{2^{10}},$$

and we conclude that G is a 2-group. □

Proof. Proof of Theorem 1.1 Let $n, k \in \mathbb{N}$ with $k \equiv_3 0$, and let $g \in B_{G_k}^{\mathrm{S}_k}(n)$. Since $\exp(A_l) = 2$ for all $l \in \mathbb{N}$, the 2^3-power of g fixes the third layer of T_n; hence,

$$\mathrm{ord}(g) \le 8 \cdot \max\{\mathrm{ord}(g^8|_v) \mid v \in \mathcal{L}_{T_k}(3)\}.$$

Now Lemma 4.4 implies

$$\pi_{G_k}^{\mathrm{S}_k}(n) \le 8 \cdot \pi_{G_{k+3}}^{\mathrm{S}_{k+3}} \left(\left\lceil \tfrac{4 \cdot n}{f(k/3)} \right\rceil + 1 \right).$$

Writing $v_k(n) = \lceil 4 \cdot n / f(k/3) \rceil + 1$ and

$$u(n) = \min\{l \in \mathbb{N} \mid v_l(v_{l-1}(\cdots(v_0(n))\cdots)) = 2\},$$

we find

$$\pi_G^{\mathrm{S}}(u(n)) \le 8^n \cdot \pi_{G_{3n}}^{\mathrm{S}_{3n}}(2).$$

Now, using the same argument as before, we see that $\pi_{G_{3n}}^{\mathrm{S}_{3n}}(2) \le 4$ by Lemma 4.2. Thus, deriving $\mathrm{tetr}_2 \precsim u(n)$ from Lemma 4.1, we obtain

$$\pi_G \precsim \exp_8 \circ \,\mathrm{slog}_2\,.$$

□

5. Lawlessness growth

Let G be a lawless group generated by a finite set S. By the definition of lawlessness, the image of the word map $w(G^m)$ is non-trivial for every reduced word $w \in F_m \setminus \{1\}$ in m letters, $m \in \mathbb{N}$. We may define the *complexity of w in G with respect to S* by

$$\chi_G^S(w) = \min\left\{ \sum_{i=1}^m \ell_S(g_i) \mid \underline{g} = (g_i)_{i=1}^m \in G^m, w(\underline{g}) \ne 1 \right\} \in \mathbb{N}.$$

Now the *lawlessness growth function* $\mathcal{A}_G^S : \mathbb{N} \to \mathbb{N}$ of G with respect to S is defined by

$$\mathcal{A}_G^S(n) = \max\{\chi_G^S(w) \mid w \in F_m \setminus \{1\} \text{ with } \ell_S(w) \le n\}.$$

This definition is due to Bradford, first given in [4], where he proves the independence of the growth type from the choice of generating set and establishes a connection to the period growth in the case of periodic p-groups.

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Proposition 5.1. [4] *Let G be a finitely generated lawless periodic p-group for some prime p and $f : \mathbb{N} \to \mathbb{N}$ some function. Then $\pi_G^S(n) \le f(n)$ implies $\mathcal{A}_G^S(f(n)) \ge n$.*

Using this, we give examples of groups with large lawlessness growth (cf. [4, Question 10.2]) by proving that the groups constructed in the previous sections are in fact lawless. As a consequence of Theorem 1.1 and Proposition 5.1, we obtain the following corollary.

Corollary. *There is a finitely generated lawless group G such that*

$$\mathcal{A}_G^S \gtrsim \mathrm{tetr}_2 \circ \log_8 .$$

It remains to prove that the group G of Theorem 1.1 is lawless. We prove that it is weakly branch, which is sufficient by [1]. Our proof is technical but also establishes that the groups K_r are weakly branch for all integers $r > 5$. To avoid some obstacles appearing for small valencies, we look at G_6 instead of $G = G_0$, for which the proof of Theorem 1.1 works verbatim, except for the number of generators. Thus, in the remainder of this section, we write G for G_6 and define the function f prescribing the valencies of the tree upon which G acts by $f(0) = 127$ and $f(n+1) = 2^{f(n)} - 1$ for $n > 0$.

Lemma 5.3. *Let $r \in \mathbb{N}_{>5}$ and let $\mathcal{G} \le \mathrm{Aut}(\widetilde{T})$ be a group that two-layer resembles K_r witnessed by b. Define*

$$N = \langle [b, e_i, e_j] \mid i, j \in [0, r), i \ne j \rangle^{\mathcal{G}} \le \mathrm{Aut}(\widetilde{T}), \quad \textit{and}$$
$$\underline{N} = \langle [b|_{1_{A_r}}, e_i, e_j] \mid i, j \in [0, r), i \ne j \rangle^{\mathcal{G}|_{1_{A_r}}} \le \mathrm{Aut}(\widetilde{T}|_{1_{A_r}})$$

Then for every $x \in \mathcal{L}_{\widetilde{T}}(1)$, we have $\mathrm{rist}_N(x) \ge \underline{N}$.

Proof. We use left-normed commutators, i.e. $[x, y, z] = [[x, y], z]$. Write $c_{i,j} = [b, e_i, e_j]$ for the (normal) generators of N. Clearly, $N \le \mathrm{St}_{\mathcal{G}}(1)$. We compute

$$c_{i,j}|_x = \begin{cases} b|_{1_{A_r}} & \text{if } x \in \{1_{A_r}, e_i, e_j, e_i e_j\}, \\ e_t & \text{if } x \in \{\overline{e_t}, \overline{e_t e_i}, \overline{e_t e_j}, \overline{e_t e_i e_j}\} \text{ and } t \in [0, r) \setminus \{i, j\}, \\ e_i e_j & \text{if } x \in \{\overline{1_{A_r}}, \overline{e_i e_j}, \overline{e_i}, \overline{e_j}\}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

Let i, j, k, m, n be pairwise distinct elements of $[0, r)$ (here we need $r > 4$). We look at $[c_{i,j}, c_{m,n}^{\overline{e_k}}]$. Since both $c_{i,j}$ and $c_{m,n}^{\overline{e_k}}$ are in $\mathrm{St}(1)$, taking the commutator commutes with taking sections. All sections except $b|_{1_{A_r}}$ commute, so we have $[c_{i,j}, c_{m,n}^{\overline{e_k}}]|_x = \mathrm{id}$ for all $x \notin \{1_{A_r}, e_i, e_j, e_i e_j, \overline{e_k}, \overline{e_k e_m}, \overline{e_k e_n}, \overline{e_k e_m e_n}\}$. Since $r > 5$, all these vertices are distinct.

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Furthermore, for the remaining cases, we calculate

$$[c_{i,j}, c_{m,n}^{\overline{e_k}}]|_x = \begin{cases} [b|_{1_{A_r}}, e_k] & \text{if } x = 1_{A_r}, \\ [e_k, b|_{1_{A_r}}] & \text{if } x = \overline{e_k}, \\ [b|_{1_{A_r}}, \mathrm{id}] = \mathrm{id} & \text{if } x \in \{e_i, e_j, e_ie_j\}, \\ [\mathrm{id}, b|_{1_{A_r}}] = \mathrm{id} & \text{if } x \in \{\overline{e_ke_m}, \overline{e_ke_n}, \overline{e_ke_me_n}\}. \end{cases}$$

Now let $l \in [0, r) \setminus \{i, j, k\}$. Then $c_{i,j}^{\overline{e_l}}|_{1_{A_r}} = e_l$ and $c_{i,j}^{\overline{e_l}}|_{\overline{e_k}} = c_{i,j}|_{e_ke_l} = \mathrm{id}$. Consequently,

$$[c_{i,j}, c_{m,n}^{\overline{e_k}}, c_{i,j}^{\overline{e_l}}]|_x = \begin{cases} [b|_{1_{A_r}}, e_k, e_l] & \text{if } x = 1_{A_r}, \\ \mathrm{id} & \text{else;} \end{cases}$$

thus, $\mathrm{rist}_N(1_{A_r}) \geq \langle [b|_{1_{A_r}}, e_i, e_j] \mid i, j \in [0, r), i \neq j \rangle$. Since $\{b^{\overline{e_i}}|_{1_{A_r}} \mid i \in [0, r)\} \cup \{b|_{1_{A_r}}\}$ generates $\mathcal{G}|_{1_{A_r}}$, for every $g \in \mathcal{G}|_{1_{A_r}}$, we find an element $\widehat{g} \in \mathrm{St}_{\mathcal{G}}(1)$ such that $\widehat{g}|_{1_{A_r}} = g$. Conjugating with these elements, we find $\mathrm{rist}_N(1_{A_r}) \geq \underline{N}$. Since $\mathcal{G}$ acts transitively on the first layer, all rigid vertex stabilizers are conjugate, and we obtain the result. □

Proposition 5.4. *Let $r \in \mathbb{N}_{>5}$. Then K_r is weakly regular branch, hence lawless.*

Proof. This follows directly from Lemma 5.3, since the two normal subgroups $N, \underline{N}$ are equal in the case of K_r. □

Lemma 5.5. *Let $k \in \mathbb{N}$ and $x \in \mathcal{L}_{T_k}(1)$. Then $\mathrm{St}_{G_k}(1)|_x \geq G_{k+1}$.*

Proof. Observe $\mathrm{E}_{k+1} = \{d_k|_x \mid x \in \mathcal{L}_{T_k}(1)\}$ and that G_k acts transitively on $L_{T_k}(1)$. □

Proposition 5.6. *The group $G = G_6$ is a weakly branch group, hence a lawless group.*

Proof. Let $k \in \mathbb{N}$ be an integer such that $k \equiv_3 0$. We adopt the following notation to better distinguish between the generators of $A_{f_3(k)}$ and $A_{f_3(k+3)}$. If $a = e_{i_0} \dots e_{i_t}$ is a non-trivial element of $A_{f_3(k)}$, we write $\underline{e}_{i_0 \dots i_t}$ for the generator $d_{k+2}|_a$ of $A_{f_3(k+3)}$. Each element of $E_{f_3(k+3)}$ appears in this way. Define

$$N_k = \langle [d_k, e_i, e_j] \mid i, j \in [0, f_3(k)), i \neq j \rangle^{G_k}, \quad \text{and}$$
$$M_k = \left\langle [[d_k, a_1], [d_k, a_2]^g] \;\middle|\; \begin{array}{l} g \in G_k, a_1 = \underline{e}_j\underline{e}_{ij}\underline{e}_l\underline{e}_{il}, a_2 = \underline{e}_n\underline{e}_{mn}\underline{e}_s\underline{e}_{ms}, \\ i, j, l, m, n, s \in [0, f_3(k-1)) \text{pairwise distinct} \end{array} \right\rangle^{G_k}.$$

The group G_k two-layer resembles $P_{f_3(k)}$; thus, Lemma 5.3 implies $\mathrm{rist}_{N_{k+1}}(u) \geq N_{k+2}$ for $u \in \mathcal{L}_{T_{k+1}}(1)$. We show that

$$\mathrm{rist}_{M_k}(w) \geq N_{k+1} \text{for } k > 0, \text{ and} \tag{†}$$

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$$\mathrm{rist}_{N_{k+2}}(v) \geq M_{k+3}. \tag{‡}$$

Using this, we see that for all $u \in \mathcal{L}_T(l)$,

$$\mathrm{rist}_G(u) \geq \begin{cases} M_l & \text{if } l \equiv_3 0, \\ N_l & \text{otherwise.} \end{cases}$$

Since N_l and M_l are non-trivial for all $l \in \mathbb{N}$, this shows that G is a weakly branch group.

In both cases, it is enough to show that the normal generators of N_{k+1}, respectively, M_{k+3}, are contained in the rigid vertex stabilizer of $1_{A_{f_3(k+1)}}$, respectively, $1_{A_{f_3(k+3)}}$. Using Lemma 5.5, we find the full normal subgroup within the rigid vertex stabilizer of $1_{A_{f_3(k)}}$, and since G_k acts spherically transitive, all rigid vertex stabilizers of the same layer are conjugate.

We first prove Equation (†). Let $k > 0$. Let $a_1, a_2 \in B_{A_{f_3(k)}}^{E_{f_3(k)}}(4)$ such that $[[d_k, a_1], [d_k, a_2]]$ is a normal generator of M_k. Calculate

$$[d_k, a_1]|_x = d_k d_k^{a_1}|_x = \begin{cases} d_{k+1} & \text{if } x \in \{1_{A_{f_3(k)}}, a_1\}, \\ e_t & \text{if } x \in \{\overline{e_t}, \overline{e_t} a_1\}, \text{ for some } t \in [0, f_3(k)), \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

We want to compute $[[d_k, a_1], [d_k, a_2]^{\overline{e}_s}]$ for arbitrary $s \in [0, f_3(k))$. The set of vertices where this element might have non-trivial sections is $\{1_{A_{f_3(k)}}, a_1, \overline{e_s}, \overline{e_s} a_2\}$.

We now prove that the sections $[d_k, a_1]|_{\overline{e_s} a_2}$ and $[d_k, a_2]^{\overline{e_s}}|_{a_1}$ are trivial, i.e. that

$$\overline{e_s} a_2 \notin \{1_{A_{f_3(k)}}, a_1, \overline{e_t}, \overline{e_t} a_1 \mid t \in [0, f_3(k)), \quad \text{and}$$
$$\overline{e_s} a_1 \notin \{1_{A_{f_3(k)}}, a_2, \overline{e_t}, \overline{e_t} a_2 \mid t \in [0, f_3(k)).$$

Now $\ell_{A_{f_3(k)}}(\overline{e_s} a_2) \geq f_3(k) - 5$; hence, $\overline{e_s} a_2$ is neither trivial nor equal to a_1 of length 4. Here we use that $f_3(k) \geq f(0) > 9$. Finally, $\overline{e_t} a_1 = \overline{e_s} a_2$ implies $a_1 e_s = a_2 e_t$, which contradicts the definition of a_1 and a_2. This proves the first, and by analogy the second, non-inclusion statement above.

Thus, we find

$$[[d_k, a_1], [d_k, a_2]^{\overline{e}_s}]|_x = \begin{cases} [d_{k+1}, e_s] & \text{if } x = 1_{A_{f_3(k)}}, \\ [e_s, d_{k+1}] & \text{if } x = \overline{e_s}, \\ \mathrm{id} & \text{otherwise.} \end{cases}$$

For every $q \in [0, f_3(k)) \setminus \{s\}$, we obtain

$$h = [[d_k, a_1], [d_k, a_2]^{\overline{e}_s}, [d_k, a_1]^{\overline{e}_q}] \in \mathrm{rist}_{M_k}(1_{A_{f_3(k)}}),$$

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such that $h|_{1_{A_{f_3(k)}}} = [d_{k+1}, e_s, e_q]$. This concludes the proof of Equation (†).

We now prove Equation (‡). Write $c_{i,j}$ for the element $[d_{k+2}, e_i, e_j] \in N_{k+2}$, where $i, j \in [0, f_3(k+2))$ are two distinct integers. Observe that

$$c_{i,j}|_{1_{A_{f_3(k+2)}}} = d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}$$

and that $c_{i,j}|_u \in A_{f_3(k+3)}$ for all $u \in \mathcal{L}_{T_{k+2}}(1)$ except the (distinct) vertices $1_{A_{f_3(k+2)}}$, e_i, e_j and e_ie_j. Thus, for $l \in [0, f_3(k+2)) \setminus \{i, j\}$, we compute

$$[c_{i,j}, c_{i,l}]|_x = \begin{cases} [d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}, d_{k+3}\underline{e}_i\underline{e}_l\underline{e}_{il}] & \text{if } x = 1_{A_{f_3(k+2)}}, \\ \text{possibly non-trivial} & \text{if } x \in \{1_{A_{f_3(k)}}, e_i, e_j, e_l, e_ie_j, e_ie_l\}, \\ \text{id} & \text{otherwise.} \end{cases}$$

By Lemma 5.5, there is an element $\widehat{g}_0 \in \mathrm{St}_{G_{k+2}}(1)$ such that $\widehat{g}_0|_{1_{A_{f_3(k+2)}}} = \underline{e}_i\underline{e}_j\underline{e}_{ij}$. Now

$$[c_{i,j}, c_{i,l}]^{\widehat{g}_0}|_{1_{A_{f_3(k+2)}}} = [d_{k+3}\underline{e}_i\underline{e}_j\underline{e}_{ij}, d_{k+3}\underline{e}_i\underline{e}_l\underline{e}_{il}]^{\underline{e}_i\underline{e}_j\underline{e}_{ij}} = [d_{k+3}, \underline{e}_j\underline{e}_l\underline{e}_{ij}\underline{e}_{il}],$$

and the set of vertices x such that $[c_{i,j}, c_{i,l}]^{g_0}|_x$ is possibly non-trivial, as for $[c_{i,j}, c_{i,l}]$, the set $\{1_{A_{f_3(k)}}, e_i, e_j, e_l, e_ie_j, e_ie_l\}$.

Let $g \in G_{k+3}$. There is an element $\widehat{g}_1 \in \mathrm{St}_{G_k}(1)$ such that $\widehat{g}_1|_{1_{A_{f_3(k)}}} = g$. We conclude that for three pairwise distinct integers $m, n, s \in [0, f_3(k+2)) \setminus \{i, j, l\}$ (which is possible since the minimum value of f_3 greater then 5)

$$[[c_{i,j}, c_{i,l}], [c_{m,n}, c_{m,s}]^{\widehat{g}}]|_{1_{A_{f_3(k)}}} = [[d_{k+2}, \underline{e}_j\underline{e}_{ij}\underline{e}_l\underline{e}_{il}], [d_{k+2}, \underline{e}_n\underline{e}_{mn}\underline{e}_s\underline{e}_{ms}]^g],$$

while all other sections are trivial; hence, $\mathrm{rist}_{N_k}(1_A) \geq M_{k+1}$. □

6. Open questions and related concepts

In [2], the authors refer to an unpublished text of Leonov [8], where he establishes a connection between the word growth and the period growth of the Grigorchuk group. It seems plausible that there is such a connection: slow word growth makes for few elements of a given length, hence for a smaller set of candidates that might have big order. Consequently, we pose the following refinement of the question of Bradford.

Q3 Is there an infinite finitely generated residually finite periodic group of exponential word growth and sublinear period growth?

To answer this, it would be sufficient to prove that the groups constructed in Theorem 1.1 and Theorem 1.2 are of exponential growth, but we doubt that this is true. In view of the numerical relation between the word and period growth in the Grigorchuk

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group, we think that the groups G and G_ϵ are interesting candidates for groups of slow intermediate word growth. Thus we ask:

Q4 Of what growth type is the word growth of G and of G_ϵ?

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