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RENORMALIZATION OF THE LOCAL TIME FOR THE d-DIMENSIONAL FRACTIONAL BROWNIAN MOTION WITH N PARAMETERS

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Abstract. We study the asymptotic behavior in Sobolev norm of the local time of the d-dimensional fractional Brownian motion with N-parameters when the space variable tends to zero, both for the fixed time case and when simultaneously time tends to infinity and space variable to zero.

§1. Introduction

Let $B^H = \{B_t^H : t \geq 0\}$ be a standard fractional Brownian motion (fBm for brevity) with Hurst parameter $H \in (0,1)$. It is well known that this process is a centered Gaussian process which admits an integral representation of the form

$$B_t^H = \int_0^t K_H(t,s)\, dW_s,$$

where W is a standard Wiener process. The kernel $K_H(t,s)$ is given, for $s < t$, by

$$K_H(t,s) = c_H (t-s)^{\mu} - \mu c_H \int_s^t (r-s)^{\mu-1} \left(1 - \left(\frac{s}{r}\right)^{-\mu}\right) dr, \tag{1}$$

with c_H being a constant and $\mu = H - \frac{1}{2}$.

The covariance function of B_t^H can be represented as

$$R_H(s,t) = \mathbb{E}(B_s^H B_t^H) = \int_0^{s\wedge t} K_H(t,r) K_H(s,r)\, dr,$$

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and has the explicit form

$$R_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}).$$

A very good survey about the fBm is the paper of Nualart [5].

For $\overline{H} = (H_1, \dots, H_N)$ the $(N,1)$-fBm is defined as

$$B_t^{\overline{H}} = \int_{[0,t]} K_{\overline{H}}(t,s)\, dW_s,$$

where $K_{\overline{H}}(t,s) = \bigotimes_{j=1}^N K_{H_j}(t_j, s_j)$, $s,t \in \mathbb{R}_+^N$ and W is a standard N-parameter Brownian motion. Its covariance function is

$$R_{\overline{H}}(s,t) = \mathbb{E}(B_s^{\overline{H}} B_t^{\overline{H}}) = \prod_{j=1}^N R_{H_j}(s_j, t_j).$$

Finally given the $N \times d$-matrix $\overline{H} = (\overline{H}_1, \dots, \overline{H}_d)$ where for $i = 1, \dots, d$ and $j = 1, \dots, N$, $\overline{H}_i = (H_{i,1}, \dots, H_{i,N})$ is a column vector and $H_{i,j} \in (0,1)$, the N-parameter, d-dimensional fractional Brownian motion ((N,d)-fBm for brevity) is defined by $B^{\overline{H}} = (B_t^{\overline{H}_1}, \dots, B_t^{\overline{H}_d})_{t \in \mathbb{R}_+^N}$ where its components are independent and for every $i = 1, \dots, d$, $B^{\overline{H}_i}$ is a $(N,1)$-fBm with Hurst parameter $\overline{H}_i$.

For any $t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d$, the local time $L(t,x)$ of the (N,d)-fBm can be defined as the density of the occupation measure μ_t, defined as

$$\mu_t(A) = \int_{[0,t]} \mathbb{1}_A(B_s^{\overline{H}})\, ds, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Formally, we can write

$$L(t,x) = \int_{[0,t]} \delta_x(B_s^{\overline{H}})\, ds$$

where δ_x denotes the Dirac function and $\delta_x(B_s^{\overline{H}})$ is therefore a distribution in the Watanabe sense (see [6]).

This local time for (N,d)-fBm has been studied by Xiao and Zhang [7], Hu and Oksendal [2] and Eddahbi et al. [1] between others.

The aim of this paper is to study the asymptotic behavior of $L(t,x)$ when $|x|$, the euclidean norm of x in $\mathbb{R}^d$, goes to 0, both for a fixed time and when the time goes to infinity, and we renormalize his Sobolev norm. We generalize the results of [3] from the (N, d)-standard Brownian motion to the (N, d)-fractional Brownian motion. In the standard Brownian motion case, the covariance function is simply $R_{\frac{1}{2}}(s, t) = s \wedge t$. Here, the control of the covariance function $R_H(s, t)$ for $H \neq \frac{1}{2}$ is the main difficulty.

Section 2 is devoted to the presentation of the problem. In particular we review from [1] the chaotic decomposition of the local time $L(t, x)$ as a functional of the (N, d)-fBm and its regularity in terms of Sobolev-Watanabe norms. In Section 3 we present a list of auxiliary lemmas. Section 4 is devoted to the presentation and proof of the main result, namely the asymptotic behavior of this local time, for fixed t, in the case $H_{i,j} = H$, $\forall\, i, j$, when $|x|$ goes to 0. In Section 5 we extend the result to the case $\underline{t} := t_1 \cdots t_N$ going to infinity.

§2. Preliminaries and statement of the problem

If F is a square integrable Brownian random variable, it can be represented by its Wiener chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $I_n(f_n)$ denotes the multiple Itô stochastic integral of the symmetric kernel $f_n \in L^2(\mathbb{R}_+^n)$ with respect to the Wiener process W.

If $\mathbf{L}$ is the Ornstein-Uhlenbeck operator

$$\mathbf{L}F = -\sum_{n=0}^{\infty} n I_n(f_n),$$

$p \in (1, \infty)$ and $\alpha \in \mathbb{R}$, we define the Sobolev-Watanabe spaces $\mathbb{D}^{\alpha,p}$ as the closure of the set of polynomial random variables with respect to the norm

$$\|F\|_{\alpha,p} = \|(\mathbf{Id} - \mathbf{L})^{\frac{\alpha}{2}} F\|_{L^p(\Omega)},$$

where $\mathbf{Id}$ stands for the identity mapping.

We denote by D the chaotic derivative operator. It acts on multiple Itô stochastic integrals as

$$D_t(I_n(f_n)) = n I_{n-1}(f_n(\,\cdot\,, t)),$$

and is continuous from $\mathbb{D}^{\alpha,p}$ into $\mathbb{D}^{\alpha-1,p}(L^2(\mathbb{R}_+))$.

It is known that a Brownian random variable F belongs to $\mathbb{D}^{\alpha,2}$ if and only if its chaotic decomposition $\sum_{n=0}^{\infty} I_n(f_n)$ satisfies

$$\sum_{n=0}^{\infty}(1+n)^{\alpha}\|I_n(f_n)\|_2^2 < \infty,$$

where $\|I_n(f_n)\|_2^2 = n!\|f_n\|_2^2$.

Set $\mathbb{D}^{\infty,2} = \bigcap_{\alpha\in\mathbb{R}} \mathbb{D}^{\alpha,2}$. If $F \in \mathbb{D}^{\infty,2}$, we can compute its chaos expansion using the Stroock formula

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(\mathbb{E}(D^n F)).$$

For a complete survey of this subjects we refer the reader to the book of Watanabe [6].

Let $p_\varepsilon(x)$ be the centered Gaussian kernel with variance $\varepsilon > 0$. Consider also, for $x \in \mathbb{R}^d$ and $\varepsilon > 0$, the Gaussian kernel on $\mathbb{R}^d$ given by

$$p_\varepsilon^d(x) = \prod_{i=1}^{d} p_\varepsilon(x_i), \quad x = (x_1, \ldots, x_d).$$

We denote by $\mathbf{H}_n$ the n-th Hermite polynomial, defined for $n \geq 1$, by

$$\mathbf{H}_n(x) = \frac{(-1)^n}{n!} \exp\Big(\frac{x^2}{2}\Big) \frac{d^n}{dx^n}\Big(\exp\Big(-\frac{x^2}{2}\Big)\Big), \quad x \in \mathbb{R}$$

and $\mathbf{H}_0(x) = 1$.

As we proved in [1] the chaotic decomposition of the local time of the (N,d)-fBm is

$$L(t,x) = \sum_{n_1,\ldots,n_d\geq 0} \int_{[0,t]} \prod_{i=1}^{d} \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i\overline{H}_i}} \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{s}^{\overline{H}_i}}\Big) I_{n_i}^i(K_{\overline{H}_i}(s,\,\cdot\,)^{\otimes n_i})\, ds,$$

provided that $\sum_{j=1}^{N} \frac{1}{H_j^*} > d$, where $t \in \mathbb{R}_+^N$, $x \in \mathbb{R}^d \backslash \{0\}$, $\underline{s} = s_1 \cdots s_N$, $\underline{s}^{\overline{H}_i} = \prod_{j=1}^{N} s_j^{H_{i,j}}$ and $H_j^* = \max\{H_{i,j},\, i = 1,\ldots,d\}$. The integrals $I_{n_i}^i$ denotes the multiple Itô stochastic integrals with respect to the independent N-parameter Wiener processes W^i.

Moreover, in [1] we proved that this functional belongs to the space $\mathbb{D}^{\alpha,2}$ if

$$\alpha < \sum_{j=1}^{N} \frac{1}{2H_j^*} - \frac{d}{2}.$$

If all $H_{i,j} = H$, this expression becomes $\alpha < \frac{N}{2H} - \frac{d}{2}$, and then a sufficient condition for the local time to be in $L^2(\Omega)$ is $N > Hd$. Observe that this sufficient condition is also founded in Xiao and Zhang [7]. From now on we will suppose always this condition.

Recall that if $H = \frac{1}{2}$, $\sum_{j=1}^{N} \frac{1}{2H_j^*} - \frac{d}{2} = N - \frac{d}{2}$, which is the same condition obtained in [3] for the N-parameter Wiener process in $\mathbb{R}^d$.

§3. Auxiliary lemmas

LEMMA 1. *If $\frac{1}{4} \le \beta \le \frac{1}{2}$ we have*

$$\sup_{x\in\mathbb{R}} \left|\sqrt{n!}\,\mathbf{H}_n(x)e^{-\beta x^2}\right| \le c(n\vee 1)^{-\frac{8\beta-1}{12}}.$$

Proof. This result is proved in [4]. ∎

Remark 2. The factor $\sqrt{n!}$ appears because we do not use the same definition of Hermite polynomials as in [4].

LEMMA 3. *Let $d \ge 1$ and $\nu \in (0,1)$. We can choose a universal constant c such that for any $m \ge 1$,*

$$\sum_{n_1+\cdots+n_d=m} \prod_{i=1}^{d} (n_i \vee 1)^{-\nu} \le c m^{d(1-\nu)-1}.$$

Proof. This result is proved in [4]. ∎

LEMMA 4. *Let γ and a be positive constants and $b \in \mathbb{R}$. Set $\alpha = \frac{b-1}{a}$. Then*

$$\int_{[0,1]^N} \exp\Big(-\frac{\gamma}{\underline{s}^a}\Big)\frac{ds}{\underline{s}^b} = \frac{1}{(N-1)!}\Big(\frac{1}{a}\Big)^N \gamma^{-\alpha} g_{N-1}(\gamma,\alpha)$$

where

$$g_{N-1}(\gamma,\alpha) := \int_\gamma^\infty t^{\alpha-1}e^{-t}\Big(\log\frac{t}{\gamma}\Big)^{N-1} dt.$$

Proof. Using the change of variables $u_1 = s_1\cdots s_N, u_2 = s_2\cdots s_N, \ldots, u_N = s_N$, with Jacobi determinant $\frac{1}{u_2\cdots u_N}$, we obtain

$$\begin{aligned}\int_{[0,1]^N} \exp\Big(-\frac{\gamma}{\underline{s}^a}\Big)\frac{ds}{\underline{s}^b} &= \int_{\{0\le u_1\le\cdots\le u_N\le 1\}} \frac{1}{u_1^b}\exp\Big(-\frac{\gamma}{u_1^a}\Big)\frac{du_N\cdots du_2}{u_N\cdots u_2}du_1\\ &= \frac{1}{(N-1)!}\int_0^1 \Big(\log\frac{1}{r}\Big)^{N-1}\frac{1}{r^b}\exp\Big(-\frac{\gamma}{r^a}\Big)dr,\end{aligned}$$

and making the change of variable $\gamma r^{-a} = t$ we get the desired result. ∎

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LEMMA 5. *The function*

$$Q_H(z) = \begin{cases} \dfrac{R_H(1,z)}{z^H} & \text{if } z \in (0,1] \\ 0 & \text{if } z = 0, \end{cases}$$

has the following properties:

1. *It is strictly increasing and it continuously maps* $[0,1]$ *onto* $[0,1]$. *Moreover,* $Q_H(1) = 1$.
2. *For fixed* $\delta \in (0,1)$ *and for any* $z \in [0, 1-\delta]$, *it satisfies the inequality*

$$Q_H(z) \leq c(H,\delta) z^G,$$

   where $G = H \wedge (1-H)$.
3. *For fixed* $\delta \in (0,1)$ *and* $\beta > 0$, *it satisfies the inequality*

$$\int_{1-\delta}^1 Q_H(z)^\beta \, dz \leq \frac{c(H,\delta)}{\beta^{\frac{1}{2H}}}.$$

Proof. The proof of parts 1 and 3 are done in [1].
For the part 2 we have

$$Q_H(z) = \frac{1-(1-z)^{2H}}{2z^H} + \frac{z^H}{2}.$$

Using Taylor expansion, $1-(1-z)^{2H} = 2H(1-\theta)^{2H-1} z$ with $0 \leq \theta \leq z$.
If $H \geq \frac{1}{2}$, we have $1-(1-z)^{2H} \leq 2Hz$, and therefore $Q_H(z) \leq Hz^{1-H} + \frac{1}{2} z^H \leq c_1 z^{1-H}$, c_1 being a positive constant.

If $H < \frac{1}{2}$ and $z \in [0, 1-\delta]$, we have $1-(1-z)^{2H} \leq 2H\delta^{2H-1} z$, and then $Q_H(z) \leq H\delta^{2H-1} z^{1-H} + \frac{1}{2} z^H \leq c_2 z^H$, c_2 being another positive constant. ∎

In what follows, for every $x > 0$ and $\gamma \geq 0$, we denote the complementary incomplete Gamma function as

$$\Gamma(x,\gamma) = \int_\gamma^\infty e^{-t} t^{x-1} \, dt.$$

In particular $\Gamma(x) := \Gamma(x,0)$ and $\Gamma(x,\gamma) \leq \Gamma(x)$.

LEMMA 6. *The function*

$$g_{N-1}(\gamma,\alpha) := \int_\gamma^\infty t^{\alpha-1}e^{-t}\Big(\log\frac{t}{\gamma}\Big)^{N-1}dt, \quad N\in\mathbb{N}$$

has the following behavior when γ tends to 0:

1. *If $\alpha>0$ and $N\geq 2$, $g_{N-1}(\gamma,\alpha) = (\log\frac{1}{\gamma})^{N-1}\Gamma(\gamma,\alpha)+\mathcal{O}((\log\frac{1}{\gamma})^{N-2})$. If $N=1$, it tends to $\Gamma(\alpha)$*
2. *If $\alpha=0$ and $N\geq 2$, $g_{N-1}(\gamma,\alpha) = e^{-\gamma}\frac{1}{N}(\log\frac{1}{\gamma})^{N}+\mathcal{O}((\log\frac{1}{\gamma})^{N-1})$. If $N=1$, it behaves as $e^{-\gamma}\ln\frac{1}{\gamma}$*
3. *If $\alpha<0$ and $N\geq 2$, $g_{N-1}(\gamma,\alpha) = \gamma^\alpha\big(\frac{\Gamma(N)}{|\alpha|^N}+o(\gamma)\big)$. If $N=1$, its behavior is as $\frac{1}{|\alpha|}\gamma^\alpha$*

Proof. The cases with $N=1$ are straightforward. For $N\geq 2$, note that

$$\Big(\log\frac{t}{\gamma}\Big)^{N-1} = \sum_{k=0}^{N-1}\binom{N-1}{k}\Big(\log\frac{1}{\gamma}\Big)^{N-1-k}(\log t)^k. \tag{2}$$

Then,

- If $\alpha>0$, the function

  $$t\longmapsto t^{\frac{\alpha}{2}-1}e^{-t}(\log t)^k$$

  is always integrable on $[0,\infty)$ for any $k\in\mathbb{N}\cup\{0\}$. Therefore,

  $$g_{N-1}(\gamma,\alpha) = \Big(\log\frac{1}{\gamma}\Big)^{N-1}\Gamma(\gamma,\alpha)+\mathcal{O}\Big(\Big(\log\frac{1}{\gamma}\Big)^{N-2}\Big).$$

- If $\alpha=0$, we need to estimate the integral

  $$g_{N-1}(\gamma,0) = \int_\gamma^\infty t^{-1}e^{-t}\Big(\log\frac{t}{\gamma}\Big)^{N-1}dt.$$

  Integrating by parts we obtain

  $$\begin{aligned} g_{N-1}(\gamma,0) &= \frac{1}{N}\int_\gamma^\infty e^{-t}\Big(\log\frac{t}{\gamma}\Big)^{N}dt \\ &= \frac{e^{-\gamma}}{N}\Big(\log\frac{1}{\gamma}\Big)^{N}+\mathcal{O}\Big(\Big(\log\frac{1}{\gamma}\Big)^{N-1}\Big). \end{aligned}$$

- If $\alpha<0$, making the change of variable $s=-\alpha\log(\frac{t}{\gamma})$, the result follows immediately. ∎

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§4. Renormalization of the local time for fixed t

The main purpose of this section is to study the asymptotic behavior of $L(t,x)$, for $t \in \mathbb{R}_+^N$ and $x \in \mathbb{R}^d \backslash \{0\}$, when $|x| \to 0$. In the case $dH \geq 1$ it has a singularity. An interesting question is to renormalize the local time, that means, to find a deterministic function $f(t,x)$ such that $f(t,x)L(t,x)$ converge to 1 in some precise sense. We will do it with respect the norm $\|\cdot\|_{\alpha,2}$. Then we will obtain a function $f(t,x)$ such that $\|f(t,x)L(t,x)\|_{\alpha,2}$ converges to 1 when $|x| \to 0$, both for fixed t and when $\underline{t} = t_1 \cdots t_N \to \infty$.

Recall the expression of the $\mathbb{D}^{\alpha,2}$-norm of the local time $L(t,x)$. For the sake of simplicity we take $t = \widetilde{1} := (1, \ldots, 1)$.

We have

$$\|L(\widetilde{1},x)\|_{\alpha,2}^2 = \sum_{m=0}^{\infty} (1+m)^{\alpha} A_m(x), \tag{3}$$

where

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \Bigg\| \int_{[0,1]^N} \prod_{i=1}^{d} \frac{p_{\underline{s}^{2\overline{H}_i}}(x_i)}{\underline{s}^{n_i \overline{H}_i}} \times \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{s}^{\overline{H}_i}}\Big) I_{n_i}^i (K_{\overline{H}_i}(s,\cdot)^{\otimes n_i})\, ds \Bigg\|_{L^2(\Omega)}^2,$$

and as

$$E(I_{n_i}^i(K_{\overline{H}_i}(u,\cdot)^{\otimes n_i}) I_{n_j}^j(K_{\overline{H}_j}(v,\cdot)^{\otimes n_j})) = \delta_{ij} n_i! \big(R_{\overline{H}_i}(u,v)\big)^{n_i},$$

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} du \int_{[0,1]^N} dv \prod_{i=1}^{d} \Bigg(\prod_{j=1}^{N} \frac{R_{H_{i,j}}(u_j,v_j)}{(u_j v_j)^{H_{i,j}}} \Bigg)^{n_i} \times n_i!\, \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{u}^{\overline{H}_i}}\Big) \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{v}^{\overline{H}_i}}\Big) p_{\underline{u}^{2\overline{H}_i}}(x_i) p_{\underline{v}^{2\overline{H}_i}}(x_i),$$

and in particular

$$A_0(x) = \Bigg(\int_{[0,1]^N} ds \prod_{i=1}^{d} \frac{1}{(2\pi \prod_{j=1}^{N} s_j^{2H_{i,j}})^{\frac{1}{2}}} \exp\Big(-\frac{x_i^2}{2\prod_{j=1}^{N} s_j^{2H_{i,j}}} \Big) \Bigg)^2.$$

In all this section we confine our attention to the situation where $H_{i,j} = H$ for all $(i,j) \in \{1,\ldots,d\} \times \{1,\ldots,N\}$, and use the notation B^H for $B^{\overline{H}}$.

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Observe that in this particular case

$$A_0(x) = \frac{1}{(2\pi)^d}\Big(\int_{[0,1]^N} \frac{1}{\underline{s}^{dH}} \exp\Big(-\frac{|x|^2}{2\underline{s}^{2H}}\Big) ds\Big)^2,$$

and

$$A_m(x) = \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N}\int_{[0,1]^N} \Big(\prod_{j=1}^N \frac{R_H(u_j,v_j)}{(u_j v_j)^H}\Big)^m$$
$$\times \prod_{i=1}^d n_i!\, \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{u}^H}\Big)\mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{v}^H}\Big) p_{\underline{u}^{2H}}(x_i) p_{\underline{v}^{2H}}(x_i)\, du dv.$$

Our main result is the following:

THEOREM 7. *Let B^H be (N, d)-fBm. Set $\lambda := d - \frac{1}{H}$. For any $\alpha < \frac{N}{2H} - \frac{d}{2}$ we have:*

1) *If $\lambda > 0$ and $N \geq 2$,*

$$\lim_{|x|\to 0} \|L(\widetilde{1}, x)\|_{\alpha,2} \left(\frac{2^{\frac{\lambda}{2}}(\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}}(N-1)!}\Big(\log\frac{2}{|x|^2}\Big)^{N-1}\Gamma\Big(\frac{\lambda}{2}\Big)\right)^{-1} = 1.$$

If $N = 1$, $\|L(1, x)\|_{\alpha,2}$ explodes when x tends to 0, and $\|L(1, x)\|_{\alpha,2} \cdot (|x|^{-\lambda})^{-1}$ is bounded by a positive constant.

2) *If $\lambda = 0$,*

$$\lim_{|x|\to 0} \|L(\widetilde{1}, x)\|_{\alpha,2}\left(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!}\Big(\log\frac{2}{|x|^2}\Big)^N\right)^{-1} = 1.$$

3) *If $\lambda < 0$,*

$$\lim_{|x|\to 0} \|L(\widetilde{1}, x)\|_{\alpha,2} = \|L(\widetilde{1}, 0)\|_{\alpha,2} = \frac{1}{(2\pi)^{\frac{d}{2}}}\Big(\frac{1}{1-Hd}\Big)^{\frac{N}{2}}$$
$$\times\left[\sum_{r=0}^\infty (1+2r)^\alpha \sum_{r_1+\cdots+r_d=r} \prod_{i=1}^d \frac{(2r_i)!}{(r_i!)^2 2^{2r_i}}\Big(\int_0^1 Q_H(z)^m \frac{dz}{z^{dH}}\Big)^N\right]^{\frac{1}{2}} < \infty.$$

Remark 8. This theorem shows that for $\lambda \geq 0$ the local time explodes at the origin and for $\lambda < 0$ it does not. Observe that for the N-Brownian process, that is $H = \frac{1}{2}$, the local time explodes at the origin if and only if $d \geq 2$, and we obtain, with another proof, the same results as Imkeller and Weisz [3].

Proof. The idea of the proof is to show that the convergence of $A_m(x)$ for any $m \geq 1$ when $|x| \to 0$, is controlled by $A_0(x)$ and then the asymptotic behavior of $L(1,x)$ coincides with the asymptotic behavior of $A_0(x)^{\frac{1}{2}}$.

Define now, for $\gamma > 0$ and $m \geq 0$,

$$B_m(\gamma) = \int_{[0,1]^N} \int_{[0,1]^N} \frac{\left(\prod_{j=1}^N R_H(u_j, v_j)\right)^m}{(\underline{u} \cdot \underline{v})^{H(m+d)}} \exp\Big(-\frac{\gamma}{\underline{u}^{2H}}\Big) \exp\Big(-\frac{\gamma}{\underline{v}^{2H}}\Big) du dv.$$

Clearly,

$$A_0(x) = \frac{1}{(2\pi)^d} B_0\Big(\frac{1}{2}|x|^2\Big).$$

For $m \geq 1$, choosing $\beta \in [\frac{1}{4}, \frac{1}{2})$, we can write

$$\begin{aligned} A_m(x) &= \frac{1}{(2\pi)^d} \sum_{n_1+\cdots+n_d=m} \int_{[0,1]^N} \int_{[0,1]^N} \Big(\prod_{j=1}^N \frac{R_H(u_j, v_j)}{(u_j v_j)^H}\Big)^m \frac{1}{(\underline{uv})^{dH}} \\ &\times \prod_{i=1}^d \sqrt{n_i!}\, \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{u}^H}\Big) \exp\Big\{-\beta \frac{x_i^2}{\underline{u}^{2H}}\Big\} \sqrt{n_i!}\, \mathbf{H}_{n_i}\Big(\frac{x_i}{\underline{v}^H}\Big) \exp\Big\{-\beta \frac{x_i^2}{\underline{v}^{2H}}\Big\} \\ &\times \exp\Big\{-\Big(\frac{1}{2} - \beta\Big) \frac{x_i^2}{\underline{u}^{2H}}\Big\} \exp\Big\{-\Big(\frac{1}{2} - \beta\Big) \frac{x_i^2}{\underline{v}^{2H}}\Big\} du dv, \end{aligned}$$

and applying Lemmas 1 and 3 we obtain

$$A_m(x) \leq c \frac{1}{(2\pi)^d} m^{d(1-\frac{8\beta-1}{6})-1} B_m\Big(\Big(\frac{1}{2} - \beta\Big)|x|^2\Big).$$

Then our problem reduces to the study of the asymptotic behavior of B_m.

As $R_H(u_j, v_j) = R_H(1, \frac{v_j}{u_j}) u_j^{2H}$, we have

$$\begin{aligned} B_m(\gamma) = 2^N \int_{[0,1]^N} \int_0^{u_N} \cdots \int_0^{u_1} \prod_{j=1}^N \frac{R_H\big(1, \frac{v_j}{u_j}\big)^m u_j^{2Hm}}{(u_j v_j)^{H(m+d)}} \\ \times \exp\Big(-\frac{\gamma}{\underline{u}^{2H}}\Big) \exp\Big(-\frac{\gamma}{\underline{v}^{2H}}\Big) dv_1 \cdots dv_N du. \end{aligned}$$

With the change $\frac{v_j}{u_j} = z_j$, $(j = 1, \ldots, N)$ and computing iteratively the

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previous integral, we find

$$B_m(\gamma) = 2^N \int_{[0,1]^N} \left(\int_{[0,1]^N} \underline{u}^{1-2Hd} \exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}} \right) du_1 \cdots du_N \right) \times \prod_{j=1}^{N} \frac{R_H(1, z_j)^m}{{z_j}^{H(m+d)}} \, dz_1 \cdots dz_N$$

where $\kappa(r) = 1 + \frac{1}{r^{2H}}$.

By Lemma 4, with $a = 2H$ and $b = 2Hd - 1$, we have

$$\begin{aligned} J_N(\gamma, \underline{z}) &= \int_{[0,1]^N} \underline{u}^{1-2Hd} \exp\left(\frac{-\kappa(\underline{z})\gamma}{\underline{u}^{2H}} \right) du \\ &= c(N, d, H)\gamma^{-\lambda} \int_\gamma^\infty e^{-s\kappa(\underline{z})} s^{\lambda-1} \left(\log \frac{s}{\gamma} \right)^{N-1} ds, \end{aligned}$$

where $\lambda = d - \frac{1}{H} = \frac{b-1}{a}$.

Therefore

$$B_m(\gamma) = c(N, H, d)\gamma^{-\lambda} \int_\gamma^\infty \int_{[0,1]^N} \prod_{j=1}^{N} \frac{R_H(1, z_j)^m}{z_j^{Hm}} \cdot \frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}} \times e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma} \right)^{N-1} dz ds.$$

First we will see that for $m > \frac{\lambda H}{G}$, we have

$$B_m(\gamma) \le c(H, d, N)\gamma^{-\lambda} g_{N-1}(\gamma, \lambda) m^{-\frac{N}{2H}}. \tag{4}$$

Indeed, controlling the exponential by 1, we obtain

$$\begin{aligned} B_m(\gamma) &\le c(N, H, d)\gamma^{-\lambda} \int_{[0,1]^N} \prod_{j=1}^{N} \frac{R_H(1, z_j)^m}{z_j^{H(m+d)}} \int_\gamma^\infty e^{-s} s^{\lambda-1} \left(\log \frac{s}{\gamma} \right)^{N-1} dz ds \\ &= c(N, H, d)\gamma^{-\lambda} g_{N-1}(\gamma, \lambda) \left(\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} \, dz \right)^N, \end{aligned}$$

where the function Q_H is introduced in Lemma 5.

Now, choosing $\delta \in (0, 1)$, we have

$$\int_0^1 Q_H(z)^m \frac{1}{z^{dH}} \, dz \le \int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}} \, dz + (1 - \delta)^{-dH} \int_{1-\delta}^1 Q_H(z)^m \, dz.$$

The second summand on the right, using part 3 of Lemma 5, is controlled by $c(H,\delta)m^{-\frac{1}{2H}}$.

For the first summand, if $m > \frac{dH-1}{G} = \frac{\lambda H}{G}$, we fix $\alpha \in (\frac{\lambda H}{G}, m)$, and write

$$\int_0^{1-\delta} Q_H(z)^m \frac{1}{z^{dH}}\, dz = \int_0^{1-\delta} Q_H(z)^{m-\alpha} Q_H(z)^{\alpha} \frac{1}{z^{dH}}\, dz.$$

Using that Q_H is an increasing function and part 2 of Lemma 5, we control this by

$$Q_H(1-\delta)^{m-\alpha} c(H,\delta,\alpha) \int_0^{1-\delta} z^{\alpha G - dH}\, dz.$$

As $\alpha > \frac{\lambda H}{G}$, the integral that appears in the last expression is a constant that depends on H, d, α and δ.

Therefore, being $Q_H(1-\delta) < 1$, we can estimate this term by

$$c(H,d,\delta,\alpha)m^{-\frac{1}{2H}},$$

and we get (4).

Note that this result is true only for $m > \frac{\lambda H}{G}$. If $\lambda \le 0$ this covers all cases. But if $\lambda > 0$ the B_m terms with $m \le \frac{\lambda H}{G}$ are not controlled yet. The following part of the proof will discuss these first terms. From now on in each expression c will be the suitable constant.

Observe that for any $0 < \epsilon < m$, being $Q_H(\cdot) \le 1$, we have

$$B_m(\gamma) \le B_\epsilon(\gamma).$$

Now we will see that for $\lambda > 0$,

$$B_\epsilon(\gamma) \le c(H,d,N)\gamma^{-\lambda}(g_{N-1}(\gamma,\lambda) + g_{N-1}(\gamma,\alpha))$$

where α is some positive constant depending also on ϵ.

Indeed, putting $c = c(N,H,d)$,

$$\begin{aligned}
B_\epsilon(\gamma) &= c\gamma^{-\lambda} \int_\gamma^\infty \int_{[0,1]^N} \prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}} e^{-s} s^{\lambda-1} \Big(\log \frac{s}{\gamma}\Big)^{N-1} dz ds \\
&= c\gamma^{-\lambda} \int_\gamma^\infty \sum_{k=0}^N \binom{N}{k} \underbrace{\int_0^{1-\delta} \cdots \int_0^{1-\delta}}_{k} \underbrace{\int_{1-\delta}^1 \cdots \int_{1-\delta}^1}_{N-k} \prod_{j=1}^N Q_H(z_j)^\epsilon \cdot \frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}} \\
&\quad \times e^{-s} s^{\lambda-1} \Big(\log \frac{s}{\gamma}\Big)^{N-1} dz_1 \cdots dz_k ds,
\end{aligned}$$

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because the function

$$\prod_{j=1}^{N} Q_H(z_j)^{\epsilon}\cdot\frac{e^{-\frac{s}{\underline{z}^{2H}}}}{\underline{z}^{Hd}},$$

is symmetric in z.

Now estimating Q_H and the exponential by 1 in the integrals between $1-\delta$ and 1, we obtain

$$\begin{aligned}B_\epsilon(\gamma)\le c\gamma^{-\lambda}\int_\gamma^\infty\Bigg(\Big(\frac{\delta}{(1-\delta)^{dH}}\Big)^N+\sum_{k=1}^N\binom{N}{k}\int_0^{1-\delta}\cdots\int_0^{1-\delta}\frac{\delta^{N-k}}{(1-\delta)^{dH(N-k)}}\\ \times\prod_{j=1}^k Q_H(z_j)^\epsilon\cdot\frac{e^{-\frac{s}{(z_1\cdots z_k)^{2H}}}}{(z_1\cdots z_k)^{Hd}}\,dz_1\cdots dz_k\Bigg)e^{-s}s^{\lambda-1}\Big(\log\frac{s}{\gamma}\Big)^{N-1}ds\\ \le c\gamma^{-\lambda}\int_\gamma^\infty\Bigg(\Big(\frac{\delta}{(1-\delta)^{dH}}\Big)^N+\sum_{k=1}^N\binom{N}{k}\int_0^{1-\delta}\cdots\int_0^{1-\delta}\Big(\frac{\delta}{(1-\delta)^{dH}}\Big)^{N-k}\\ \times c(H,\delta)^{k\epsilon}\prod_{j=1}^k z_j^{\epsilon G-dH}e^{-\frac{s}{(z_1\cdots z_k)^{2H}}}\,dz_1\cdots dz_k\Bigg)e^{-s}s^{\lambda-1}\Big(\log\frac{s}{\gamma}\Big)^{N-1}ds,\end{aligned}$$

where we have used part 2 of Lemma 5.

Now, choosing $\epsilon<\frac{dH}{G}$, we can use Lemma 4 with $a=2H$, $b=-\epsilon G+dH$, $\gamma=s$, $N=k$ and $\alpha=\frac{dH-\epsilon G-1}{2H}$, to bound the right hand side of the last inequality by

$$\begin{aligned}c\gamma^{-\lambda}\Bigg\{g_{N-1}(\gamma,\lambda)+\int_\gamma^\infty\sum_{k=1}^N\int_s^\infty\Big(\log\frac{t}{s}\Big)^{k-1}s^{\frac{\lambda}{2}+\frac{\epsilon G}{2H}-1}\,t^{\frac{\lambda}{2}-\frac{\epsilon G}{2H}-1}\\ \times e^{-t}e^{-s}\Big(\log\frac{s}{\gamma}\Big)^{N-1}dtds\Bigg\},\end{aligned}$$

where c is a constant that depends on H, d, N, ϵ, δ.

Using the fact that for any $n\ge 1$ and for $t\ge s$ we have

$$\log\frac{t}{s}\le n\Big(\frac{t}{s}\Big)^{\frac{1}{n}},$$

and taking $n=M(k-1)$ for a big M, we obtain

$$\begin{aligned}B_\epsilon(\gamma)\le c\gamma^{-\lambda}\Bigg\{g_{N-1}(\gamma,\lambda)+\int_\gamma^\infty\int_s^\infty s^{\frac{\lambda}{2}+\frac{\epsilon G}{2H}-1-\frac{1}{M}}\,t^{\frac{\lambda}{2}-\frac{\epsilon G}{2H}-1+\frac{1}{M}}\\ \times e^{-t}e^{-s}\Big(\log\frac{s}{\gamma}\Big)^{N-1}dtds\Bigg\},\end{aligned}$$

where now c depends also on M.

As $\epsilon < m < \frac{\lambda H}{G}$, we have $\frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M} > 0$ and

$$B_\epsilon(\gamma) \leq c\gamma^{-\lambda}\Big\{g_{N-1}(\gamma,\lambda) + \int_\gamma^\infty s^{\frac{\lambda}{2}+\frac{\epsilon G}{2H}-1-\frac{1}{M}} e^{-s}\Gamma\Big(\frac{\lambda}{2} - \frac{\epsilon G}{2H} + \frac{1}{M}, s\Big) \times \Big(\log\frac{s}{\gamma}\Big)^{N-1} ds\Big\}.$$

Controlling the complementary incomplete Gamma function by the corresponding Gamma function we obtain

$$\begin{aligned} B_\epsilon(\gamma) &\leq c\gamma^{-\lambda}\Big\{g_{N-1}(\gamma,\lambda) + \int_\gamma^\infty s^{\frac{\lambda}{2}+\frac{\epsilon G}{2H}-1-\frac{1}{M}} e^{-s}\Big(\log\frac{s}{\gamma}\Big)^{N-1} ds\Big\} \\ &= c\gamma^{-\lambda}\Big\{g_{N-1}(\gamma,\lambda) + g_{N-1}\Big(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}\Big)\Big\}. \end{aligned}$$

Observe that for M sufficiently large

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

Finally for the $m = 0$ case, using Lemma 4, we have immediately, as $\alpha = \frac{\lambda}{2}$

$$B_0(\gamma) = \frac{1}{((N-1)!)^2}\frac{1}{(2H)^{2N}}\gamma^{-\lambda} g_{N-1}\Big(\gamma, \frac{\lambda}{2}\Big)^2.$$

Therefore we have to separate the cases $\lambda \geq 0$ and $\lambda < 0$.

For $\lambda \geq 0$ we have

$$\|L(\widetilde{1}, x)\|_{\alpha,2}^2 = \sum_{m=0}^\infty (1+m)^\alpha A_m(x)$$

- The terms $A_m(x)$ with $m = 1, \dots, [\frac{\lambda H}{G}]$ are controlled by

$$c\gamma^{-\lambda}\Big\{g_{N-1}(\gamma,\lambda) + g_{N-1}\Big(\gamma, \frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M}\Big)\Big\} m^{d(1-\frac{8\beta-1}{6})-1}$$

  where $\gamma = (\frac{1}{2} - \beta)|x|^2$, and ε and M satisfy

$$\frac{\lambda}{2} + \frac{\epsilon G}{2H} - \frac{1}{M} > 0.$$

  Then, by Lemma 6, part 1, this behaves asymptotically, when $\gamma \downarrow 0$, as $c\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{N-1}$ if $\lambda > 0$, and as $c\big(\log(\frac{1}{\gamma})\big)^N$ if $\lambda = 0$.

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- The terms $A_m(x)$ with $m \geq [\frac{\lambda H}{G}] + 1$ are controlled by

  $$cm^{d(1-\frac{8\beta-1}{6})-1}\gamma^{-\lambda} g_{N-1}(\gamma,\lambda) m^{-\frac{N}{2H}}.$$

  Then

  $$\sum_{m>\frac{\lambda H}{d}} (1+m)^{\alpha} A_m(x) \leq c\Big[\sum_{m>\frac{\lambda H}{d}} m^{d(1-\frac{8\beta-1}{6})-1} m^{-\frac{N}{2H}}(1+m)^{\alpha}\Big] \times \gamma^{-\lambda} g_{N-1}(\gamma,\lambda),$$

  and using the fact that $\alpha < \frac{N}{2H} - \frac{d}{2}$, we have that the series between keys is convergent and the asymptotic behavior of the last expression is, by Lemma 6, as $c\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{N-1}$ if $\lambda > 0$, and $c\big(\log\frac{1}{\gamma}\big)^{N}$ if $\lambda = 0$. Finally,

  $$\begin{aligned} A_0(x) &= \frac{1}{(2\pi)^d} B_0\Big(\frac{1}{2}|x|^2\Big) \\ &= \frac{1}{(2\pi)^d}\frac{1}{((N-1)!)^2}\frac{1}{(2H)^{2N}}\gamma^{-\lambda} g_{N-1}\Big(\gamma,\frac{\lambda}{2}\Big)^2, \end{aligned}$$

  where $\gamma = \frac{|x|^2}{2}$.
  Summarizing, if $N \geq 2$ and $\lambda > 0$, $A_0(x)$ goes to ∞ when $\gamma \downarrow 0$, as $c\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{2N-2}$, and this first term dominates the asymptotical behavior because all the rest goes to ∞ more slowly than $\gamma^{-\lambda}\big(\log\frac{1}{\gamma}\big)^{(N-1)}$.
  If $N = 1$ and $\lambda > 0$, as the logarithm disappears, the proof only says that $\|L(1,x)\|_{\alpha,2}(|x|^{-\lambda})^{-1}$ is bounded by a positive constant.
  If $\lambda = 0$, $A_0(x)$ goes to ∞ when $\gamma \downarrow 0$, as $c\big(\log\frac{1}{\gamma}\big)^{2N}$, and the rest as $c\big(\log\frac{1}{\gamma}\big)^{N}$, so also the first term dominates the asymptotical behavior.
  Note that we consider $A_0^{\frac{1}{2}}$ in place of A_0 to get the functions that appear in the theorem.

The $\lambda < 0$ case follows directly. As we have seen before,

$$\sum_{m\geq 1} (1+m)^{\alpha} A_m(x)$$

is controlled by $\gamma^{-\lambda} g_{N-1}(\gamma,\lambda)$, and by Lemma 6, part 3, this term goes to a constant when $\gamma \downarrow 0$.

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In this case the norm $\|L(t,x)\|_{\alpha,2}$ is continuous. Therefore we don't have an explosion, and

$$\lim_{|x|\to 0} \|L(\widetilde{1},x)\|_{\alpha,2} = \|L(\widetilde{1},0)\|_{\alpha,2} = \left(\sum_{m=0}^{\infty}(1+m)^{\alpha}A_m(0)\right)^{\frac{1}{2}},$$

where

$$A_m(0) = \frac{1}{(2\pi)^d}\left(\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d} n_i!\,\mathbf{H}_{n_i}(0)^2\right)B_m(0),$$

and

$$\begin{aligned}
B_m(0) &= 2^N \int_{[0,1]^N}\left(\int_{[0,1]^N} \underline{u}^{1-2Hd}\,du_1\cdots du_N\right)\prod_{j=1}^{N}\frac{R_H(1,z_j)^m}{z_j{}^{H(m+d)}}\,dz_1\cdots dz_N \\
&= 2^N\left(\int_0^1 u^{1-2Hd}\,du\right)^N\left(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\right)^N \\
&= \left(\frac{1}{1-Hd}\right)^N\left(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\right)^N.
\end{aligned}$$

Note that as $\lambda < 0$, $1-2Hd > -1$.

Finally,

$$\begin{aligned}
\|L(\widetilde{1},0)\|^2_{\alpha,2} &= \frac{1}{(2\pi)^d}\left(\frac{1}{1-Hd}\right)^N\sum_{m=0}^{\infty}(1+m)^{\alpha} \\
&\qquad\times\left(\sum_{n_1+\cdots+n_d=m}\prod_{i=1}^{d} n_i!\cdot\mathbf{H}_{n_i}(0)^2\right)\left(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\right)^N \\
&= \frac{1}{(2\pi)^d}\left(\frac{1}{1-Hd}\right)^N\sum_{r=0}^{\infty}(1+2r)^{\alpha} \\
&\qquad\times\left(\sum_{r_1+\cdots+r_d=r}\prod_{i=1}^{d}\frac{(2r_i)!}{(r_i!)^2 2^{2r_i}}\right)\left(\int_0^1 Q_H(z)^m\frac{dz}{z^{dH}}\right)^N,
\end{aligned}$$

because $\mathbf{H}_{2n}(0) = \frac{1}{2^n n!}$ and $\mathbf{H}_{2n+1}(0) = 0$.

By the continuity of the norm, it is not necessary to prove the convergence of this series.

Remark 9. Xiao and Zhang [7] proved that when $Hd < 1$, that is $\lambda < 0$, B^H has a jointly continuous local time.

§5. Renormalization of the local time when the time tends to infinity

We can also deduce the behavior of the local time $L(t,x)$ when $\underline{t} = t_1 \cdots t_N \to \infty$ and $|x| \to 0$. We also have to distinguish the three cases, namely $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

The precise result is the following:

THEOREM 10. *Let $\{L(t,x) : t \in [0,\infty)^N, x \in \mathbb{R}^d \backslash \{0\}\}$ be the local time of the (N,d)-fBm B^H. Let $\lambda = d - \frac{1}{H}$. Then the following limits hold for any $\alpha < \frac{N}{2H} - \frac{d}{2}$:*

1) *For $\lambda > 0$ and $N \geq 2$,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \left(\frac{2^{\frac{\lambda}{2}} (\frac{1}{2H})^N |x|^{-\lambda}}{(2\pi)^{\frac{d}{2}} (N-1)!} \left(\log \frac{2\underline{t}^{2H}}{|x|^2} \right)^{N-1} \Gamma\left(\frac{\lambda}{2}\right) \right)^{-1} = 1.$$

If $N = 1$, $\|L(t,x)\|_{\alpha,2}$ explodes when x tends to 0 and t to ∞, and independently of $t \in \mathbb{R}_+$, $\|L(t,x)\|_{\alpha,2} \cdot (|x|^{-\lambda})^{-1}$ is bounded by a positive constant.

2) *For $\lambda = 0$,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \left(\frac{(\frac{1}{2H})^N}{(2\pi)^{\frac{d}{2}} N!} \left(\log \frac{2\underline{t}^{2H}}{|x|^2} \right)^{N} \right)^{-1} = 1.$$

3) *For $\lambda < 0$,*

$$\lim_{\underline{t}\to\infty,|x|\to 0} \|L(t,x)\|_{\alpha,2} \left(\underline{t}^{(1-dH)} \|L(\widetilde{1},0)\|_{\alpha,2} \right)^{-1} = 1.$$

Proof. From the Wiener chaos expansion of $L(t,x)$ proved in [1], we can obtain that

$$\|L(t,x)\|_{\alpha,2}^2 = \underline{t}^{2(1-dH)} \|L(\widetilde{1}, \underline{t}^{-H} x)\|_{\alpha,2}^2,$$

and the conclusion follows from the results of the previous section. □

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