Glasgow Math. J. **59** (2017) 743–751.
doi:10.1017/S0017089516000537.

EIGENVALUES OF GEOMETRIC OPERATORS RELATED TO THE WITTEN LAPLACIAN UNDER THE RICCI FLOW

SHOUWEN FANG

School of Mathematical Science, Yangzhou University, Yangzhou, Jiangsu 225002, P. R. China
e-mail: shwfang@163.com

FEI YANG

School of Mathematics and physics, China University of Geosciences, Wuhan 430074, P. R. China
e-mail: yangfei810712@163.com

and PENG ZHU

School of Mathematics and physics, Jiangsu University of Technology, Changzhou, Jiangsu 213001, P. R. China
e-mail: zhupeng2004@126.com

(Received 14 December 2015; accepted 12 July 2016; first published online 27 February 2017)

Abstract. Let $(M, g(t))$ be a compact Riemannian manifold and the metric $g(t)$ evolve by the Ricci flow. In the paper, we prove that the eigenvalues of geometric operator $-\Delta_\phi + \frac{R}{2}$ are non-decreasing under the Ricci flow for manifold M with some curvature conditions, where Δ_ϕ is the Witten Laplacian operator, $\phi \in C^2(M)$, and R is the scalar curvature with respect to the metric $g(t)$. We also derive the evolution of eigenvalues under the normalized Ricci flow. As a consequence, we show that compact steady Ricci breather with these curvature conditions must be trivial.

2010 *Mathematics Subject Classification.* 53C21, 53C44.

1. Introduction. It is well-known that the eigenvalue problems have received a lot of attention in various areas in mathematics. The eigenvalues of geometric operators also have become a powerful tool in the study of geometry and topology of manifolds. Recently, there occur many interesting results on the eigenvalue problems under different geometric flows, especially the Ricci flow. In a seminal preprint [**14**], Perelman introduced the so-called F-entropy functional and proved that it is non-decreasing along the Ricci flow coupled to a backward heat-type equation. The non-decreasing of the functional F implies the monotonicity of the first eigenvalue of $-4\Delta + R$ along the Ricci flow. With his entropy and the monotonicity of the first eigenvalue, Perelman was able to rule out non-trivial steady or expanding breathers on compact manifolds. In [**13**], Ma obtained the monotonicity of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition along the Ricci flow. Cao [**2**] considered the eigenvalues of $-\Delta + \frac{R}{2}$, showed that they are non-decreasing under the Ricci flow for manifolds with non-negative curvature operator, and got a new proof of non-existence for non-trivial steady Ricci solitons which had been proved by Hamilton [**6**, **7**] and Perelman [**14**]. Li got the monotonicity of eigenvalues of the operator $-4\Delta + kR$

and ruled out compact steady Ricci breathers by using their monotonicity [**9**]. Later, Cao [**3**] also improved his own previous results and proved that the first eigenvalues of $-\Delta + cR(c \geq \frac{1}{4})$ are non-decreasing under the Ricci flow on the manifolds without curvature assumption. Ling studied the first non-zero eigenvalue under the normalized Ricci flow, gave a Faber–Krahn type of comparison theorem and a sharp bound [**11**], and constructed a class of monotonic quantities on closed n-dimensional manifolds [**12**]. Moreover, Zhao [**16**] got the evolution equation for the first eigenvalue of the Laplacian operator along the Yamabe flow and gave some monotonic quantities under the Yamabe flow. Guo and his collaborators [**5**] derived an explicit formula for the evolution of the lowest eigenvalue of the Laplace–Beltrami operator with potential in abstract geometric flows. The first author, Xu and Zhu [**4**] proved the monotonicity of eigenvalues of $-\Delta_\phi + cR(c > \frac{1}{4})$ along the system of Ricci flow coupled to a heat equation.

In this paper, we consider an n-dimensional compact Riemannian manifold M with a time-dependent Riemannian metric $g(t)$. $(M, g(t))$ is a smooth solution to the Ricci flow equation:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}(t). \tag{1}$$

Let ∇ be the Levi–Civita connection on (M, g), Δ the Laplace–Beltrami operator, $d\nu$ the Riemannian volume measure, and $d\mu$ the weight volume measure on (M, g), i.e.,

$$d\mu = e^{-\phi(x)} d\nu,$$

where $\phi \in C^2(M)$. Then, the Witten Laplacian (also called symmetric diffusion operator)

$$\Delta_\phi = \Delta - \nabla\phi \cdot \nabla$$

is a symmetric operator on $L^2(M, \mu)$, and satisfies the following integration by parts formula:

$$\int_M (\nabla u, \nabla v) d\mu = -\int_M \Delta_\phi u v d\mu = -\int_M \Delta_\phi v u d\mu, \forall u, v \in C_0^\infty(M). \tag{2}$$

When ϕ is a constant function, the Witten Laplacian operator is just the Laplace–Beltrami operator. As an extension of the Laplace–Beltrami operator, many classical results in Riemannian geometry asserted in terms of the Laplace–Beltrami operator have been extended to the analogous ones on the Witten Laplacian operator. For example, we can see these results ([**10**] and [**15**]). Inspired by Perelman [**14**] and Cao [**2**], we study the eigenvalues of the geometric operator $-\Delta_\phi + \frac{R}{2}$ under the Ricci flow and the normalized Ricci flow. The purpose of this paper is to prove the monotonicity of eigenvalues of the operator along the Ricci flow on compact Riemannian manifolds under some curvature assumptions. As an application, we can prove that compact steady Ricci breathers must be trivial.

The following theorem is our main result.

THEOREM 1.1. *Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow (1) on a compact manifold M^n with non-negative curvature operator. Suppose that the Ricci curvature*

https://doi.org/10.1017/S0017089516000537 Published online by Cambridge University Press

satisfies

$$|Rc| \geq |\nabla\nabla\phi|$$

for all times $t \in [0, T)$*, where* $|\cdot|$ *is the length of a two-tensor* S_{ij} *which is defined by* $|S_{ij}| = \sqrt{|S_{ij}|^2} = \sqrt{S_{ij}S_{kl}g^{ik}g^{jl}}$ *. Then, the eigenvalues of the operator*

$$-\Delta_\phi + \frac{R}{2}$$

are non-decreasing under the Ricci flow.

The rest of this paper is organized as follows. In Section 2, we will derive the evolution equation of eigenvalues under the Ricci flow. In Section 3, we will prove Theorem 1.1 using the evolution equation of eigenvalues under the Ricci flow. As application, some corollaries will be obtained. In Section 4, we will derive the evolution equation of eigenvalues under the normalized Ricci flow.

2. Evolution equation of eigenvalues. In this section, we establish the evolution equation of eigenvalues of the geometry operator $-\Delta_\phi + \frac{R}{2}$ under the Ricci flow.

Let $(M, g(t))$ be a compact Riemannian manifold with non-negative curvature operator, and $(M, g(t))$, $t \in [0, T)$ be a smooth solution to the Ricci flow equation (1). Let λ be an eigenvalue of the operator $-\Delta_\phi + \frac{R}{2}$ at time t_0 where $0 \leq t_0 < T$, and f the corresponding eigenfunction, i.e.,

$$-\Delta_\phi f + \frac{R}{2} f = \lambda f, \tag{3}$$

with the normalization

$$\int_M f^2 d\mu = 1.$$

We assume that $f(x, t)$ is a C^1-family of smooth functions on M, and satisfies the following condition:

$$\frac{d}{dt}\left[\int_M f^2 d\mu\right] = 0.$$

Hence, we have

$$\int_M f\,[f_t d\mu + (f\,d\mu)_t] = 0, \tag{4}$$

where $f_t = \frac{\partial f}{\partial t}$.

We also need to define a functional

$$\lambda(f, t) = \int_M \left(-f\,\Delta_\phi f + \frac{R}{2} f^2\right) d\mu = \int_M \left(-\Delta_\phi f + \frac{R}{2} f\right) f\,d\mu,$$

where f satisfies the equality (4). At time t, if f is the eigenfunction of λ, then

$$\lambda(f, t) = \lambda(t).$$

Let us first derive the evolution equation of the above functional under the general geometric flow.

LEMMA 2.1. *Suppose that λ is an eigenvalue of the operator $-\Delta_\phi + \frac{R}{2}$, f is the eigenfunction of λ at time t_0, and the metric $g(t)$ evolves by*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij},$$

where v_{ij} is a symmetric two-tensor. Then, we have

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j + \frac{1}{2}\frac{\partial R}{\partial t} f \right) f\, d\mu + \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f\, d\mu, \tag{5}$$

where $V = Tr(v)$.

Proof. The proof is only a direct computation. Notice that

$$\frac{\partial}{\partial t}\Delta_\phi = \Delta_\phi \frac{\partial}{\partial t} - v_{ij}\nabla_i\nabla_j - \frac{1}{2} g^{kl}(2(\mathbf{div} v)_k - \nabla_k V)\nabla_l + v_{ij}\nabla_i\phi\nabla_j.$$

Hence, we have

$$\begin{aligned}
\frac{d}{dt}\lambda(f,t) &= \frac{d}{dt}\int_M \left(-\Delta_\phi f + \frac{R}{2} f \right) f\, d\mu \\
&= \int_M \left(v_{ij} f_{ij} + \frac{1}{2} g^{kl}(2v_{ki,i} - V_k) f_l - v_{ij}\phi_i f_j + \frac{1}{2}\frac{\partial R}{\partial t} f \right) f\, d\mu \\
&\quad + \int_M \left(-\Delta_\phi f_t + \frac{R}{2} f_t \right) f\, d\mu + \int_M \left(-\Delta_\phi f + \frac{R}{2} f \right) \frac{d}{dt}(f\, d\mu) \\
&= \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j + \frac{1}{2}\frac{\partial R}{\partial t} f \right) f\, d\mu + \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f\, d\mu \\
&\quad + \int_M \left(-\Delta_\phi f + \frac{R}{2} f \right) [f_t d\mu + (f\, d\mu)_t],
\end{aligned}$$

where we used (2) in the last equality. At time t_0, f is the eigenfunction of λ, i.e., the equality (3) holds. Combining (3) with (4), the last term in the above evolution equation vanishes. So we get

$$\frac{d}{dt}\lambda(f,t)|_{t=t_0} = \int_M \left(v_{ij} f_{ij} - v_{ij}\phi_i f_j + \frac{1}{2}\frac{\partial R}{\partial t} f \right) f\, d\mu + \int_M \left(v_{ij,i} - \frac{1}{2} V_j \right) f_j f\, d\mu.$$

□

REMARK 2.1. In fact, Lemma 2.1 also gives us the evolution of eigenvalues. From the above proof, it is easy to see that the evolution equation (5) does not depend on the evolution equation of f, as long as f satisfies (4). Hence, we have

$$\frac{d}{dt}\lambda(t) = \frac{d}{dt}\lambda(f,t) \tag{6}$$

for any time t, when f is the eigenfunction of λ at time t.

https://doi.org/10.1017/S0017089516000537 Published online by Cambridge University Press

Now we can calculate the evolution equation of eigenvalues of the geometric operator under the Ricci flow. In Lemma 2.1, when the symmetric two-tensor $v_{ij} = -2R_{ij}$, we get the following result.

THEOREM 2.1. *Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow (1) on a compact manifold M^n. Assume that there is a C^1-family of smooth functions $f(x, t)$ which satisfy*

$$-\Delta_\phi f(x,t) + \frac{R}{2} f(x,t) = \lambda(t) f(x,t),$$

and the normalization

$$\int_M f(x,t)^2 d\mu = 1.$$

Then, the eigenvalue $\lambda(t)$ satisfies

$$\frac{d}{dt}\lambda(t) = \int_M \left(2R_{ij}f_if_j - 2R_{ij}\phi_i f_j f + R_{ij}\phi_i\phi_j f^2 + |Rc|^2 f^2 - R_{ij}\phi_{ij}f^2\right) d\mu. \tag{7}$$

Proof. The proof also follows from a direct computation. Note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2,$$

and

$$\mathbf{div} Rc = \frac{1}{2}\nabla R.$$

Using (6) and substituting $v_{ij} = -2R_{ij}$ into the equality (5), we have

$$\frac{d}{dt}\lambda(t) = \int_M \left(-2R_{ij}f_{ij}f + 2R_{ij}\phi_i f_j f + \frac{1}{2}\Delta R f^2 + |Rc|^2 f^2\right) d\mu.$$

Using integration by parts for the third term, we get

$$\frac{1}{2}\int_M \Delta R f^2 d\mu = \int_M \left(2R_{ij}f_if_j + 2R_{ij}f_{ij}f - 4R_{ij}\phi_i f_j f + R_{ij}\phi_i\phi_j f^2 - R_{ij}\phi_{ij}f^2\right) d\mu.$$

Thus, we can easily see that (7) holds from the above two formulas. □

3. Proof of Theorem 1.1. In this section, we prove Theorem 1.1 using the evolution equation of eigenvalues under the Ricci flow. Moreover, from Theorem 1.1, we obtain some corollaries including the study of compact steady Ricci breathers.

Proof of Theorem 1.1. Notice that the non-negativity of the curvature operator is preserved by the Ricci flow [**1**], and this implies that the Ricci curvature is also non-negative at all time $t \in [0, T)$. Let $\lambda(t)$ be the eigenvalue of $-\Delta_\phi + \frac{R}{2}$, and $f(x, t)$

its eigenfunction with the normalization at time t. From Theorem 2.1, we have

$$\begin{aligned}\frac{d}{dt}\lambda(t) &= \int_M \left(2R_{ij}f_if_j - 2R_{ij}\phi_if_jf + R_{ij}\phi_i\phi_jf^2 + |Rc|^2f^2 - R_{ij}\phi_{ij}f^2\right)d\mu \\ &= \int_M R_{ij}f_if_jd\mu + \int_M R_{ij}(f\phi_i - f_i)(f\phi_j - f_j)d\mu + \int_M (|Rc|^2 - R_{ij}\phi_{ij})f^2d\mu \\ &\geq \int_M R_{ij}f_if_jd\mu + \int_M R_{ij}(f\phi_i - f_i)(f\phi_j - f_j)d\mu + \frac{1}{2}\int_M (|Rc|^2 - |\phi_{ij}|^2)f^2d\mu \\ &\geq 0.\end{aligned}$$

The last inequality follows from the non-negativity and lower boundedness of Ricci curvature, i.e. by the non-negativity we can get $R_{ij}f_if_j \geq 0$ and $R_{ij}(f\phi_i - f_i)(f\phi_j - f_j) \geq 0$, and the lower boundedness condition $|Rc| \geq |\nabla\nabla\phi|$ in Theorem 1.1 implies the fact that $|Rc|^2 - |\phi_{ij}|^2 \geq 0$. Hence, $\lambda(t)$ is non-decreasing under the Ricci flow. □

In fact, we use the non-negativity of the Ricci curvature in the above proof. Therefore, our theorem will also hold if the Ricci curvature is non-negative, but in general, the non-negativity of the Ricci curvature is not preserved unless the dimension of M is two or three [**6**, **7**]. In view of this, we have the following result in dimension two and three.

COROLLARY 3.1.

(1) *In dimension two, if a compact Riemannian manifold has non-negative scalar curvature and $R \geq \Delta\phi$, the eigenvalues of the operator*

$$-\Delta_\phi + \frac{R}{2}$$

are non-decreasing under the Ricci flow.

(2) *In dimension three, if a compact Riemannian manifold has non-negative Ricci curvature and $|Rc| \geq |\nabla\nabla\phi|$, the eigenvalues of the operator*

$$-\Delta_\phi + \frac{R}{2}$$

are non-decreasing under the Ricci flow.

REMARK 3.1. In Theorem 1.1, if we choose ϕ be a constant function on M, our theorem reduce to Cao's Theorem 1 in [**2**]. So our result is an extension version of Cao's.

COROLLARY 3.2 (Cao [**2**]). *On a Riemannian manifold with non-negative curvature operator, the eigenvalues of the operator*

$$-\Delta + \frac{R}{2}$$

are non-decreasing under the Ricci flow.

Next, we are ready to consider compact steady Ricci breathers. We recall the definition of Ricci breathers, see original definition in [**14**] and [**8**].

DEFINITION 3.1. A metric $g(t)$ evolving by the Ricci flow is called a breather, if there exist times $t_1 < t_2$ and $\alpha > 0$, such that the metrics $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1, \alpha < 1, \alpha > 1$ correspond to steady, shrinking, or expanding breathers, respectively.

Using the monotonicity of eigenvalues in Theorem 1.1, we can rule out non-trivial steady Ricci breathers on a compact manifold. We refer the reader to Theorem 3 in **[2]** and Corollary 5.5 in **[9]** for analogous details of the proof.

COROLLARY 3.3. *If there exists a C^2 function ϕ on a compact steady Ricci breather with non-negative curvature operator, such that the Ricci curvature satisfies $|Rc| \geq |\nabla\nabla\phi|$, then the compact steady Ricci breather is Ricci-flat and ϕ is a constant.*

REMARK 3.2. In fact, when ϕ is a constant, the condition of the Ricci curvature in the above corollary is trivial. Cao had gotten a same result with non-negative curvature operator **[2]**. A similar result without the curvature assumption was obtained by both Ivey **[8]** and Li **[9]**.

Ricci solitons are special Ricci breathers, for which the metrics $g(t_1)$ and $g(t_2)$ only differ by a diffeomorphism and scaling for each pair of t_1 and t_2. Perelman **[14]** proved that a steady breather is necessarily a steady soliton. Hence, we actually give a result on compact steady Ricci solitons.

COROLLARY 3.4. *If there exists a C^2 function ϕ on a compact steady Ricci soliton with non-negative curvature operator, such that the Ricci curvature satisfies $|Rc| \geq |\nabla\nabla\phi|$, then the compact steady Ricci soliton is Ricci-flat and ϕ is a constant.*

REMARK 3.3. A similar result without the curvature assumption was obtained by both Hamilton **[6, 7]** and Perelman **[14]**.

4. Eigenvalues under the normalized Ricci flow. In the last section, we come to consider the normalized Ricci flow, i.e.,

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \frac{2r}{n} g_{ij}, \tag{8}$$

where $r = \frac{\int_M R dv}{\int_M dv}$ is the average scalar curvature. In Lemma 2.1, if we evolve the metric by the normalized Ricci flow, we can get the evolution of eigenvalues of the geometric operator $-\Delta_\phi + \frac{R}{2}$ under the normalized Ricci flow.

THEOREM 4.1. *Let $g(t)$, $t \in [0, T)$, be a solution to the normalized Ricci flow (8) on a compact manifold M^n. Assume that there is a C^1-family of smooth functions $f(x, t)$ which satisfy*

$$-\Delta_\phi f(x, t) + \frac{R}{2} f(x, t) = \lambda(t) f(x, t),$$

and the normalization

$$\int_M f(x, t)^2 d\mu = 1.$$

Then, the eigenvalue $\lambda(t)$ satisfies

$$\frac{d}{dt}\lambda(t) = -\frac{2r\lambda}{n} + \int_M \left(2R_{ij} f_i f_j - 2R_{ij}\phi_i f_j f + R_{ij}\phi_i\phi_j f^2 + |Rc|^2 f^2 - R_{ij}\phi_{ij} f^2\right) d\mu.$$

https://doi.org/10.1017/S0017089516000537 Published online by Cambridge University Press

Proof. We note that the evolution of scalar curvature is

$$\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 - \frac{2r}{n}R,$$

and

$$v_{ij} = -2R_{ij} + \frac{2r}{n}g_{ij}.$$

The proof can be obtained from the same calculation with Theorem 2.1. So it is easy to get the extra term $-\frac{2r\lambda}{n}$. □

When M is a two-dimensional surface, r is a constant. We have the following corollary.

COROLLARY 4.1. *Assume that a two-dimensional compact Riemannian manifold has non-negative scalar curvature and $R \geq \Delta\phi$. If $\lambda(t)$ is the eigenvalue of the geometric operator $-\Delta_\phi + \frac{R}{2}$, then $e^{rt}\lambda(t)$ is non-decreasing under the normalized Ricci flow.*

ACKNOWLEDGEMENTS. The authors would like to thank Professor Kefeng Liu and Professor Hongyu Wang for constant support and encouragement. The authors are also grateful to the anonymous referees and the editor for their careful reading and helpful comments. The work is supported by PRC grant NSFC 11401514, 11471145, 11371309, 11326076, and 11371310, the University Science Research Project of Jiangsu Province 13KJB110029, and Qing Lan Project.

REFERENCES

1. H. D. Cao and X. P. Zhu, A complete proof of the Poincaré and Geometrization conjectures—Application of the Hamilton-Perelman theory of the Ricci flow, *Asian J. Math.* **10** (2006), 165–492.

2. X. D. Cao, Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator, *Math. Ann.* **337** (2007), 435–441.

3. X. D. Cao, First eigenvalues geometric operators under the Ricci flow, *Proc. Amer. Math. Soc.* **136** (2008), 4075–4078.

4. S. W. Fang, H. F. Xu and P. Zhu, Evolution and monotonicity of eigenvalues under the Ricci flow, *Sci. China Math.* **58** (2015), 1737–1744.

5. H. X. Guo, R. Philipowski and A. Thalmaier, Entropy and lowest eigenvalue on evolving manifolds, *Pacific J. Math.* **264** (2013), 61–82.

6. R. Hamilton, Three-manifolds with positive Ricci curvature, *J. Diff. Geom.* **17** (1982), 255–306.

7. R. Hamilton, Four-manifolds with positive curvature operator, *J. Diff. Geom.* **24** (1986), 153–179.

8. T. Ivey, Ricci solitons on compact three-manifolds, *Differ. Geom. Appl.* **3** (1993), 301–307.

9. J. F. Li, Eigenvalues and energy functionals with monotonicity formulae under Ricci flow, *Math. Ann.* **338** (2007), 927–946.

10. X. D. Li, Liouville theorems for symmetric diffusion operators on complete Riemannian manifolds, *J. Math. Pures Appl.* **84** (2005), 1295–1361.

11. J. Ling, A comparison theorem and a sharp bound via the Ricci flow, http://arxiv.org/abs/0710.2574.

12. J. Ling, A class of monotonic quantities along the Ricci flow, http://arxiv.org/abs/0710.4291.

13. L. Ma, Eigenvalue monotonicity for the Ricci-Hamilton flow, *Ann. Global Anal. Geom.* **29** (2006), 287–292.

14. G. Perelman, The entropy formula for the Ricci flow and its geometric applications, http://arxiv.org/abs/math/0211159.

15. C. Y. Xia and H. W. Xu, Inequalities for eigenvalues of the drifting Laplacian on Riemannian manifolds, *Ann. Global Anal. Geom.* **45** (2014), 155–166.

16. L. Zhao, The first eigenvalue of the Laplace operator under Yamabe flow, *Mathematica Applicata* **24** (2011), 274–278.