INTEGRALS INVOLVING PRODUCTS OF BESSEL FUNCTIONS

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1. Introductory. In this paper certain infinite integrals involving products of four Bessel functions of different arguments are evaluated in terms of Appell's function F_4 by the methods of the operational calculus. The results obtained are believed to be new.

As usual, the conventional notation $\phi(p) \doteqdot h(t)$ will be used to denote the classical Laplace integral relation

$$\phi(p) = p\int_0^\infty e^{-pt}h(t)\,dt. \tag{1}$$

In the proofs of the formulae the following results will be required [**1**, pp. 281, 284], [**3**, pp. 78, 79].

$$\exp\left(-\frac{\gamma+\delta}{p}\right)I_\mu\left(\frac{2\sqrt{(\gamma\delta)}}{p}\right) \doteqdot J_\mu(2\sqrt{(\gamma t)})J_\mu(2\sqrt{(\delta t)}), \tag{2}$$

where $R(\mu) > -1$.

$$\exp\left(\frac{\gamma+\delta}{p}\right)I_\mu\left(\frac{2\sqrt{(\gamma\delta)}}{p}\right) \doteqdot I_\mu(2\sqrt{(\gamma t)})I_\mu(2\sqrt{(\delta t)}), \tag{3}$$

where $R(\mu) > -1$ and $R(p) > 0$.

$$p^{1-\lambda}e^{-\gamma/p}I_\mu(\gamma/p) \doteqdot \frac{\gamma^\mu t^{\lambda+\mu-1}}{2^\mu\Gamma(\mu+1)\Gamma(\lambda+\mu)}\,{}_1F_2(\mu+\tfrac{1}{2};\,2\mu+1,\,\lambda+\mu;\,-2\gamma t), \tag{4}$$

where $R(\lambda+\mu) > 0$.

$$2pK_\nu(2\sqrt{(\alpha p)})\,I_\nu(2\sqrt{(\beta p)}) \doteqdot \frac{1}{t}\exp\left(-\frac{\alpha+\beta}{t}\right)I_\nu\left(\frac{2\sqrt{(\alpha\beta)}}{t}\right), \tag{5}$$

where $R(\alpha) > 0$, $R(\beta) > 0$ and $R(p) > 0$.

$$K_\nu(z) = \tfrac{1}{2}\sum_{\nu,\,-\nu}\Gamma(-\nu)\Gamma(1+\nu)I_\nu(z), \tag{6}$$

and

$$K_{-\nu}(z) = K_\nu(z). \tag{7}$$

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2. Integrals. The first of the integrals to be established here is

$$\int_0^\infty tK_\nu(\alpha t)I_\nu(\beta t)J_\mu(\gamma t)J_\mu(\delta t)\,dt = \frac{(\alpha\beta)^\nu(\gamma\delta)^\mu\Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^{\mu+\nu+1}\Gamma(\nu+1)\Gamma(\mu+1)}$$

$$\times F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}\right], \tag{8}$$

where $R(\mu+\nu+1) > 0$, $R(\alpha) > | R(\beta) | + | \operatorname{Im} \gamma | + | \operatorname{Im} \delta |$.

To prove this we use the operational pairs (2) and (5) in the Goldstein form of the Parseval relation [**2**, p. 105] and then apply the formula [**1**, p. 196, (13)].

The following results can be derived in the same way from the pair of formulae (4), (5) and (3), (5) respectively.

$$\int_0^\infty t^{2\lambda+2\mu-1}K_\nu(\alpha t)I_\nu(\beta t)\,{}_1F_2(\mu+\tfrac{1}{2}; 2\mu+1, \lambda+\mu; -\gamma^2 t^2)\,dt$$

$$= \frac{2^{2(\lambda+\mu-1)}(\alpha\beta)^\nu\Gamma(\lambda+\mu+\nu)\Gamma(\lambda+\mu)}{(\alpha^2+\beta^2+2\gamma^2)^{\lambda+\mu+\nu}\Gamma(\nu+1)}$$

$$\times F_4\left[\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+2\gamma^2)^2}, \frac{4\gamma^4}{(\alpha^2+\beta^2+2\gamma^2)^2}\right], \tag{9}$$

where $R(\lambda+\mu+\nu) > 0$, $R(\lambda+\mu) > 0$ and $R(\alpha) > | R(\beta) | + | \operatorname{Im} \gamma |$.

$$\int_0^\infty tK_\nu(\alpha t)I_\nu(\beta t)I_\mu(\gamma t)I_\mu(\delta t)\,dt = \frac{(\alpha\beta)^\nu(\gamma\delta)^\mu\Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^{\mu+\nu+1}\Gamma(\nu+1)\Gamma(\mu+1)}$$

$$\times F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^2}\right], \tag{10}$$

where $R(\mu+\nu+1) > 0$, $R(\mu+1) > 0$ and $R(\alpha) > | R(\beta) | + | R(\gamma) | + | R(\delta) |$.

On applying (6) to (8), (9) and (10) and using (7), we find that

$$\int_0^\infty tK_\nu(\alpha t)K_\nu(\beta t)J_\mu(\gamma t)J_\mu(\delta t)\,dt = \frac{(\gamma\delta)^\mu}{2\Gamma(\mu+1)}\sum_{\nu,-\nu}\frac{(\alpha\beta)^\nu\Gamma(-\nu)\Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^{\mu+\nu+1}}$$

$$\times F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2}; \nu+1, \mu+1; \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2+\gamma^2+\delta^2)^2}\right], \tag{11}$$

where $R(1+\mu\pm\nu) > 0$, $R(\alpha+\beta) > | \operatorname{Im} \gamma | + | \operatorname{Im} \delta |$;

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$$\int_0^\infty t^{2\lambda+2\mu-1} K_\nu(\alpha t) K_\nu(\beta t)\, {}_1F_2(\mu+\tfrac{1}{2};\, 2\mu+1,\, \lambda+\mu;\, -\gamma^2 t^2)\, dt$$

$$= 2^{2\lambda+2\mu-3}\Gamma(\lambda+\mu) \sum_{\nu,\,-\nu} \frac{(\alpha\beta)^\nu \Gamma(-\nu)\Gamma(\lambda+\mu+\nu)}{(\alpha^2+\beta^2+2\gamma^2)^{\lambda+\mu+\nu}}$$

$$\times F_4\left[\frac{\lambda+\mu+\nu}{2}, \frac{\lambda+\mu+\nu+1}{2};\, \nu+1,\, \mu+1;\, \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2+2\gamma^2)^2}, \frac{4\gamma^4}{(\alpha^2+\beta^2+2\gamma^2)^2}\right], \qquad (12)$$

where $R(\lambda+\mu\pm\nu) > 0$, $R(\alpha+\beta) > |\operatorname{Im}\gamma|$; and

$$\int_0^\infty t K_\nu(\alpha t) K_\nu(\beta t) I_\mu(\gamma t) I_\mu(\delta t)\, dt = \frac{(\gamma\delta)^\mu}{2\Gamma(\mu+1)} \sum_{\nu,\,-\nu} \frac{(\alpha\beta)^\nu \Gamma(-\nu)\Gamma(\mu+\nu+1)}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^{\mu+\nu+1}}$$

$$\times F_4\left[\frac{\mu+\nu+1}{2}, \frac{\mu+\nu+2}{2};\, \nu+1,\, \mu+1;\, \frac{4\alpha^2\beta^2}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^2}, \frac{4\gamma^2\delta^2}{(\alpha^2+\beta^2-\gamma^2-\delta^2)^2}\right], \qquad (13)$$

where $R(1+\mu\pm\nu) > 0$, $R(\alpha+\beta) > |R(\gamma)| + |R(\delta)|$.

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