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On Limit Multiplicities for Spaces of Automorphic Forms

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Abstract. Let Γ be a rank-one arithmetic subgroup of a semisimple Lie group G. For fixed K-Type, the spectral side of the Selberg trace formula defines a distribution on the space of infinitesimal characters of G, whose discrete part encodes the dimensions of the spaces of square-integrable Γ-automorphic forms. It is shown that this distribution converges to the Plancherel measure of G when Γ shrinks to the trivial group in a certain restricted way. The analogous assertion for cocompact lattices Γ follows from results of DeGeorge-Wallach and Delorme.

Introduction

Let G be a semisimple Lie group and (Γ_j) a tower of lattices. This means that every Γ_j is a lattice in G, is normal in Γ_1, we have $\Gamma_1 \supset \Gamma_2 \supset \cdots$ and $\bigcap_j \Gamma_j = \{e\}$. For any irreducible unitary representation π of G let $N_{\Gamma_j}(\pi)$ be the multiplicity of π in the unitary G-representation $L^2(\Gamma_j \setminus G)$. In the case that the Γ_j are cocompact, DeGeorge and Wallach [11] showed that

$$\lim_{j\to\infty} \frac{N_{\Gamma_j}(\pi)}{\mathrm{vol}(\Gamma_j \setminus G)} = \mu(\{\pi\}),$$

where μ denotes the Plancherel measure on $\hat{G}$. Using results of Rohlfs and Speh, Savin [33] proved the same assertion for towers of non-cocompact congruence subgroups. See also [36] for an alternate approach. The above limit will be nonzero only for π in the discrete series of G. This is unsatisfactory since it says nothing about other subsets of the unitary dual. Actually, one should recover the entire Plancherel measure from the numbers $N_{\Gamma_j}(\pi)$. In this spirit, DeGeorge and Wallach attached a measure on $\hat{G}$ to any cocompact group Γ:

$$\mu_\Gamma := \sum_{\pi\in\hat{G}} N_\Gamma(\pi)\delta_\pi.$$

In the rank one case, DeGeorge and Wallach proved and in general they conjectured that the sequence of measures

$$\frac{\mu_{\Gamma_j}}{\mathrm{vol}(\Gamma_j \setminus G)}$$

converges vaguely to the Plancherel measure provided the Γ_j in the tower are cocompact. Delorme later proved the conjecture of DeGeorge and Wallach [13]. He also determined in

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certain cases the asymptotics of $\mu_\Gamma(\Omega_j)$ when Γ is fixed, but Ω_j runs through an expanding family of subsets of $\hat{G}$. The latter result is a variation of Weyl's asymptotic law, and the two types of asymptotics have many features in common.

Of course, Weyl's law gives less information because one only counts the eigenvalues of the Laplacian in a bundle over the locally-symmetric space $\Gamma \backslash G/K$ or, what amounts to the same, the eigenvalues of the Casimir element in a K-isotypical component of $L^2(\Gamma \backslash G)$. We shall take an intermediate point of view by fixing a K-type τ and counting the multiplicities of infinitesimal characters in the corresponding K-isotypical component, which defines a measure on the parameter space for these infinitesimal characters. This simplification allows us to concentrate on the difficulties which arise when $\Gamma \setminus G$ is noncompact.

In the non-cocompact case one can define a measure $\mu_\Gamma^{\mathrm{dis}}$ on $\hat{G}$ by counting the multiplicities in the discretely decomposable subspace of $L^2(\Gamma_j \setminus G)$. But it is not *a priori* clear whether this is the right object for studying limit multiplicities.

To begin with, let us look at the case $G = \mathrm{PSL}_2(\mathbb{R})$. Here we can let the lattice Γ vary in Teichmüller space, and the most natural measure should vary continuously with the group. Phillips and Sarnak showed in [28], [29] that for the measure $\mu_\Gamma^{\mathrm{dis}}$ this is not the case. On the other hand, one can define another measure $\mu_\Gamma^{\mathrm{con}}$ (which is not necessarily positive) using the winding number of scattering determinant. This measure can be written as

$$\mu_\Gamma^{\mathrm{con}} = c_\Gamma d\nu + \sum_\lambda N_\Gamma(\lambda)\mu_\lambda,$$

where c_Γ is a constant, λ runs through the poles of the scattering matrix and each μ_λ is a certain absolutely continuous measure on the principal series $\{\pi_\nu \mid \nu \in i\mathbb{R}\}$, which tends to the delta measure at π_ν if $\lambda \to \nu \in i\mathbb{R}$. Selberg showed, using his trace formula, that the analog of Weyl's asymptotic law is true for the measure $\mu_\Gamma := \mu_\Gamma^{\mathrm{dis}} + \mu_\Gamma^{\mathrm{con}}$ (see [34, p. 668], and [15, Ch. 6, Prop. 3.17]), and in [16] it was proved that this measure depends continuously on Γ.

In this paper we consider the analog $\mu_{\tau,\Gamma}$ of this measure on the set of infinitesimal characters of representations having a fixed K-type τ. Restricting ourselves to certain types of towers in $\mathbb{Q}$-rank one groups, called local towers of bounded depth in this paper, we are able to show that

$$\frac{\mu_{\tau,\Gamma_j}}{\mathrm{vol}(\Gamma_j \setminus G)}$$

converges to the Plancherel measure as a distribution of order one. We also show that this sequence converges vaguely to the Plancherel measure on the non-tempered set.

So far we do not know whether the restriction to congruence subgroups really is necessary for the study of limit multiplicities in the noncompact case. If it were the fact that μ_Γ depends continuously on Γ would be irrelevant for the given problem. It is because this measure naturally occurs in the Selberg trace formula that we chose it. We conjecture that for towers of bounded depth the convergence assertion is true for the discrete part alone. The analogous conjecture about the Weyl asymptotics has been made in [32]. In that case, the only approach known so far is to find the asymptotics for μ_Γ as a whole and then to show that the continuous contribution is of smaller order. This requires estimating the order of growth of the logarithmic derivative of the scattering matrix, which can be expressed, at least in principal, in terms of automorphic L-functions. In our case, the situation is similar, but more complicated, since we need such bounds uniformly in the level. We can prove our conjecture in the special case of principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, where the necessary estimates are known.

The paper is organized as follows. In the first section we consider Arthur's trace formula, which is an equality of two expansions for the same distribution $J(f)$. For a tower (Γ_j) of congruence subgroups we consider test functions of the form $f_j = \mathrm{Pr}_j \otimes \varphi$, where φ is smooth and of compact support on $G = \mathcal{G}(\mathbb{R})$, $\mathcal{G}$ defined over $\mathbb{Q}$. Further Pr_j is the function that projects to the Γ_j-invariants. Using the geometric expansion, we then show that, for local towers, $\big(\mathrm{vol}(\Gamma_j \setminus G)\big)^{-1} J(f_j)$ converges to $\varphi(e)$. The idea then is to switch to the spectral side and plug in functions for which the trace formula is not necessarily true but the convergence of $J(f_j)$ still holds. To create suitable test functions, we introduce the central functional calculus in Section 2. Previously, the main tool was Lemma 9.3 from [10] and its generalization, Lemma 3.7 in [13], which shows that the Fourier transforms of functions in $\mathcal{C}^0(G)$ are dense in a certain Schwartz space. We show here that test functions in $C_c^\infty(G)$ suffice for the same purpose. While it may be possible to prove the convergence of the geometric side of the trace formula for $\varphi \in \mathcal{C}^0(G)$ in the $\mathbb{Q}$-rank one case, to which we finally specialize, our approximation result is a significant relief in the general case. In Section 3 we prove the convergence of $J(f_j)$ for the extended class of test functions. For this purpose, we have to assume that the $\mathbb{Q}$-rank of $\mathcal{G}$ is one, because we depend on an argument of Müller which seems to work only in this case. The spectral estimates we proved in [12] are crucial for our method. In Section 4 we show the vanishing of the non-tempered contribution if the $\mathbb{R}$-rank of $\mathcal{G}$ is also equal to one. In Section 5 we consider principal congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$, in which case we can show the continuous contribution to vanish for $j \to \infty$.

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1 The Geometric Side

In this section we show that for test functions which are of compact support on the group G, the trace formula distribution converges to the Plancherel measure.

1.1

Let $\mathcal{G}$ be a semisimple simply connected linear group over the rational numbers. We embed the group of rational points $\mathcal{G}(\mathbb{Q})$ diagonally into the group of adelic points $\mathcal{G}(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of $\mathbb{Q}$. Further consider the group $\mathcal{G}(\mathbb{A}_{\mathrm{f}})$ of points over the finite adeles $\mathbb{A}_{\mathrm{f}}$. Reserve the letter G for the semisimple real Lie group $G = \mathcal{G}(\mathbb{R})$. So we have $\mathcal{G}(\mathbb{A}) = \mathcal{G}(\mathbb{A}_{\mathrm{f}}) \times G$. Fix a rational invariant top degree differential form on $\mathcal{G}$ and the corresponding Tamagawa measure on the groups $\mathcal{G}(\mathbb{A}_{\mathrm{f}})$, G and $\mathcal{G}(\mathbb{A})$.

Choose a maximal compact subgroup $K_{\max} = \prod_v K_{\max,v}$ such that $K_{\max,\mathrm{f}} := K_{\max} \cap \mathcal{G}(\mathbb{A}_{\mathrm{f}})$ is open in $\mathcal{G}(\mathbb{A}_{\mathrm{f}})$ and that $K_{\max,v}$ is a special maximal compact subgroup of $\mathcal{G}(\mathbb{Q}_v)$ for each finite place v. Then we have $\mathcal{G}(\mathbb{A}) = \mathcal{P}(\mathbb{A}) K_{\max}$ for any parabolic $\mathbb{Q}$-subgroup $\mathcal{P}$ of $\mathcal{G}$. Write $K = K_{\max,\infty} \subset G$, so that $K_{\max} = K_{\max,\mathrm{f}} \times K$.

1.2

A *tower of subgroups* of a given group G is a sequence $(\Gamma_j)_{j\in\mathbb{N}}$ of subgroups of G such that

(i) the sequence decreases to the trivial group, *i.e.*, $\Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \cdots$ and $\bigcap_j \Gamma_j = \{1\}$ and
(ii) each Γ_j is normal of finite index in Γ_1.

1.3

A subgroup Γ of $\mathcal{G}(\mathbb{Q})$ is called a *congruence subgroup* if there is a compact open subgroup K_Γ of $\mathcal{G}(\mathbb{A}_{\mathrm{f}})$ such that $\Gamma = \mathcal{G}(\mathbb{Q}) \cap K_\Gamma$. Whenever we consider a tower (Γ_j) of congruence subgroups we will tacitly assume that Γ_1 is maximal, *i.e.*, $\Gamma_1 = \mathcal{G}(\mathbb{Q}) \cap K_{\max,\mathrm{f}}$. A tower of congruence subgroups (Γ_j) will be called *local* if

(i) there is a finite set S of places, containing ∞, such that each K_{Γ_j} contains $\prod_{v\notin S} K_{\max,v}$.
(ii) There is finite a place $v \in S$ such that for the projection $K_{\Gamma_j,v}$ of K_{Γ_j} to $\mathcal{G}(\mathbb{Q}_v)$ we have $\bigcap_j K_{\Gamma_j,v} = \{1\}$.

1.4

The principal congruence subgroup of level N in $\mathrm{GL}_n(\mathbb{Z})$ is defined as the kernel $\Gamma_n(N)$ of the residue map $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$. A tower Γ_j of subgroups of $\mathcal{G}(\mathbb{Q})$ will be called a *tower of bounded depth* (with respect to a faithful $\mathbb{Q}$-rational representation $\eta\colon \mathcal{G} \to \mathrm{GL}_n$) if there exists a sequence of natural numbers N_j and a natural number D such that $\Gamma_n(N_j) \cap \eta\big(\mathcal{G}(\mathbb{Q})\big)$ is a subgroup of $\eta(\Gamma_j)$ of index not exceeding D. It is easy to see that this notion is independent of the choice of η.

1.5 Remarks

The term tower seems contradictory at the first glance since this tower points downwards; the notion stems from the tower of coverings (G/Γ_j) of the manifold G/Γ_1.

Note that in the case of local towers the condition ii) implies that the $K_{\Gamma_j,v}$ form a basis of neighborhoods of the unit element in $\mathcal{G}(\mathbb{Q}_v)$. To see this, take any open neighborhood U of the unit element in $\mathcal{G}(\mathbb{Q}_v)$, then $B_j := K_{\Gamma_j,v} - U$ is a decreasing sequence of compact subsets with $\bigcap_j B_j = \varnothing$. By the finite intersection property there exists a j_0 such that $B_j = \varnothing$ for all $j \leq j_0$. This means that for all $j \leq j_0$ we have $K_{\Gamma_j,v} \subset U$.

Another remark belongs here: We could also consider a group $\mathcal{G}$ defined over a number field $F/\mathbb{Q}$. This would give a slightly more general notion of a local tower, but we chose to work over the rationals in order to keep the presentation as simple as possible.

1.6

We fix a tower (Γ_j) of congruence subgroups. Let $\varphi \in C_c^\infty(G)$ and define for each j a function $f_j = f_{j,\varphi}$ by

$$f_j := \frac{1}{\mathrm{vol}(K_\Gamma)} \mathbf{1}_{K_{\Gamma_j}} \otimes \varphi.$$

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Note that this could also be written as $f_j = \frac{1}{[\Gamma_1:\Gamma_j]} \cdot \frac{1}{\mathrm{vol}(K_{\Gamma_j})} \mathbf{1}_{K_{\Gamma_j}} \otimes \varphi$.

Apply Arthur's trace formula [2], [3] to f_j (notation as in *loc. cit.*):

$$\sum_{o \in \mathcal{O}} J_o(f_j) = \sum_{\chi \in \mathbf{X}} J_\chi(f_j).$$

For simplicity we will write $J(f_j)$ for either side of the trace formula.

Proposition 1.7 *Assume the tower (Γ_j) is local. As $j \to \infty$, the geometric side of the trace formula (and hence the spectral one, too) converges to*

$$\mathrm{vol}(\Gamma \setminus G)\varphi(e),$$

where $\Gamma = \Gamma_1$.

Proof For unexplained notation see [2]. Recall the definition of the geometric terms [2]: On $\mathcal{G}(\mathbb{Q})$ we have the equivalence relation: $x \sim y$ iff the semisimple parts of x and y are conjugate in $\mathcal{G}(\mathbb{Q})$. Let $\mathcal{O}$ be the set of equivalence classes in $\mathcal{G}(\mathbb{Q})$. For $f \in C_c^\infty\big(\mathcal{G}(\mathbb{A})\big)$, each parabolic $\mathcal{P} \supset \mathcal{P}_0$ and each $o \in \mathcal{O}$ let

$$K_{\mathcal{P},o,f}(x,y) := \sum_{\gamma \in \mathcal{M}_{\mathcal{P}}(\mathbb{Q}) \cap o} \int_{\mathcal{N}_{\mathcal{P}}(\mathbb{A})} f(x^{-1}\gamma n y)\, dn$$

and

$$k_o^T(x,f) := \sum_{\mathcal{P} \supset \mathcal{P}_0} (-1)^{\dim \mathcal{A}_{\mathcal{P}}} \sum_{\delta \in \mathcal{P}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{Q})} K_{\mathcal{P},o,f}(\delta x, \delta x) \hat{\tau}_{\mathcal{P}}\big(H_{\mathcal{P}}(\delta x) - T_{\mathcal{P}}\big),$$

then for T sufficiently regular

$$J_o^T(f) = \int_{\mathcal{G}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A})} k_o^T(x,f)\, dx.$$

In [4, p. 18], Arthur shows that $J_o^T(f)$ is a polynomial in T, and he defines $J_o(f) := J_o^{T_0}(f)$ for a special point T_0.

Let $o_u \in \mathcal{O}$ be the class of unipotent elements. To prove the proposition we first consider the sum

$$\sum_{o \neq o_u} J_o^T(f_j)$$

and show that it tends to zero for any sufficiently regular T. Following Arthur in the proof of Theorem 7.1 in [2] this sum can be estimated by

$$\sum_{\mathcal{P}_1 \subset \mathcal{P}_2} \int_{\mathcal{P}_1(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{A})} F^1(x,T)\sigma_1^2\big(H_0(x) - T\big) k_{\mathcal{P}_1,\mathcal{P}_2}(x, f_j)\, dx,$$

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where $k_{\mathcal{P}_1,\mathcal{P}_2}(x, f_j)$ equals

$$\sum_{\gamma\in E\subset\mathcal{M}_1(\mathbb{Q})-o_u}\ \sum_{\zeta\in\mathfrak{n}_1^2(\mathbb{Q})'}\left|\int_{\mathfrak{n}_1(\mathbb{A})} f_j\big(x^{-1}\gamma e(X)x\big)\psi(\langle X,\zeta\rangle)\,dX\right|.$$

The sum over γ is finite and can be taken outside. Since γ is not unipotent, it has eigenvalues different from 1 and so has $\gamma e(X)$ for any X. Let $p\in S$ be as in condition (ii). Let $A\subset\mathcal{G}(\mathbb{Q}_p)$ be the closure of the set of all $\mathcal{G}(\mathbb{Q}_p)$-conjugates of all elements of the form $\gamma e(X)$, $X\in\mathfrak{n}_1(\mathbb{Q}_p)$. Then the unit element does not lie in A, and since the $K_{\Gamma_j,v}$ form a basis of neighborhoods of the unit element in $\mathcal{G}(\mathbb{Q}_v)$, they do not intersect A for sufficiently large j. This shows that we have

$$\sum_{o\neq o_u} J_o^T(f_j)=0$$

for $j\geq j_0$, which also implies the same assertion without the superscript T.

Now we are left with the unipotent contribution $J_{o_u}(f)$. Recall from Theorem 8.1 in [7] that this is a linear combination of the distributions $J_{\mathcal{M}}(u,f)$ and from Corollary 6.2 in [8] that the latter is an integral over u^G relative to a measure which is absolutely continuous with respect to the invariant measure class. According to Corollary 4.4 and Corollary 8.4 in [7] the contribution with $u=e$ gives us

$$f_j(e)\operatorname{vol}\big(\mathcal{G}(\mathbb{Q})\setminus\mathcal{G}(\mathbb{A})\big)=\varphi(e)\operatorname{vol}(\Gamma\setminus G).$$

So it remains to show that $J_{\mathcal{M}}(u,f_j)$ tends to zero for $u\neq e$ unipotent. Write

$$J_{\mathcal{M}}(u,f)=\int_{u^G} f(x)\,dm(x).$$

By condition (ii) the set $\operatorname{supp}(f_j)\cap u^G$ shrinks to the empty set as $j\to\infty$. Therefore the above integral tends to zero for $u\neq e$. The proposition is proven. ■

Conjecture 1.8 *The above proposition holds for arbitrary towers of congruence subgroups.*

2 The Functional Calculus

In this section we are going to construct test functions on the group with prescribed Fourier transform on the unitary dual. The reader may compare this to the Paley-Wiener theorems by Clozel and Delorme [9] and Arthur [6]. Our construction differs from those in that our functions depend on different parameters, namely the infinitesimal character and a K-type, whereas the usual Paley-Wiener functions depend on induction parameters. The induction parameters give a more complete result but are not as easy to handle as ours.

2.1

We will construct test functions by a functional calculus on the center $\mathfrak{z}$ of the universal enveloping algebra $U(\mathfrak{g})$, where $\mathfrak{g}$ is the complex Lie algebra of G. On $U(\mathfrak{g})$ we have the

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antilinear involution $X \mapsto X^*$, defined on the real form $\mathfrak{g}_0 = \mathrm{Lie}_{\mathbb{R}} G$ by $X \mapsto -X$. Then by definition we have for any $\pi \in \hat{G}$ that $\pi(X^*) = \pi(X)^*$ for $X \in U(\mathfrak{g})$. Clearly $*$ preserves $\mathfrak{z}$ and hence defines an involution on $\mathfrak{z}$.

For any Levi subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ we denote the center of $U(\mathfrak{m})$ by $\mathfrak{z}_{\mathfrak{m}}$. The (relative) Harish-Chandra homomorphism $\mathfrak{z} \to \mathfrak{z}_{\mathfrak{m}}$, $T \mapsto T_{\mathfrak{m}}$, is characterized by the following fact: Let $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{n}$ be any parabolic subgroup of $\mathfrak{g}$ with Levi component $\mathfrak{m}$ and $\mathfrak{p}^- = \mathfrak{m} \oplus \mathfrak{n}^-$ the opposite parabolic. Let ρ be the linear functional on $\mathfrak{m}$ defined by $2\rho(X) = \mathrm{tr}\big(\mathrm{ad}_{\mathfrak{n}}(X)\big)$ and let t_ρ be the automorphism of $U(\mathfrak{m})$ given for $X \in \mathfrak{m}$ by $t_\rho(X) := X - \rho(X)$. Then we have for any $T \in \mathfrak{z}$ that $T - t_\rho(T_{\mathfrak{m}}) \in U(\mathfrak{g})\mathfrak{n}$ or equivalently $T - t_\rho(T_{\mathfrak{m}}) \in \mathfrak{n}^- U(\mathfrak{g})$. The Harish-Chandra homomorphism is injective, and $\mathfrak{z}_{\mathfrak{m}}$ becomes a finitely generated module over $\mathfrak{z}$. In particular, if $\mathfrak{m} = \mathfrak{h}$ is a Cartan subalgebra, we get the Harish-Chandra isomorphism $\mathfrak{z} \to U(\mathfrak{h})^W$, where W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

Now let $\mathcal{M}$ be a Levi subgroup of $\mathcal{G}$ defined over the reals, $\mathfrak{m}$ the complexified Lie algebra of $M := \mathcal{M}(\mathbb{R})$. Then $\mathfrak{m}$ is stable under the involution $*$, further $\mathfrak{p}^*$ is again a parabolic subalgebra with corresponding $\rho_{\mathfrak{p}^*}$. We have $\rho_{\mathfrak{p}^*} = -\rho^*$, where $\rho^*(X) = \overline{\rho(X^*)}$. It follows $T^* - t_{\rho_{\mathfrak{p}^*}}\big((T_{\mathfrak{m}})^*\big) = \big(T - t_{-\rho}(T_{\mathfrak{m}})\big)^* \in U(\mathfrak{g})\mathfrak{n}^*$, thus the Harish-Chandra homomorphism commutes with the involution $*$.

Lemma 2.2 *The $\mathbb{C}$-algebra $\mathfrak{z}$ has a set of algebraically independent homogeneous selfadjoint generators $T_1, T_2, \ldots, T_r$. Moreover, for T_1 one can choose the Casimir element C.*

Proof Let H be a Cartan subgroup of G and write $\mathfrak{h}$ for its complex Lie algebra. We have the Harish-Chandra isomorphism $\mathfrak{z} \to U(\mathfrak{h})^W$, which commutes with the involution $*$. The natural gradation on $U(\mathfrak{h}) \cong S(\mathfrak{h})$, the symmetric algebra, gives a gradation on $\mathfrak{z}$:

$$\mathfrak{z} = \bigoplus_{k=0}^{\infty} \mathrm{Gr}_k(\mathfrak{z}).$$

which is preserved by the involution $*$ and so is the filtration:

$$F_k(\mathfrak{z}) := \bigoplus_{l \le k} \mathrm{Gr}_l(\mathfrak{z}),$$

so $\big(F_k(\mathfrak{z})\big)^* = F_k(\mathfrak{z})$. There is a set of algebraically independent generators $C = S_1, \ldots S_r$, which are homogeneous with respect to $(\mathrm{Gr}_.)$ [17]. We will show that, whenever S_j is not selfadjoint, we can replace it either with $\frac{1}{2}(S_j + S_j^*)$ or with $\frac{i}{2}(S_j - S_j^*)$ to get the desired set of generators.

We will proceed by induction on k and consider all generators inside $F_k(\mathfrak{z})$. For $k = 0$ we have $F_k(\mathfrak{z}) = \mathbb{C}$ and we are done. Now suppose, all generators inside $F_k(\mathfrak{z})$ are chosen selfadjoint already. Let B denote the algebra generated by $F_k(\mathfrak{z})$ and all selfadjoint elements in $\{S_1, \ldots, S_r\}$. Let $S_{j_1}, \ldots, S_{j_t}$ denote the non selfadjoint generators which lie in $Gr_{k+1}(\mathfrak{z})$. We then have

$$\begin{aligned} F_{k+1}(\mathfrak{z}) &= B \cap F_{k+1}(\mathfrak{z}) + \mathrm{Span}(S_{j_1}, \ldots, S_{j_t}) \\ &= B \cap F_{k+1}(\mathfrak{z}) + \mathrm{Span}(S_{j_1} + S_{j_1}^*, S_{j_1} - S_{j_1}^*, \ldots, S_{j_t} + S_{j_t}^*, S_{j_t} - S_{j_t}^*). \end{aligned}$$

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By linear algebra it becomes clear that we can choose R_i with either $R_i = S_{j_i} + S_{j_i}^*$ or $R_i = i(S_{j_i} - S_{j_i}^*)$ such that

$$F_{k+1}(\mathfrak{z}) = B \cap F_{k+1}(\mathfrak{z}) + \operatorname{Span}(R_1, \dots, R_t).$$

This shows that the algebra generated by $F_k(\mathfrak{z})$ and $R_1, \dots, R_t$ contains $S_{j_1}, \dots, S_{j_t}$, so we can replace the $S_{j_1}, \dots, S_{j_t}$ by the selfadjoint elements $R_1, \dots, R_t$. By induction the claim follows. ■

2.3

The involution $*$ on $\mathfrak{z}$ defines a real structure on the affine algebraic variety Spec $\mathfrak{z}$. We have the set $V_{\mathbb{C}} := \operatorname{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{z}, \mathbb{C})$ of $\mathbb{C}$-valued points and the set $V = \{\chi \in V_{\mathbb{C}} \mid \chi(T^*) = \overline{\chi(T)}\}$ of $\mathbb{R}$-valued points, and the ideal $\mathfrak{z} \cap \mathfrak{g}U(\mathfrak{g})$ defines a point $0 \in V$. The infinitesimal character χ_π of an irreducible unitary representation π will lie in V.

The fact that $\mathfrak{z}$ is a polynomial ring implies that Spec $\mathfrak{z}$ has the structure of an affine space and $V_{\mathbb{C}}$ that of a complex vector space. By the preceding lemma, V is a real subspace and $V_{\mathbb{C}}$ its complexification. However, these linear structures are noncanonical. By choosing self-adjoint generators of $\mathfrak{z}$, we fix an isomorphism $V \to \mathbb{R}^r$, which allows us to pull the standard Euclidean norm $\|.\|$ back to V. If $\|.\|'$ comes from another choice of generators, we have $\|x\|' \leq C(1 + \|x\|)^N$ for positive constants C and N.

Thus, we can unambiguously define the space $\mathcal{S}$ of Schwartz functions f on V by the seminorms

$$|f|_{n,D} = \sup_{x \in V} |(1 + \|x\|)^n Df(x)|,$$

where n runs through $\mathbb{N}$ and D through the algebra $\mathcal{D}$ of differential operators on V with polynomial coefficients. It is, of course, sufficient to take for D the monomials in the derivations with respect to a fixed basis. Since the order of any $D \in \mathcal{D}$ is independent of the choice of generators for $\mathfrak{z}$, it makes sense to define the space $\mathcal{S}^d \subset C^d(V)$ by requiring the finiteness of $|f|_{n,D}$ for $\operatorname{ord} D \leq d$ only.

2.4

If ρ is an arbitrary unitary representation of G acting on a Hilbert space H, then H decomposes as an integral over $\hat{G}$:

$$H = \int_{\hat{G}} H(\pi)\, dm(\pi),$$

where m is some measure depending on H and $H(\pi)$ a multiple of the representation space H_π of π [14]. For any continuous function f on V we define the operator

$$\rho(f) := \int_{\hat{G}} f(\chi_\pi)\, \mathrm{Id}_{H(\pi)}\; dm(\pi),$$

where χ_π is the infinitesimal character of π. This is a densely defined operator on the Hilbert space H. If f is the polynomial function on V defined by some $T \in \mathfrak{z}$, then $\rho(f) = \rho(T)$.

2.5

Now we fix an irreducible unitary representation τ of K acting on a finite dimensional Hilbert space H_τ.

We have an obvious representation ρ_τ of the algebra $\mathfrak{z}$ on the space of K-invariants $(H \otimes \tau)^K := (H \otimes H_\tau)^K$. Let I_τ be the intersection of $\mathfrak{z}$ with the left ideal in $U(\mathfrak{g})$ generated by the annihilator of $\check{\tau}$ in $U(\mathfrak{k})$. Since I_τ acts trivially, ρ_τ factors through the quotient algebra $\mathfrak{z}_\tau := \mathfrak{z}/I_\tau$. For f as before, we likewise define the operator

$$\rho_\tau(f) := \int_{\hat{G}} f(\chi_\pi)\,\mathrm{Id}_{(H(\pi)\otimes\tau)^K}\; dm(\pi).$$

If $(H_\pi \otimes \tau)^K \neq 0$, then χ_π vanishes on I_τ. Thus, if f vanishes on the subvariety $V_{\mathbb{C},\tau}$ defined by I_τ, then $\rho_\tau(f) = 0$. In other words, $\rho_\tau(f)$ depends only on the restriction of f to $V \cap V_{\mathbb{C},\tau}$. (A description of $V_{\mathbb{C},\tau}$ can be read off from Prop. 1.11 of [37], but we do not need it.)

In fact, $\rho_\tau(f)$ depends only on the restriction to an even smaller subset of V. Let $C_K \in U(\mathfrak{k})$ denote the Casimir element induced by the restriction of the Killing form of $\mathfrak{g}$; then $\tau(C_K) = \lambda_\tau \,\mathrm{Id}$ with $\lambda_\tau \geq 0$. Let $\hat{G}(\tau)$ be the set of all $\pi \in \hat{G}$ for which τ occurs in $\pi|_K$.

Lemma 2.6 *Let $T \in \mathfrak{z}$ of degree k. Then there exists a positive constant c such that for each $\pi \in \hat{G}(\tau)$ we have $\lambda_\tau - \chi_\pi(C) \geq 0$ and $|\chi_\pi(T)| \leq c\big(2\lambda_\tau - \chi_\pi(C)\big)^{k/2}$.*

Proof If $X \in \mathfrak{g}$ is a unit vector for the inner product $X \mapsto -B(X, \theta X)$, where B denotes the Killing form and θ the Cartan involution defined by K, then there is an orthonormal basis $X_1, \ldots, X_n$ with $X = X_1$, and $\sum_{i=1}^n X_i^2 = C - 2C_K$. Let $\pi \in \hat{G}(\tau)$ and take a nonzero smooth vector v in the τ-isotypical component of H_π. Then

$$\|Xv\|^2 \leq \sum_{i=1}^n \|X_i v\|^2 = \big((2C_K - C)v, v\big) = \big(2\lambda_\tau - \chi_\pi(C)\big)\|v\|^2.$$

Considering a basis for the Killing-orthogonal complement of $\mathfrak{k}$ only, one obtains in the same way that $0 \leq \big(\lambda_\tau - \chi_\pi(C)\big)\|v\|^2$, which proves the first assertion. We deduce from these two inequalities that for each $T \in U(\mathfrak{g})$ of degree k there exists a positive constant c independent of π such that

$$\|Tv\| \leq c\big(2\lambda_\tau - \chi_\pi(C)\big)^{k/2}\|v\|.$$

Specializing to $T \in \mathfrak{z}$, we get our second assertion. ■

2.7

We want to prove that there are enough $f \in \mathcal{S}$ for which $\rho_\tau(f)$ is represented by a smooth compactly supported kernel. Let $C_c^\infty(G, \tau)$ be the algebra of smooth compactly supported

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functions F on G with values in $\mathrm{End}(H_\tau)$ satisfying $F(k_1xk_2) = \tau(k_1^{-1})F(x)\tau(k_2)$. The representation of this algebra in $H \otimes \tau$ given by

$$v \otimes w \mapsto \int_G \rho(x)v \otimes F(x)w\, dx$$

restricts to a representation in $(H \otimes \tau)^K$, which we also denote by ρ_τ.

Proposition 2.8 *Let $\mathcal{B}_\tau$ be the set of all continuous functions f on V for which there is a function $F \in C_c^\infty(G, \tau)$ such that $\rho_\tau(f) = \rho_\tau(F)$ for all unitary representations ρ of G. Then $\mathcal{B}_\tau \cap \mathcal{S}$ is dense in $\mathcal{S}$.*

The proof will be given in several steps. First we consider functions f of the form $f(\chi) = h\bigl(-\chi(C)\bigr)$ for some $h \in \mathcal{S}(\mathbb{R})$. In this case $\rho_\tau(f) = h\bigl(-\rho_\tau(C)\bigr)$, where the right-hand side is defined by the usual functional calculus.

Lemma 2.9 *Let $h \in \mathcal{S}(\mathbb{R})$, $c \in \mathbb{R}$, and suppose that there exists an even Paley-Wiener function g such that $h(x^2 - c) = g(x)$ for all $x \in \mathbb{C}$ with $x^2 - c \in \mathbb{R}$. If we define $f \in \mathcal{S}$ by $f(\chi) = h\bigl(-\chi(C)\bigr)$, then $f \in \mathcal{B}_\tau$.*

The assumption on h is in fact independent of c. Indeed, if g is an even Paley-Wiener function, then so is $g_1(z) := g(\sqrt{z^2 + c})$. This curious fact follows directly from the definition, since $(\operatorname{Im} z)^2 - c \le (\operatorname{Im} \sqrt{z^2 + c})^2 \le (\operatorname{Im} z)^2$ for $c > 0$. Our lemma can also be deduced from the Paley-Wiener theorem of [6], but we give an independent proof.

Proof First we consider the case when ρ is the right regular representation of G. The measure m is then, of course, the Plancherel measure μ. Let $E_\tau = G \times_K H_\tau$ be the homogeneous vector bundle over the symmetric space $X = G/K$ corresponding to τ. The space of smooth sections of E_τ may be identified with the space of K-invariants $\bigl(C^\infty(G) \otimes H_\tau\bigr)^K$, where K acts on $C^\infty(G)$ by right translations. Analogously the Hilbert space of L^2-sections may be identified with $\bigl(L^2(G) \otimes H_\tau\bigr)^K$. Therefore the operator $\rho_\tau(f)$ may be viewed as an operator on $L^2(X, E_\tau)$.

By the above identification, the representation ρ_τ of $\mathfrak{z}$ embeds $\mathfrak{z}_\tau$ into the algebra of G-invariant differential operators in E_τ (see [25]). If $C \in \mathfrak{z}$ is the Casimir element as before, then $\Delta_\tau := \rho_\tau(-C) + \lambda_\tau \,\mathrm{Id}$ is the Bochner-Laplace operator in E_τ (see [24]). For simplicity of notation we choose $c = \lambda_\tau$, *i.e.*, $\rho_\tau(f) = g(\sqrt{\Delta_\tau})$.

The operator $\Delta_\tau + 1$ is strictly positive. It is clear that $\operatorname{im} h(\Delta_\tau) \subset \operatorname{im}(\Delta_\tau + 1)^{-N}$ for any $N \in \mathbb{N}$. Since the operators $(\Delta_\tau + 1)^{-N}$ become more and more regular as N grows it follows that $h(\Delta_\tau)$ is a smoothing operator, *i.e.*, an integral operator whose kernel is a smooth section of $E_\tau \boxtimes E_\tau^*$. By the above identification, we may view this kernel as a function $\Phi(x, y)$ on $G \times G$ with values in $\mathrm{End}(H_\tau)$ satisfying $\Phi(xk_1, yk_2) = \tau(k_1^{-1})\Phi(x, y)\tau(k_2)$ for $k_1, k_2 \in K$. Since $f(\Delta_\tau)$ commutes with the action of G by left translations, we have $\Phi(gx, gy) = \Phi(x, y)$, hence $\Phi(x, y) = F(y^{-1}x)$ for some smooth function F, and the action of $f(\Delta_\tau)$ on compactly supported sections is given by $\rho_\tau(F)$.

By functional calculus we have

$$\rho_\tau(f) = g(\sqrt{\Delta_\tau}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{g}(t) \cos(t\sqrt{\Delta_\tau})\, dt,$$

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where $\hat{g}$ denotes the Fourier transform of g. It is known that the Cauchy problem for the wave equation in the bundle E_τ,

$$\ddot{v} = -\Delta_\tau v, \quad v(0) = v_0, \quad \dot{v}(0) = 0$$

with smooth initial datum v_0 has a unique smooth solution for all times, which for square integrable sections v_0 is given by $v(t) = \cos(t\sqrt{\Delta_\tau})v_0$. The solution operator $\cos(t\sqrt{\Delta_\tau})$ extends to a bounded operator in the Fréchet space $C^\infty(X, E_\tau)$, which allows us to extend $f(\Delta_\tau)$ by the above formula. Moreover, the solutions have propagation speed one (see for example [35, Chs. 2 and 6]. Thus, if g is supported in $[-r, r]$, then $f(\Delta_\tau)$ can spread the support of v_0 only over a distance r. Consequently, F has compact support, and $\rho_\tau(F)$ extends to a bounded operator on $L^2(X, E_\tau)$ and $C^\infty(X, E_\tau)$. The equality $f(\Delta_\tau) = \rho_\tau(F)$ remains true by continuity.

Now we consider the case of $\pi \in \hat{G}$. We fix a continuous linear functional l on H_π and consider for each $v \in H_\pi$ the matrix coefficient $c_v(g) := l\bigl(\pi(g)v\bigr)$. Then $v \mapsto c_v$ is a continuous linear map from H_π to the Fréchet space $C(G)$, which intertwines π with the right regular representation ρ and preserves smooth vectors. For nonzero l this map is nonzero, hence injective by the irreducibility of π. Tensoring with H_τ and taking K-invariants, we get an embedding

$$e_\tau \colon (H_\pi \otimes \tau)^K \to \bigl(C^\infty(G) \otimes \tau\bigr)^K \cong C^\infty(X, E_\tau),$$

which obviously intertwines the representations π_τ and ρ_τ of $C_c^\infty(G, \tau)$. Since $e^{it\pi_\tau\sqrt{\lambda_\tau - C}}$ is determined by an ordinary differential equation just as $e^{it\sqrt{\Delta_\tau}}$ was, it is easy to deduce that $e_\tau\pi_\tau(f) = \rho_\tau(f)e_\tau$. Now the assertion of the proposition for π follows from the known assertion for ρ. The general case of a reducible representation follows by functional calculus. ∎

Lemma 2.10 *Let $r, k \in \mathbb{N}$, $U = \{x \in \mathbb{R}^r \mid x_1 \geq 1, \|x\| \leq x_1^k\}$, where x_1 denotes the first coordinate of x. Denote by $\mathcal{B}$ the set of all functions $f \in \mathcal{S}(\mathbb{R}^r)$ whose restriction to U is of the form $p(x)g(\sqrt{x_1})$ for a polynomial function p on $\mathbb{R}^r$ and an even Paley-Wiener function g on $\mathbb{R}$. Then $\mathcal{B}$ is dense in $\mathcal{S}(\mathbb{R}^r)$.*

Proof We choose a smooth homogeneous function η on $\mathbb{R} \times \mathbb{R}_+$ such that $\eta(u, v) = u$ for $v \leq u$ and $\eta(u, v) = v$ for $v \geq 2u$. If we now define a function ν on $\{x \in \mathbb{R}^r \mid \|x\| > 1\}$ by $\nu(x) = \eta(x_1, \|x\|^{1/k})$, then ν extends to a strictly positive smooth function on $\mathbb{R}^r$ such that $\nu(x) = x_1$ for $x \in B$. Its partial derivatives of any positive order are bounded, and $c_1^{-1}(1 + \|x\|)^{1/k} \leq \nu(x) \leq c_1(1 + \|x\|)$ for some $c_1 > 0$. Thus, the map $h \mapsto h \circ \nu$ is continuous from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R}^r)$, and $h\bigl(\nu(x)\bigr) = h(x_1)$ for $x \in B$. Remember that multiplication by a polynomial function is a continuous endomorphism of $\mathcal{S}(\mathbb{R}^r)$.

The set of all $h \in \mathcal{S}(\mathbb{R})$ such that $h(x_1) = g(\sqrt{x_1})$ for an even Paley-Wiener function g and all $x_1 \geq c_1^{-1}$ is clearly dense in $\mathcal{S}(\mathbb{R})$. Therefore, the closure of $\mathcal{B}$ contains all functions of the form $p(x)h\bigl(\nu(x)\bigr)$, where p is a polynomial and $h \in \mathcal{S}(\mathbb{R})$.

If Ω is a compact subset of $\mathbb{R}^r$, then there exists $h \in C_c^\infty(\mathbb{R})$ such that $h\bigl(\nu(x)\bigr) = 1$ for $x \in \Omega$. Denote $\Omega' = \operatorname{supp}(h \circ \nu)$. Let us topologize $C^\infty(\Omega')$ by the seminorms $\sup_{\Omega'} |Df|$,

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where D runs through the differential operators with constant coefficients. Then multiplication by $h \circ \nu$ defines a continuous map $C^\infty(\Omega') \to C_c^\infty(\mathbb{R}^r)$, which is the identity on functions with support in Ω. It is well known that the set of polynomial functions is dense in $C^\infty(\Omega')$. Therefore the closure of $\mathcal{B}$ contains all smooth functions with support in Ω. Now the assertion follows from the density of $C_c^\infty(\mathbb{R}^r)$ in $\mathcal{S}(\mathbb{R}^r)$. ■

Proof of Proposition 2.8 It is clear that the relation defining $\mathcal{B}_\tau$ needs only be checked for $\rho = \pi \in \hat{G}(\check{\tau})$. We know by Lemma 2.9 that certain functions f belong to $\mathcal{B}_\tau$. Let F be the corresponding kernel on G. If $T \in \mathfrak{z}$, then it is easy to check that $\rho_\tau(T)\rho_\tau(f) = \rho_\tau(TF)$. Since $TF \in C_c^\infty(G,\tau)$, we have $pf \in \mathcal{B}_\tau$, where p is the polynomial function on V corresponding to T. Using the generators from Lemma 2.2, we identify V with $\mathbb{R}^r$. Choose $k \in \mathbb{N}$ such that the degrees of the generators are bounded by $2k$. After replacing T_1 by $c - C$, $c \gg 0$, and possibly rescaling the other generators, Lemma 2.6 shows that $\{\chi_\pi \mid \pi \in \hat{G}(\check{\tau})\}$ is contained in the set U figuring in Lemma 2.10. Thus, $\mathcal{B} \subset \mathcal{B}_\tau$, and the density of $\mathcal{B}$ in $\mathcal{S}$ proved in Lemma 2.10 implies the assertion. ■

For the trace formula we need scalar test functions on G. These can be obtained as usual from the τ-spherical functions constructed above.

Lemma 2.11 *If $f \in \mathcal{B}_\tau$, then there exists a smooth compactly supported function φ on G such that*

$$\pi(\varphi) = (\dim \tau)^{-1} f(\chi_\pi)\,\mathrm{Pr}_{\check{\tau}},$$

for each $\pi \in \hat{G}$, where $\mathrm{Pr}_{\check{\tau}}$ is the projection to the K-type $\check{\tau}$.

Proof For each unitary representation ρ of G on a space H we have a bounded linear map

$$i_\tau\colon (H \otimes H_\tau)^K \otimes H_\tau^* \to H$$

induced by the convolution $H_\tau \otimes H_\tau^* \to \mathbb{C}$. Its image is the K-isotypical component of type τ. If $F \in C_c^\infty(G,\tau)$ and $\varphi(g) = \operatorname{tr} F(g)$, then an easy calculation shows that $i_\tau \circ \big(\rho_\tau(F) \otimes \mathrm{Id}\big) = \dim(\tau)\rho(\varphi) \circ i_\tau$. Moreover, $\rho(\varphi)$ vanishes on all other K-isotypical components. If we take for ρ some $\pi \in \hat{G}$ and if F and f are related as in the Proposition, then $\rho_\tau(F) = f(\chi_\pi)$, and the assertion follows. ■

2.12

Next we check that the Schwartz space behaves well under the Harish-Chandra homomorphism. Let $\mathcal{M}$ be a Levi subgroup of $\mathcal{G}$ defined over $\mathbb{R}$ and define $V_{\mathcal{M},\mathbb{C}} := \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{z}_{\mathfrak{m}}, \mathbb{C})$. The involution $*$ defines a real structure and a corresponding real subspace $V_{\mathcal{M}}$. Introduce $\mathcal{D}_{\mathcal{M}}$, $\mathcal{S}_{\mathcal{M}}$ and $\mathcal{S}_{\mathcal{M}}^d$ by analogy to the case $\mathcal{M} = \mathcal{G}$. Since the Harish-Chandra homomorphism $\mathfrak{z} \to \mathfrak{z}_{\mathfrak{m}}$ commutes with $*$, it defines a finite morphism $r_{\mathcal{M}} = r_{\mathcal{M}}^{\mathcal{G}}\colon V_{\mathcal{M},\mathbb{C}} \to V_{\mathbb{C}}$ satisfying $r_{\mathcal{M}}(V_{\mathcal{M}}) \subset V$ and $r_{\mathcal{M}}(0) = 0$. For Levi subgroups $\mathcal{M}' \subset \mathcal{M} \subset \mathcal{G}$ we have $(T_{\mathfrak{m}})_{\mathfrak{m}'} = T_{\mathfrak{m}'}$, hence $r_{\mathcal{M}'}^{\mathcal{M}} r_{\mathcal{M}} = r_{\mathcal{M}'}$.

Lemma 2.13 *The pullback by $r_{\mathcal{M}}$ defines continuous maps $r_{\mathcal{M}}^*\colon \mathcal{S} \to \mathcal{S}_{\mathcal{M}}$, $\mathcal{S}^d \to \mathcal{S}^d_{\mathcal{M}}$. More precisely, for each $n \in \mathbb{N}$ and each $D \in \mathcal{D}_{\mathcal{M}}$ there are finitely many $n_i \in \mathbb{N}$ and $D_i \in \mathcal{D}$ with* $\operatorname{ord} D_i \leq \operatorname{ord} D$ *such that*

$$|r_{\mathcal{M}}^* f|_{n,D} \leq \sum_{i=1}^{s} |f|_{n_i,D_i}.$$

Proof If $x \in V_{\mathbb{C}}$ and $r_{\mathcal{M}}(x) = 0$, then $x = 0$. In fact, by the transitivity of the maps $r_{\mathcal{M}}$ it is enough to check this if $\mathcal{M} = \mathcal{H}$ is a Cartan subgroup, in which case it follows from the standard identification $V_{\mathbb{C}} = W \setminus \mathfrak{h}^*$. Now the continuous function $\|r_{\mathcal{M}}(x)\|$ is seen to be bounded from zero for $\|x\| = 1$. If we attach suitable positive weights to the generators of $\mathfrak{z}$ and $\mathfrak{z}_{\mathfrak{m}}$, the maps $r_{\mathcal{H}}$ and $r_{\mathcal{H}}^{\mathcal{M}}$ (and hence $r_{\mathcal{M}}$) become homogeneous. Thus we get

$$\|x\| \leq C\big(1 + \|r_{\mathcal{M}}(x)\|\big)^N$$

for some $C > 0$, $N \in \mathbb{N}$. The asserted estimate can easily be deduced from this inequality and the chain rule. ∎

2.14

Finally, let us fix some notation in the case that the Levi subgroup $\mathcal{M}$ is defined over $\mathbb{Q}$. Write $X(\mathcal{M})_{\mathbb{Q}}$ for the group of its rational characters defined over $\mathbb{Q}$. Let $M^1 = \{m \in M : |\psi(x)| = 1\, \forall \psi \in X(\mathcal{M})_{\mathbb{Q}}\}$ and denote the maximal $\mathbb{Q}$-split torus in the center of $\mathcal{M}$ by $\mathcal{A}_{\mathcal{M}}$. Then $M = M^1 A_{\mathcal{M}}$ and correspondingly $\mathfrak{m} = \mathfrak{m}^1 \oplus \mathfrak{a}_{\mathcal{M},\mathbb{C}}$, where $A_{\mathcal{M}} := \mathcal{A}_{\mathcal{M}}(\mathbb{R})^0$ and $\mathfrak{a}_{\mathcal{M}}$ is its Lie algebra. This defines a decomposition $V_{\mathcal{M},\mathbb{C}} = V^1_{\mathcal{M},\mathbb{C}} \times \mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$ and $V_{\mathcal{M}} = V^1_{\mathcal{M}} \times i\mathfrak{a}^*_{\mathcal{M}}$.

3 The Convergence Theorem

3.1

Let $\mathcal{HS}$ be the usual Hilbert field over the unitary dual $\hat{G}$ whose fiber over $\pi \in \hat{G}$ is the space $\mathrm{HS}(\pi)$ of Hilbert-Schmidt operators in H_π. The scalar product on $\mathrm{HS}(\pi)$ is $\langle T, S\rangle = \operatorname{tr}(TS^*)$. Now $\varphi \in C_c^\infty(G)$ defines a section

$$\hat{\varphi}\colon \hat{G} \to \mathcal{HS}$$

by $\hat{\varphi}(\pi) = \pi(\varphi)$. Furthermore, $\pi(\varphi)$ is of trace class and

$$\varphi(e) = \int_{\hat{G}} \operatorname{tr} \hat{\varphi}(\pi)\, d\mu(\pi),$$

where μ is the Plancherel measure. The spectral side of the trace formula gives an explicit expansion of the distribution $J(\mathbf{1}_{K_\Gamma} \otimes \varphi)$ in terms of the section $\hat{\varphi}$. In particular, if

$$\hat{\varphi}(\pi) = (\dim \tau)^{-1} f(\chi_\pi)\, \mathrm{Pr}_{\check{\tau}}$$

for all $\pi \in \hat{G}$ as in Lemma 2.11, we may define

$$J_{\tau,j}(f) := J\left(\frac{1}{\operatorname{vol}(K_\Gamma)}\mathbf{1}_{K_{\Gamma_j}} \otimes \varphi\right).$$

The convergence assertion of Proposition 1.7 then reads

$$J_{\tau,j}(f) \overset{j\to\infty}{\longrightarrow} \operatorname{vol}(\Gamma \setminus G) \int_{\hat{G}} [\pi : \check{\tau}] f(\chi_\pi)\, d\mu(\pi).$$

However, from our previous discussion this follows only for f in a dense subspace $\mathcal{B}_\tau$ of $\mathcal{S}$. In this section we will show that $J_{\tau,j}$ extends continuously to $\mathcal{S}$ and that the convergence assertion still holds, provided $\mathcal{G}$ has $\mathbb{Q}$-rank one and the tower (Γ_j) satisfies a certain mild assumption. Note that we consider functions for which the geometric expansion in the trace formula does not converge any more. Thus, from now on we will assume that

$$\operatorname{rank}_{\mathbb{Q}} \mathcal{G} = 1.$$

3.2

First we introduce the necessary notation to write down the spectral side of the trace formula in this special case. Fix a parabolic $\mathbb{Q}$-subgroup $\mathcal{P} \neq \mathcal{G}$ with Levi decomposition $\mathcal{P} = \mathcal{M}\mathcal{N}$, and let $\chi \in V^1_{\mathcal{M}}$. We consider the space $\mathcal{H}_{\mathcal{P}}(\chi)$ of all functions ϕ on $\mathcal{N}(\mathbb{A})\mathcal{M}(\mathbb{Q})A_{\mathcal{M}} \setminus \mathcal{G}(\mathbb{A})$ whose pullback to $\mathcal{M}(\mathbb{Q}) \setminus \mathcal{M}(\mathbb{A})^1 \times K_{\max}$ is square integrable and which satisfy $\phi(Tx) = \chi(T)\phi(x)$ for $T \in \mathfrak{z}_{\mathfrak{m}}$ in the distributional sense. Thus, we use χ as in [1], but not as in [2]. Given $\phi \in \mathcal{H}_{\mathcal{P}}(\chi)$ and $\lambda \in \mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$, put $\phi_\lambda(x) = e^{\langle \lambda+\rho, H_{\mathcal{P}}(x)\rangle}\phi(x)$, where $2\rho(H) = \operatorname{tr}\operatorname{ad}_{\mathfrak{n}}(H)$ and $H_{\mathcal{P}}\colon \mathcal{G}(\mathbb{A}) \to \mathfrak{a}_{\mathcal{M}}$ is defined by $\langle \psi, H_{\mathcal{P}}(nmk)\rangle = |\psi(m)|$ for all $\psi \in X(\mathcal{M})_{\mathbb{Q}}$, $n \in \mathcal{N}(\mathbb{A})$, $m \in \mathcal{M}(\mathbb{A})$ and $k \in K_{\max}$. Then we get a representation $\mathcal{I}_{\mathcal{P}}(\chi,\lambda)$ of $\mathcal{G}(\mathbb{A})$ on $\mathcal{H}_{\mathcal{P}}(\chi)$ by

$$\big(\mathcal{I}_{\mathcal{P}}(\chi,\lambda,y)\phi\big)_\lambda(x) = \phi_\lambda(xy)$$

for $x, y \in \mathcal{G}(\mathbb{A})$. It is easy to see that $\mathcal{I}_{\mathcal{P}}(\chi,\lambda,T) = r_{\mathcal{M}}(\chi,\lambda)(T)\operatorname{Id}_{\mathcal{H}_{\mathcal{P}}(\chi)}$ for $T \in \mathfrak{z}$.

In the theory of Eisenstein series one considers the operator $M(\chi,\lambda)$ from the subspace of $K_{\max}$-finite vectors in $\mathcal{H}_{\mathcal{P}}(\chi)$ to that in $\mathcal{H}_{\mathcal{P}}(w\chi)$ which is defined for $\operatorname{Re}\lambda - \rho$ positive with respect to $\mathcal{P}$ by

$$\big(M(\chi,\lambda)\phi\big)_{-\lambda}(x) = \int_{\mathcal{N}(\mathbb{A})} \phi_\lambda(\tilde{w}nx)\, dn$$

and has a meromorphic continuation to $\mathfrak{a}^*_{\mathcal{M},\mathbb{C}}$. Here w denotes the only nontrivial element of the Weyl group of $(\mathcal{G},\mathcal{A})$ and $\tilde{w} \in \mathcal{G}(\mathbb{Q})$ any of its representatives. Let us write $\mathcal{H}_{\mathcal{P}}(\chi,\tau,j)$ for the $\check{\tau}$-isotypical subspace in the space of K_{Γ_j}-fixed vectors in $\mathcal{H}_{\mathcal{P}}(\chi)$. This is a finite-dimensional space of analytic functions. The operator $M(\chi,\lambda)$ intertwines $\mathcal{I}_{\mathcal{P}}(\chi,\lambda)$ with $\mathcal{I}_{\mathcal{P}}(w\chi,w\lambda)$, hence maps $\mathcal{H}_{\mathcal{P}}(\chi,\tau,j)$ to $\mathcal{H}_{\mathcal{P}}(w\chi,\tau,j)$. Moreover, it satisfies

$$M(\chi,\lambda)M(w\chi,w\lambda) = \operatorname{Id}, \quad M(\chi,\lambda)^* = M(\chi,\bar{\lambda})$$

(equality of meromorphic functions). Note that $w\lambda = -\lambda$. If $T \in \mathfrak{a}_{\mathcal{M}}$ is a truncation parameter, we put $M_T(\chi,\lambda) = e^{-2\lambda(T)}M(\chi,\lambda)$. We denote the restriction of $M(\chi,\lambda)$ to $\mathcal{H}_{\mathcal{P}}(\chi,\tau,j)$ by $M(\chi,\lambda,\tau,j)$ and write $N_{\Gamma_j}(\pi)$ for the multiplicity of π in $L^2(\Gamma_j \setminus G)$.

3.3

The spectral side of the trace formula $J^T_{\tau,j}(f)$ is given by

$$\frac{1}{[\Gamma_1:\Gamma_j]}\Bigg(\sum_{\pi\in\hat{G}(\check{\tau})} N_{\Gamma_j}(\pi)[\pi:\check{\tau}]f(\chi_\pi)$$
$$-\frac{1}{\dim\tau}\cdot\frac{1}{4\pi}\sum_{\chi}\int_{\mathfrak{a}^*_{\mathcal{M}}}\operatorname{tr}\big(M_T(w\chi,-i\lambda,\tau,j)M'_T(\chi,i\lambda,\tau,j)\big)r^*_{\mathcal{M}}(f)(\chi,i\lambda)\,d\lambda$$
$$+\frac{1}{\dim\tau}\cdot\frac{1}{4}\sum_{\chi=w\chi}\operatorname{tr}\big(M(\chi,0,\tau,j)\big)r^*_{\mathcal{M}}(f)(\chi,0)\Bigg).$$

Here we have identified $\mathfrak{a}^*_{\mathcal{M}}$ with $\mathbb{R}$, which explains the meaning of the complex derivative $M'_T(\chi,\lambda,\tau,j)=\frac{d}{d\lambda}M_T(\chi,\lambda,\tau,j)$ and the measure $d\lambda$. This formula is a special case of Theorem 8.2 in [5], where groups of arbitrary rank are treated. If one wants to avoid going through the general case, one can also extract it from [1]. In that earlier paper, a different kind of truncation was used, for which the spectral side is only asymptotic to $J^T_{\tau,j}(f)$ as $T\to\infty$. In both references it is proved that all sums and integrals occurring in the formula are absolutely convergent. For the second term this is meant in an iterated sense, and the same assertion will follow from the proof of Lemma 3.7 below. The stronger assertion that the integral over $(\chi,\lambda)\in\{1,w\}\setminus V_{\mathcal{M}}$ is absolutely convergent can be deduced from an argument of Langlands (see [18, Theorem 4.2]).

If we define the winding number of the scattering determinant

$$\phi_T(\chi,\Lambda,\tau,j)=\int_0^\Lambda\operatorname{tr}\big(M_T(\chi,-i\lambda,\tau,j)M'_T(\chi,i\lambda,\tau,j)\big)\,d\lambda,$$

an odd real-valued function of the real variable Λ by the functional equations, then the continuous contribution to $J^T_{\tau,j}(f)$ can be written as a Stielties integral

$$-\frac{1}{[\Gamma_1:\Gamma_j]\dim\tau}\cdot\frac{1}{2\pi}\sum_{\chi}\int_0^\infty r^*_{\mathcal{M}}(f)(\chi,i\lambda)\,d\phi_T(\chi,\lambda,\tau,j).$$

3.4

We need to sharpen a result of Müller [26, Theorem 7.1 and Corollary 3.25]. The restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}^1$ defines a Casimir element $C_{\mathcal{M}}\in\mathfrak{z}_{\mathfrak{m}^1}$, and we put $\mu_\chi=\chi(-C_{\mathcal{M}})+\lambda_\tau$. It is easy to see that $\mu_\chi\ge 0$. Let us normalize the identification of $\mathfrak{a}^*_{\mathcal{M}}$ with $\mathbb{R}$ in such a way that the restriction of the Killing form becomes multiplication. Given $\mathcal{O}=\{\chi,w\chi\}$, let $\mathcal{H}_{\mathcal{P}}(\mathcal{O})=\sum_{\chi'\in\mathcal{O}}\mathcal{H}_{\mathcal{P}}(\chi')$. If $w\chi=\chi$, this is just $\mathcal{H}_{\mathcal{P}}(\chi)$. We write $\mu_{\mathcal{O}}:=\mu_\chi=\mu_{w\chi}$ and $\phi_T(\mathcal{O},\lambda,\tau,j):=\sum_{\chi'\in\mathcal{O}}\phi_T(\chi',\lambda,\tau,j)$. The maps $M_T(\chi,\lambda)$ and $M_T(w\chi,\lambda)$ define an operator $M_T(\mathcal{O},\lambda)$ in $\mathcal{H}_{\mathcal{P}}(\mathcal{O})$, and

$$\phi_T(\mathcal{O},\Lambda,\tau,j)=-i\int_0^\Lambda\frac{d}{d\lambda}\log\det M_T(\mathcal{O},i\lambda,\tau,j)\,d\lambda.$$

An expression for this winding number in terms of the poles of the function $M_T(\mathcal{O}, i\lambda, \tau, j)$ is given in Theorem 6.9 of [26]. In a certain sence, it plays the role of a spectral counting function for the continuous spectrum.

Theorem 3.5 *Fix τ, $T \gg 0$ be as before and let (Γ_j) be a tower of bounded depth as defined in section 1. Then there exist constants $C > 0$, $\epsilon > 0$ such that for all $X \geq 0$ and all $j \in \mathbb{N}$ the following estimates hold:*

$$\sum_{\mathcal{O}:\rho^2+\mu_{\mathcal{O}}\leq X} |\phi_T(\mathcal{O}, \Lambda_{\mathcal{O}}, \tau, j)| \leq C[\Gamma_1 : \Gamma_j](1+X)^{n/2},$$

$$\sum_{\mathcal{O}:\rho^2+\mu_{\mathcal{O}}\leq X} \sum_{\mu\in]0,\rho]} \operatorname{rank} \operatorname{Res}_{\lambda=\mu} M_T(\mathcal{O}, \lambda, \tau, j) \leq C[\Gamma_1 : \Gamma_j](1+X)^{n/2},$$

$$\sum_{\mathcal{O}:\rho^2+\mu_{\mathcal{O}}\leq X} \dim \mathcal{H}_{\mathcal{P}}(\mathcal{O}, \tau, j) \leq C\big([\Gamma_1 : \Gamma_j](1+X)^{n/2}\big)^{1-\epsilon},$$

where $n = \dim X$, $\mathcal{O}$ runs through $\{1, w\} \setminus V^1_{\mathcal{M}}$, and $\Lambda^2_{\mathcal{O}} + \rho^2 + \mu_{\mathcal{O}} \leq X$ for each $\mathcal{O}$.

Proof Given $\phi \in \mathcal{H}_{\mathcal{P}}(\mathcal{O}, \tau, j)$ and $\delta \in \mathcal{G}(\mathbb{Q})$, we have a function $\phi_\delta(x) := \phi(\delta x)$ on $\delta^{-1}NA_{\mathcal{M}}\delta(\Gamma_j \cap \delta^{-1}P\delta) \setminus G$. Assigning to each ϕ the tuple $(\phi_\delta)_{\delta\in\mathcal{P}(\mathbb{Q})\setminus\mathcal{G}(\mathbb{Q})/\Gamma_j}$, we get an isomorphism of $\mathcal{H}_{\mathcal{P}}(\mathcal{O}, \tau, j)$ with the space $\mathcal{E}(\sigma, \mathcal{O})$ in [26] for $\Gamma = \Gamma_j$ and $\sigma = \tau$, and one can check (*cf.* [27, Ch. 2]) that our operator $M_T(\mathcal{O}, \lambda, \tau, j)$ corresponds to the operator $C_t(s)$ defined in [26, p. 485]. Müller proved the above estimate for a fixed group Γ and without summing over $\mathcal{O}$ (with a somewhat stronger growth in $\mathcal{O}$). We will now check how the constants in his estimates depend on j, using the notation of [26] freely.

Fix $T \gg 0$ and let Δ^Γ_T be the cut-off Laplacian with coefficients in the bundle $\Gamma \setminus E_\tau$ as defined in section 1 of [12]. This operator is nonnegative and was denoted by $\tilde{\Delta}_T$ in [26, p. 489]. Let $N^\Gamma_T(X)$ be the number of its eigenvalues (counted with multiplicity) not exceeding X. In [12, Corollary 3], we proved that there exists a constant $C_1 > 0$ such that for all $j \in \mathbb{N}$ and $X > 0$ we have

$$N^{\Gamma_j}_T(X) \leq C_1[\Gamma_1 : \Gamma_j](1+X)^{n/2}$$

provided (Γ_j) is a tower of bounded depth. This is just the necessary generalization of Theorem 3.23 in [26], on which all the subsequent estimates depend. However, those estimates are devaluated for our present purposes by the appearance of the factor $d = d_{\mathcal{O}} = \dim \mathcal{E}(\sigma, \mathcal{O})$ (*cf.* the proofs of Corollary 3.25 and Theorem 7.1), which also grows with j. Fortunately, all one needs in order to avoid this factor is the following remark: If $\Phi^{(1)}, \ldots, \Phi^{(d')} \in \mathcal{E}(\sigma, \mathcal{O})$ are linearly independent and $s \neq 0$, then the functions $\Lambda^T E(\Phi^{(k)}, s)$ are linearly independent, too.

Indeed, suppose some nontrivial linear combination vanishes:

$$\sum_{k=1}^{d'} c_k \Lambda^T E(\Phi^{(k)}, s) = 0.$$

Since the truncation operator Λ^T leaves functions unchanged on an open subset of G and since the Eisenstein series are analytic, the same relation is valid without Λ^T. If we now take the constant term along a cuspidal parabolic P_l evaluated at ax with varying $a \in A_l$, then from (3.2) and (2.1) of [26] we get

$$\sum_{k=1}^{d'} c_k \Big(e^{st} \Phi_l^{(k)} + e^{-st} \big(C(s) \Phi^{(k)} \big)_l \Big) = 0$$

for all l and all $t \in \mathbb{R}$. (Note that the additional subscript i is not necessary in the rank one case.) Writing this for two suitable values of t and taking an appropriate linear combination, we obtain $\sum_{k=1}^{d'} c_k \Phi_l^{(k)} = 0$ for all l, contradicting the choice of the $\Phi^{(k)}$.

To prove our first estimate, let us go through the proof of Theorem 7.1 in [26]. For each fixed $\mathcal{O}$, the integral in question is bounded by the number $2\pi \sum_{k=1}^{d_{\mathcal{O}}} \big(n_{\mathcal{O},k}(\Lambda_{\mathcal{O}}) + 1 \big)$. Here, $n_{\mathcal{O},k}(\Lambda_{\mathcal{O}})$ denotes the number of points $w \in \,]0, \Lambda_{\mathcal{O}}]$ with $e^{i\beta_k(w)} = -1$, where $\beta_1(w), \dots, \beta_{d_{\mathcal{O}}}(w)$ are real-valued real-analytic functions such that $e^{i\beta_k(w)}$ are the eigenvalues of the operator $C_t(iw)$ (which depends on $\mathcal{O}$). If Φ is an eigenfunction of $C_t(iw)$, then by [26, Lemma 3.14], $\Lambda^T E(\Phi, iw)$ is an eigenfunction of Δ_T^Γ with eigenvalue $w^2 + \rho^2 + \chi(-C_{\mathcal{M}}) + \lambda_\tau = w^2 + \rho^2 + \mu_{\mathcal{O}}$.

Functions Φ coming from different orbits $\mathcal{O}$ yield linearly independent functions $\Lambda^T E(\Phi, iw)$, because the constant terms of the latter are orthogonal. For fixed $\mathcal{O}$, it is clear that eigenfunctions with different eigenvalues are linearly independent, while for those with the same parameter $w > 0$ the linear independence follows from the preceding remark. Thus we know that the values $\mathcal{O}$ and $w \in \,]0, \Lambda_{\mathcal{O}}]$ with $e^{i\beta_k(w)} = -1$ for some k produce $\sum_{\mathcal{O}} \sum_k n_{\mathcal{O},k}(\Lambda_{\mathcal{O}})$ linearly independent eigenfunctions of Δ_T^Γ. We deduce from the spectral estimate that

$$\sum_{\mathcal{O}: \rho^2 + \mu_{\mathcal{O}} \le X} \sum_{k=1}^{d_{\mathcal{O}}} n_{\mathcal{O},k}(\Lambda_{\mathcal{O}}) \le C_2 [\Gamma_1 : \Gamma_j] (1 + X)^{n/2}.$$

(The upshot is that in equation (7.2) of [26], we need not pass from the sum to the maximum times d.) The last assertion of the theorem allows us to replace $n_{\mathcal{O},k}(\Lambda_{\mathcal{O}})$ by $n_{\mathcal{O},k}(\Lambda_{\mathcal{O}}) + 1$.

For Corollary 3.25 the argument is similar. For fixed $\mathcal{O}$, the eigenvalues of $C_t(u)$ for $u \ge 0$ are given by real-valued real-analytic functions $\lambda_k(u)$, $k = 1, \dots, d_{\mathcal{O}}$, defined on the complement of a finite set M_k depending on k, such that $(u - u_0)\lambda_k(u)$ is real-analytic and positive at each $u_0 \in M_k$ (see [26, Prop. 3.6]). Let $n_{\mathcal{O},k}$ be the number of values $u > 0$ with $\lambda_k(u) = -1$. For each $\mathcal{O}$, the sum of the ranks of the residues we have to estimate equals $\sum_{k=1}^{d} \#(M_k)$, which does not exceed $\sum_{k=1}^{d} (n_{\mathcal{O},k} + 1)$ (*cf.* [26, p. 487]). By the same argument as above we conclude that

$$\sum_{\mathcal{O}: \rho^2 + \mu_{\mathcal{O}} \le X} \sum_{k=1}^{d_{\mathcal{O}}} n_{\mathcal{O},k} \le C_2 [\Gamma_1 : \Gamma_j] (1 + X)^{n/2}.$$

We mention that, if $w\chi \ne \chi$, the meromorphic function $M(\chi, \lambda)$ has actually no poles in the right half-plane (see [1, Lemma 3.4]).

It remains to prove the last inequality. If we denote the projection of $\Gamma_j \cap \delta^{-1}P\delta$ (resp. $K \cap \delta^{-1}P\delta$) on $\delta^{-1}M\delta$ by $\Gamma_{j,\delta}$ (resp. K_δ), then $\mathcal{H}_\mathcal{P}(\chi, \tau, j)$ is isomorphic to the χ-eigenspace in $\bigoplus_{\delta \in \mathcal{P}(\mathbb{Q})\setminus\mathcal{G}(\mathbb{Q})/\Gamma_j} \big(L^2(\Gamma_{j,\delta} \setminus \delta^{-1}M^1\delta) \otimes \tau\big)^{K_\delta}$. Since $-C_\mathcal{M}$ acts in this space as generalized Laplacian and $\mu_\chi = \chi(-C_\mathcal{M}) + \lambda_\tau$, we conclude from [12, Corollary 4] (in the simple cocompact case), that there exists $C_3 > 0$ such that for all j and all $X > 0$

$$\sum_{\chi:\rho^2+\mu_\chi \le X} \dim \mathcal{H}_\mathcal{P}(\chi, \tau, j) \le C_3 \sum_{\delta \in \mathcal{P}(\mathbb{Q})\setminus\mathcal{G}(\mathbb{Q})/\Gamma_j} \operatorname{vol}(\Gamma_{j,\delta} \setminus \delta^{-1}M^1\delta/K_\delta)(1 + X)^{n'/2},$$

where $n' = \dim(\delta^{-1}M^1\delta/K_\delta)$ and the various left-invariant measures are compatible in the obvious sense. We want to show that the sum of volumes is bounded by $C_4[\Gamma_1 : \Gamma_j]^{1-\epsilon}$ for some $\epsilon > 0$. If (Γ'_j) is another tower with $\Gamma'_j \subset \Gamma_j$ and $[\Gamma_j : \Gamma'_j] \le D$ for all j, then $[\Gamma_{j,\delta} : \Gamma'_{j,\delta}] \le D$. Thus we may suppose that our tower has strictly bounded depth in the sense of [12, Section 2]. The disjoint union of the manifolds $\Gamma_j \cap \delta^{-1}P\delta \setminus \delta^{-1}P\delta/K_\delta$ with δ running through $\mathcal{P}(\mathbb{Q}) \setminus \mathcal{G}(\mathbb{Q})/\Gamma_j$ is a covering of the analogous space with $j = 1$ of degree $[\Gamma_1 : \Gamma_j]$, and therefore

$$\sum_{\delta \in \mathcal{P}(\mathbb{Q})\setminus\mathcal{G}(\mathbb{Q})/\Gamma_j} [\Gamma_1 \cap \delta^{-1}N\delta : \Gamma_j \cap \delta^{-1}N\delta] \operatorname{vol}(\Gamma_{j,\delta} \setminus \delta^{-1}M^1\delta/K_\delta) = C_5[\Gamma_1 : \Gamma_j].$$

Since (Γ_j) has strictly bounded depth, there are positive constants ϵ, ϵ_1 such that $[\Gamma_1 \cap \delta^{-1}N\delta : \Gamma_j \cap \delta^{-1}N\delta] \ge \epsilon_1[\Gamma_1 : \Gamma_j]^\epsilon$ for all j and δ (*cf.* [12, Lemma 2]), and the assertion follows. ■

3.6

Define

$$J_{\tau,\infty}(f) := \operatorname{vol}(\Gamma \setminus G) \int_{\hat{G}} [\pi : \check{\tau}] f(\chi_\pi) \, d\mu(\pi),$$

where μ denotes the Plancherel measure, and let $J_{\tau,j} = J^{T_0}_{\tau,j}$ with T_0 as in Section 1.

Lemma 3.7 *Suppose that* $\operatorname{rank}_\mathbb{Q} \mathcal{G} = 1$ *and that* (Γ_j) *is a tower of bounded depth. Then* $J_{\tau,j}$ *extends to a continuous linear functional on* $\mathcal{S}^1$. *Moreover, there are* $C > 0$, $n_i \in \mathbb{N}$ *and* $D_i \in \mathcal{D}$ *of order* ≤ 1 *such that for* $j = 1, \ldots, \infty$ *we have*

$$|J_{\tau,j}(f)| \le C \sum_{i=1}^{s} |f|_{n_i,D_i}.$$

We write $\|f\| := \sum_{i=1}^{s} |f|_{n_i,D_i}$.

Proof For $j = \infty$ this follows from Harish-Chandra's formula for the Plancherel measure.

Now we consider the terms in $J_{\tau,j}(f)$ for $j < \infty$. One easily sees that, for truncation parameters $T \gg T'$,

$$J^T_{\tau,j}(f) - J^{T'}_{\tau,j}(f)$$

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equals

$$\frac{|T-T'|}{[\Gamma_1:\Gamma_j]\dim\tau}\cdot\frac{1}{2\pi}\sum_{\chi}\dim\mathcal{H}_P(\chi,\tau,j)\int_{\mathfrak{a}_{\mathcal{M}}^*} r_{\mathcal{M}}^*(f)(\chi,i\lambda)\,d\lambda,$$

which tends to zero as $j\to\infty$ by the third inequality in Theorem 3.5. Thus it suffices to consider any T. Let $f\in\mathcal{S}^0$ and $m\in\mathbb{N}$. Multiplying f by the polynomial function $\chi\mapsto\big(1+\lambda_\tau+\chi(-C)\big)^m$, where C denotes the Casimir element, we get another function $f_1\in\mathcal{S}^0$, and there are $m'\in\mathbb{N}$ and $C_1>0$ with $|f_1|_{0,1}\le C_1|f|_{m',1}$. Let us write $N_\Gamma(\pi)=N_{\Gamma,\mathrm{cus}}(\pi)+N_{\Gamma,\mathrm{res}}(\pi)$ in the obvious way. Then

$$\begin{aligned}\sum_{\pi\in\hat{G}(\check{\tau})} N_{\Gamma_j,\mathrm{cus}}(\pi)[\pi:\check{\tau}]|f(\chi_\pi)| &\le |f_1|_{0,1}\sum_{\pi\in\hat{G}(\check{\tau})} N_{\Gamma_j,\mathrm{cus}}(\pi)[\pi:\check{\tau}]\big(1+\lambda_\tau+\chi(-C)\big)^{-m}\\ &\le C_1|f|_{m',1}\operatorname{tr}\big((1+\Delta_{\mathrm{cus}}^{\Gamma_j})^{-m}\big),\end{aligned}$$

where $\Delta_{\mathrm{cus}}^{\Gamma}$ denotes the restriction of the Bochner-Laplace operator Δ^Γ in $\Gamma\setminus E_\tau$ to the cuspidal subspace. In [12, Corollary 4], we have proved that the spectral counting function $N_{\mathrm{cus}}^{\Gamma_j}(X)$ of $\Delta_{\mathrm{cus}}^{\Gamma_j}$ satisfies

$$N_{\mathrm{cus}}^{\Gamma_j}(X)\le C_2[\Gamma_1:\Gamma_j](1+X)^{n/2}$$

with a constant C_2 independent of $j\in\mathbb{N}$ and $X>0$. Thus, if we choose $m>n/2$, then $[\Gamma_1:\Gamma_j]^{-1}\operatorname{tr}\big((1+\Delta_{\mathrm{cus}}^{\Gamma_j})^{-m}\big)$ is bounded uniformly in j.

Next we come to the residual contribution. For $\chi=w\chi$ and $\mu\in\,]0,\rho]$, $\operatorname{Res}_{\lambda=\mu}M(\chi,\lambda,\tau,j)$ is positive semidefinite and defines an inner product on the quotient of $\mathcal{H}_{\mathcal{P}}(\chi,\tau,j)$ by the corresponding nullspace. The direct sum of these quotients over all μ and χ is isometric to the residual subspace of the $\check{\tau}$-isotypical component of $L^2(\Gamma\setminus G)$ (see [1, Section 2]). Therefore,

$$\sum_{\pi\in\hat{G}(\check{\tau})} N_{\Gamma_j,\mathrm{res}}(\pi)[\pi:\check{\tau}]|f(\chi_\pi)|$$

is less than or equal to

$$(\dim\tau)^{-1}\sum_{\chi=w\chi}\sum_{\mu\in]0,\rho]}\operatorname{rank}\operatorname{Res}_{\lambda=\mu}M(\chi,\lambda,\tau,j)|r_{\mathbb{C}}M^*(f)(\chi,\mu)|.$$

Since $r_{\mathcal{M}}(\chi,\lambda)(-C)=\chi(-C_{\mathcal{M}})+\rho^2-\lambda^2$, we have

$$r_{\mathcal{M}}^*(f)(\mathcal{O}\chi,\lambda)=(1-\lambda^2+\rho^2+\mu_\chi)^{-m}r_{\mathcal{M}}^*(f_1)(\chi,\lambda),$$

and the previous expression is bounded by

$$|r_{\mathcal{M}}^*(f_1)|_{0,1}(\dim\tau)^{-1}\sum_{\chi=w\chi}(1+\mu_\chi)^{-m}\sum_{\mu\in]0,\rho]}\operatorname{rank}\operatorname{Res}_{\lambda=\mu}M(\chi,\lambda,\tau,j).$$

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Using the second estimate from Theorem 3.5 and Lemma 2.13, we see that this expression, divided by $[\Gamma_1 : \Gamma_j]$, is bounded by $C_3|f|_{m'',1}$ for some $m'' \in \mathbb{N}$ and $C_3 > 0$ independent of j. Thus we have shown that the discrete part of the spectral distribution extends continuously to $\mathcal{S}^0$.

For the continuous part we have to take $f \in \mathcal{S}^1$. Suppressing the dependence of the winding number on T, τ and j for the moment, we can write, for $\Lambda \geq 0$,

$$\int_{-\Lambda}^{\Lambda} \frac{d}{d\lambda}\phi(\chi,\lambda) r^*_{\mathcal{M}}(f)(\chi, i\lambda)\, d\lambda = 2\phi(\chi,\Lambda) r^*_{\mathcal{M}}(f)(\chi, i\Lambda) - 2\int_0^{\Lambda} \phi(\chi,\lambda)\frac{d}{d\lambda} r^*_{\mathcal{M}}(f)(\chi, i\lambda)\, d\lambda.$$

Here we want to estimate the terms on the right-hand side. Expressing $r^*_{\mathcal{M}}(f)$ in terms of $r^*_{\mathcal{M}}(f_1)$ as above, we obtain with Lemma 2.13

$$|r^*_{\mathcal{M}}(f)(\chi, i\lambda)| \leq C_4 |f_1|_{0,1} (1+\lambda^2+\rho^2+\mu_\chi)^{-m},$$

$$\begin{aligned}\left|\frac{d}{d\Lambda} r^*_{\mathcal{M}}(f)(\chi, i\lambda)\right| &\leq C_4\big(|f_1|_{0,D}(1+\lambda^2+\rho^2+\mu_\chi)^{-m} + |f_1|_{0,1}\,|\lambda|(1+\lambda^2+\rho^2+\mu_\chi)^{-m-1}\big)\\ &\leq C_4\big(|f_1|_{0,D}+|f_1|_{0,1}\big)(1+\lambda^2+\rho^2+\mu_\chi)^{-m}\end{aligned}$$

for some $C_4 > 0$ and $D \in \mathcal{D}$ of order 1. The seminorms of f_1 occurring here can be expressed by a linear combination of seminorms of f, which we denote by $\|f\|$ as in the statement of the lemma. Inserting these bounds, we get

$$\sum_{\chi\in\mathcal{O}} \left|\int_{-\Lambda}^{\Lambda} \frac{d}{d\lambda}\phi(\chi,\lambda) r^*_{\mathcal{M}}(f)(\chi, i\lambda)\, d\lambda\right| \leq C_5\|f\|\left(|\phi(\mathcal{O},\Lambda)|(1+\Lambda^2+\rho^2+\mu_{\mathcal{O}})^{-m} + \int_0^{\Lambda} |\phi(\mathcal{O},\lambda)|(1+\lambda^2+\rho^2+\mu_{\mathcal{O}})^{-m}\, d\lambda\right).$$

For each natural number $\nu \geq 1$, the sum of these expressions over those $\mathcal{O}$ with $\nu - 1 \leq \rho^2 + \mu_{\mathcal{O}} \leq \nu$ is bounded by

$$C_5\|f\| \sum_{\nu-1\leq\rho^2+\mu_{\mathcal{O}}\leq\nu} \left(|\phi(\mathcal{O},\Lambda)|(\Lambda^2+\nu)^{-m} + \int_0^{\Lambda} |\phi(\mathcal{O},\lambda)|(\lambda^2+\nu)^{-m}\, d\lambda\right),$$

but since

$$\sum_{\mathcal{O}:\rho^2+\mu_{\mathcal{O}}\leq\nu} |\phi_T(\mathcal{O},\lambda,\tau,j)| \leq C[\Gamma_1:\Gamma_j](1+\lambda^2+\nu)^{n/2}$$

by Theorem 3.5, this is bounded by

$$CC_5\|f\|[\Gamma_1:\Gamma_j]2^{n/2}\left((\Lambda^2+\nu)^{n/2-m} + \int_0^{\Lambda} (\lambda^2+\nu)^{n/2-m}\, d\lambda\right),$$

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which tends to $C_6\|f\|[\Gamma_1 : \Gamma_j]\nu^{(n-1)/2-m}$ as $\Lambda \to \infty$, provided we choose $m > (n+1)/2$. Summing over ν, we obtain, as desired,

$$\sum_{\chi}\left|\int_{-\infty}^{\infty}\frac{d}{d\lambda}\phi_T(\chi,\lambda,\tau,j)r^*_{\mathcal{M}}(f)(\chi,i\lambda)\,d\lambda\right| \le C_7\|f\|[\Gamma_1 : \Gamma_j].$$

It remains to consider the last summand of the spectral distribution. We have

$$\sum_{\chi=w\chi}|\mathrm{tr}\big(M(\chi,0,\tau,j)\big)r^*_{\mathcal{M}}(f)(\chi,0)| \le \sum_{\chi=w\chi}\|M(\chi,0,\tau,j)\|_1\,|r^*_{\mathcal{M}}(f)(\chi,0)|,$$

where $\|\,.\,\|_1$ means trace norm. Since $M(\chi,0,\tau,j)^2 = 1$, we know that $\|M(\chi,0,\tau,j)\|_1 \le \dim \mathcal{H}_{\mathcal{P}}(\chi,\tau,j)$, and the third inequality of Theorem 3.5 suffices to complete the proof. ∎

Now we come to the main result of the present paper. As before, let $\mathcal{S}$ denote the space of Schwartz functions on V and $\mathcal{S}^1$ the larger space introduced in the previous section.

Theorem 3.8 *Suppose that* $\mathrm{rank}_{\mathbb{Q}}\,\mathcal{G} = 1$ *and that* (Γ_j) *is a local tower of bounded depth. For any* $f \in \mathcal{S}^1$ *we have*

$$J_{\tau,j}(f) \overset{j\to\infty}{\longrightarrow} \mathrm{vol}(\Gamma\setminus G)\int_{\hat{G}}[\pi:\check{\tau}]f(\chi_\pi)\,d\mu(\pi) = J_{\tau,\infty}(f).$$

Proof Let $f \in \mathcal{S}$ and $\epsilon > 0$ be arbitrary. By Proposition 2.8 there is some function $g \in \mathcal{B}_\tau$ with $\|f-g\| < \epsilon$, and Lemma 2.11 yields a function $\varphi \in C_c^\infty(G)$ which can be inserted in the trace formula. It follows that

$$|J_{\tau,j}(f) - J_{\tau,\infty}(f)| \le |J_{\tau,j}(f) - J_{\tau,j}(g)| + |J_{\tau,j}(g) - J_{\tau,\infty}(g)| + |J_{\tau,\infty}(g) - J_{\tau,\infty}(f)|.$$

The first and the third summands on the right-hand side are less than ϵC. The second summand converges to zero as $j \to \infty$ by Proposition 1.7. The theorem follows. ∎

3.9

The third inequality of Theorem 3.5 shows that the last term in $J_{\tau,j}(f)$ tends to zero as $j \to \infty$. The other two terms constitute the measure μ_{τ,Γ_j} on V alluded to in the introduction. Since the bounds for both of them are deduced from the spectral bounds of the cut-off Laplacian, one cannot prove along the same lines that the continuous part is of smaller order than the discrete part as $j \to \infty$.

It would be desirable to prove the analogue of our theorem for measures on $\hat{G}$ in the spirit of [13].

For any lattice $\Gamma \subset G$ define the measure

$$\mu_\Gamma^{\mathrm{dis}} := \frac{1}{\mathrm{vol}(\Gamma\setminus G)}\sum_{\pi\in\hat{G}}N_\Gamma(\pi)\delta_\pi.$$

In Section 5 we will give some evidence for the following:

Conjecture 3.10 *Let* (Γ_j) *be a tower of bounded depth. Then the sequence of measures* $\mu_\Gamma^{\mathrm{dis}}/\,\mathrm{vol}(\Gamma_j\setminus G)$ *tends vaguely to the Plancherel measure* μ.

4 The Non-Tempered Spectrum

4.1

Let $\hat{G}_{\text{temp}} \subset \hat{G}$ be the tempered dual, this is a closed subset. Consider the natural map $\hat{G} \to V$, $\pi \mapsto \chi_\pi$. Let V_{temp} be the image of $\hat{G}_{\text{temp}}$ then V_{temp} is a closed subset of V. Let $V_{\text{nt}} = V - V_{\text{temp}}$ be its complement.

4.2 Example

Consider the special case $\mathcal{G} = \mathrm{SL}_2$. Then $\mathfrak{z} = \mathbb{C}[C]$ and the map $\chi \mapsto -\chi(C)$ maps $V_{\mathbb{C}}$ isomorphically to $\mathbb{C}$ and V to $\mathbb{R}$. An inspection shows:

$$V_{\text{temp}} = \left\{ \frac{1-n^2}{4} \;\middle|\; n = 0, 1, \dots \right\} \cup \left] \frac{1}{4}, \infty \right[.$$

We now bring the K-type into the picture. Let $\hat{G}(\tau)$ be the subset of $\hat{G}$ of all $\pi \in \hat{G}$ with $[\pi, \tau] \neq 0$. Let $V_{\text{temp}}(\tau)$ be the image in V of $\hat{G}_{\text{temp}}(\tau) = \hat{G}(\tau) \cap \hat{G}_{\text{temp}}$.

4.3 Example

Again $\mathcal{G} = \mathrm{SL}_2$ and now $\tau = 1$, the trivial K-type. We get

$$V_{\text{temp}}(1) = \left[\frac{1}{4}, \infty \right[.$$

In this setting we have the

Selberg's conjecture For any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Q})$: if $\pi \in \hat{G}(1)$ occurs in $L^2(\Gamma \setminus G)$ then $\chi_\pi \in V_{\text{temp}}(1)$.

Recently Luo, Rudnick and Sarnak [23] have proven:

$$\chi_\pi \in \left[\frac{21}{100}, \infty \right[.$$

Theorem 4.4 *Suppose* $\mathrm{rank}_{\mathbb{R}}\, \mathcal{G} = 1$. *Fix a K-type τ and let U be an open and relatively compact subset of $V_{\text{nt}}(\tau)$. Let for $j \in \mathbb{N}$:*

$$N_j(U) := \sum_{\substack{\pi \in \hat{G} \\ \chi_\pi \in U}} N_{\Gamma_j}(\pi)[\pi : \tau],$$

then the quotient $\frac{N_j(U)}{[\Gamma_1 : \Gamma_j]}$ tends to zero as $j \to \infty$.

Proof Since we have real rank one here, the intertwining terms in $J_{\tau,j}$ will be supported in the tempered spectrum $V_{\text{temp}}(\tau)$. Now apply Theorem 3.8 to any compactly supported positive smooth function f on V which is zero on $V_{\text{temp}}(\tau)$ and 1 on U. ■

Corollary 4.5 *Choose $\epsilon > 0$ and let for a tower (Γ_j) of congruence subgroups in $\mathrm{SL}_2(\mathbb{Q})$ the number $N_j(\epsilon)$ be defined by*

$$N_j(\epsilon) := \sum_{\pi \in \hat{G}(1), \chi_\pi < \frac{1}{4} - \epsilon} N_{\Gamma_j}(\pi),$$

then $\frac{N_j(\epsilon)}{[\Gamma_1:\Gamma_j]}$ tends to zero as j tends to infinity.

Proof Apply the theorem to $U = \left]-T, \frac{1}{4} - \epsilon\right[$ for any $T > 0$. ■

This corollary also follows from the density theorem of [20]. Note that for towers of Hecke congruence subgroups, which are not covered by our definition of towers, a much stronger density theorem has been obtained in [22].

5 A Special Case

5.1

In this section we consider the group $\mathcal{G} = \mathrm{PSL}_2$ and its principal congruence subgroups, *i.e.*, the projections of $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$. Since $\dim M^1 = 0$, we may omit $\chi \in V^1_{\mathcal{M}}$ from the notation. The scattering matrix for the group $\Gamma(N)$ and the trivial K-type $\tau = 1$ will be denoted by $M(\lambda, N)$, where we identify $\mathfrak{a}^*$ with $\mathbb{R}$ in such a way that ρ corresponds to 1. Huxley [19] has found the following explicit formula in terms of Dirichlet L-functions:

$$\det M(\lambda, N)$$

equals

$$(-1)^{\frac{h-h_0}{2}} \left(\frac{N}{\pi}\right)^{-h\lambda} \left(\frac{\Gamma(\frac{1-\lambda}{2})}{\Gamma(\frac{1+\lambda}{2})}\right) \prod (m_1 m_2 q_1)^{-\lambda} \frac{L(1-\lambda, \bar{\chi}_1\bar{\chi}_2\omega_{m_1m_2})}{L(1+\lambda, \chi_1\chi_2\omega_{m_1m_2})}.$$

Here $h = [\Gamma(1) : \{\pm 1\}\Gamma(N)]/N$ is the number of cusps (which is equal to $\frac{N^2}{2}\prod_{p|N}(1-\frac{1}{p^2})$ for $N \geq 3$), $h_0 = N\prod_{p|N}(1+\frac{1}{p}) = -\operatorname{tr} M(0,N)$, and ω_m denotes the trivial Dirichlet character modulo m. The product is taken over all quadruples $(m_1, m_2, \chi_1, \chi_2)$ where m_1, m_2 are coprime natural numbers and the χ_i are Dirichlet characters with conductor q_i such that $m_i q_i | N$ for $1 = 1, 2$ and $\chi_1(-1) = \chi_2(-1)$. The number of such quadruples equals h. Of course, we may replace $\bar{\chi}_i$ by χ_i in the numerator.

We deduce from [30, Satz 7.1], that for each $\epsilon > 0$ there exists $C > 0$ such that for all $k \in \mathbb{N}$, all Dirichlet characters χ modulo k and all $t \in \mathbb{R}$ we have

$$\left|\frac{L'(1-it,\chi)}{L(1-it,\chi)} + \frac{L'(1+it,\chi)}{L(1+it,\chi)}\right| \leq C\left(\log k(|t|+2) + \frac{k^{-\epsilon}}{k^{-2\epsilon}+t^2}\right).$$

Here the second term in the bound is only necessary for at most one character χ modulo each k. In Huxley's formula, only characters modulo k with $k|N$ occur, and one easily deduces

Lemma 5.2 *For each $\epsilon > 0$ there is a constant C such that, for all $N \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$,*

$$[\Gamma(1) : \Gamma(N)]^{-1} |\operatorname{tr} M(i\lambda, N)^{-1} M'(i\lambda, N)| \leq CN^{-1}\big(N^\epsilon + \log(|\lambda| + 2)\big).$$

5.3

For any lattice $\Gamma \subset G$ define the measure μ_Γ^{dis} on $\hat{G}$ by

$$\mu_\Gamma^{\text{dis}} := \frac{1}{\text{vol}(\Gamma \setminus G)} \sum_{\pi \in \hat{G}} N_\Gamma(\pi)\delta_\pi.$$

We can now easily deduce the following result, which is also contained in a preprint by Sarnak from 1983 ([31], also compare [21, end of Section 3]).

Theorem 5.4 *Let N_j be an increasing sequence of natural numbers and Γ_j the image of $\Gamma(N_j)$ in $G = \mathrm{PSL}_2(\mathbb{R})$. For the tower Γ_j, the sequence of measures $\mu_{\Gamma_j}^{\text{dis}}$ converges vaguely to the Plancherel measure μ.*

Proof For the restriction of the measures to the discrete series $\hat{G}_{\text{dis}}$, this is known (see [33]). Thus we need only consider the complement, which is $\hat{G}(1)$. The preceding lemma shows that the assertion of Lemma 3.7 extends to $f \in \mathcal{S}^0$ and, moreover, that the continuous part of $J_{\tau,j}(f)$ tends to zero as $j \to \infty$. The assertion of Theorem 3.8 now reads

$$\int_{\hat{G}(1)} f(\chi_\pi)\, d\mu_{\Gamma_j}(\pi) \stackrel{j\to\infty}{\longrightarrow} \int_{\hat{G}(1)} f(\chi_\pi)\, d\mu(\pi).$$

On $\hat{G}(1)$ the map $\pi \to \chi_\pi$ is injective. Now we can approximate any element of $C_c(V)$ by Schwartz functions to get the claim. ■

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