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Weak Factorizations of the Hardy Space $H^1(\mathbb{R}^n)$ in Terms of Multilinear Riesz Transforms

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Abstract. This paper provides a constructive proof of the weak factorization of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms. As a direct application, we obtain a new proof of the characterization of $\mathrm{BMO}(\mathbb{R}^n)$ (the dual of $H^1(\mathbb{R}^n)$) via commutators of the multilinear Riesz transforms.

1 Introduction and Statement of Main Results

The real-variable Hardy space theory on n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 1$) plays an important role in harmonic analysis and has been systematically developed. There are many equivalent ways to define the Hardy space, but for the purposes of this paper we will use the atomic decomposition. Namely, the space $H^1(\mathbb{R}^n)$ is the set of functions of the form $f = \sum_{j=1}^{\infty} \lambda_j a_j$ with $\{\lambda_j\} \in \ell^1$ and a_j an atom, meaning that it is supported on a ball B, has mean value zero $\int_B a(x)dx = 0$, and has a size condition $\|a\|_{L^\infty(\mathbb{R}^n)} \leq |B|^{-1}$. One norms this space of functions by

$$\|f\|_{H^1(\mathbb{R}^n)} := \inf\Big\{\sum_{j=1}^{\infty} |\lambda_j| : \{\lambda_j\} \in \ell^1, f = \sum_{j=1}^{\infty} \lambda_j a_j, a_j \text{ an atom}\Big\}$$

with the infimum taken over all possible representations of f via its atomic decomposition.

An important result about the Hardy space is the weak factorization obtained by Coifman, Rochberg, and Weiss [2]. This factorization proves that all $H^1(\mathbb{R}^n)$ functions can be written in terms of bilinear forms associated with the Riesz transforms, with the basic building blocks being

$$\Pi_j(f,g) = fR_j g + gR_j f,$$

with R_j the j-th Riesz transform

$$R_j f(x) = \text{p.v. } c_n \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}}\, dy.$$

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This result follows as a corollary of the characterization of the function space $\mathrm{BMO}(\mathbb{R}^n)$ in terms of the boundedness of the commutators $[b, R_j](f) = bR_jf - R_j(bf)$.

The main goal of this paper is to provide a constructive proof of the weak factorization of the classical Hardy space $H^1(\mathbb{R}^n)$ in terms of multilinear Riesz transforms. As a direct corollary, we obtain a full characterization of $\mathrm{BMO}(\mathbb{R}^n)$ (the dual of $H^1(\mathbb{R}^n)$) via commutators of the multilinear Riesz transforms. Recall that $\mathrm{BMO}(\mathbb{R}^n)$ is the set of functions for which

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|}\int_Q \big|b(y) - \mathrm{Avg}_Q(b)\big|\,dy < \infty.$$

Our strategy and approach will be to modify the direct constructive proof of Uchiyama in [12] for the weak factorization of the Hardy spaces.

We now recall the definition of multilinear Calderón–Zygmund operators (see, for example, the standard statements in [5, pp. 127–129]). Let $K(y_0, y_1, \dots, y_m)$, $y_i \in \mathbb{R}^n$, $i = 0, 1, \dots, m$, be a locally integrable function defined away from the diagonal $\{y_0 = y_1 = \cdots = y_m\}$. Then K is said to be an m-linear Calderón–Zygmund kernel if there exist positive constants A and η such that

$$|K(y_0, y_1, \dots, y_m)| \le \frac{A}{\big(\sum_{k,l=0}^m |y_k - y_l|\big)^{mn}}$$

and

$$|K(y_0, y_1, \dots, y_j, \dots, y_m) - K(y_0, y_1, \dots, y_j', \dots, y_m)| \le \frac{A|y_j - y_j'|^\eta}{\big(\sum_{k,l=0}^m |y_k - y_l|\big)^{mn+\eta}} \tag{1.1}$$

for all $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2}\max_{0\le k\le m}|y_j - y_k|$.

Suppose T is an m-linear operator mapping from $\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$, where we denote by $\mathcal{S}(\mathbb{R}^n)$ the spaces of all Schwartz functions on $\mathbb{R}^n$ and by $\mathcal{S}'(\mathbb{R}^n)$ its dual space, *i.e.,* the set of all tempered distributions on $\mathbb{R}^n$. We further assume that T is associated with the m-linear Calderón–Zygmund kernel K defined as above, *i.e.,*

$$T(f_1, \dots, f_m)(x) := \int_{\mathbb{R}^{mn}} K(x, y_1, \dots, y_m) \prod_{j=1}^m f_j(y_j)\,dy_1\cdots dy_m,$$

whenever $f_1, \dots, f_m \in \mathcal{S}(\mathbb{R}^n)$ with compact support and $x \notin \cap_{j=1}^m \mathrm{supp}(f_j)$.

If

$$T\colon L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n)$$

for some $1 < p_1, \dots, p_m < \infty$ and p with $p^{-1} = \sum_{j=1}^m p_j^{-1}$, then we say T is an m-linear Calderón–Zygmund operator. According to [5, Theorem 3], T can be extended to a bounded operator from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_m < \infty$ and p with $p^{-1} = \sum_{j=1}^m p_j^{-1}$. See also the boundedness of T when some $p_j = 1$ or when all $p_j = \infty$ in [5, Theorem 3].

We also recall the j-th transpose T^{*j} of T, defined via

$$\big\langle T^{*j}(f_1, \dots, f_m), h\big\rangle = \big\langle T(f_1, \dots, f_{j-1}, h, f_{j+1}, \dots, f_m), f_j\big\rangle \tag{1.2}$$

for all $f_1, \ldots, f_m, h \in \mathcal{S}(\mathbb{R}^n)$ (see [5, pp. 127–128]). It is easy to see that the kernel K^{*j} of T^{*j} is related to the kernel K of T via

(1.3) $$K^{*j}(x, y_1, \ldots, y_{j-1}, y_j, y_{j+1}, \ldots, y_m) = K(y_j, y_1, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_m).$$

We now introduce a property of the multilinear operator T. We say that T is *mn-homogeneous* if T satisfies

$$|T(\chi_{B_1}, \ldots, \chi_{B_m})(x)| \geq \frac{C}{M^{mn}} \quad \forall x \in B_0(x_0, r)$$

for m pairwise disjoint balls $B_0 = B_0(x_0, r), \ldots, B_m = B_m(x_m, r)$ satisfying the condition that $|y_0 - y_l| \approx Mr$ for all $y_0 \in B_0$ and $y_l \in B_l$, $l = 1, 2, \ldots, m$, where $r > 0$ and $M > 10$.

Another, stronger, version of mn-homogeneity is as follows:

$$K(x_0, \ldots, x_m) \geq \frac{C}{M^{mn}} \quad \text{or} \quad K(x_0, \ldots, x_m) \leq -\frac{C}{M^{mn}}$$

for $x_i \in B_i$, $i = 0, \ldots, m$, where $B_0 := B_0(\overline{x}_0, r), \ldots, B_m := B_m(\overline{x}_m, r)$ are $m+1$ pairwise disjoint balls satisfying the condition that $|y_0 - y_l| \approx Mr$ for all $y_0 \in B_0$ and $y_l \in B_l$, $l = 1, 2, \ldots, m$, where $r > 0$ and $M > 10$. It is easy to see that this stronger version implies the version above.

In analogy with the linear case, we define the l-th partial multilinear commutators of the m-linear Calderón–Zygmund operator T as follows.

Definition 1.1 Suppose T is an m-linear Calderón–Zygmund operator as defined above. For $l = 1, 2, \ldots, m$, we set

$$[b, T]_l(f_1, \ldots, f_m)(x) := T(f_1, \ldots, bf_l, \ldots, f_m)(x) - bT(f_1, \ldots, f_m)(x).$$

This is simply measuring the commutation properties in each linear coordinate separately.

Dual to the multilinear commutator, in both language and via a formal computation, we define the multilinear "multiplication" operators Π_l:

Definition 1.2 Suppose T is an m-linear Calderón–Zygmund operator as defined above. For $l = 1, 2, \ldots, m$, associate with T the operator

(1.4) $$\Pi_l(g, h_1, \ldots, h_m)(x) := h_l(x)\, T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) - g(x)\, T(h_1, \ldots, h_m)(x),$$

where T^{*l} is the l-th partial adjoint of T defined as in (1.3).

Our main result is then the following factorization result for $H^1(\mathbb{R}^n)$ in terms of the multilinear operators Π_l. Again, this is in direct analogy with the result in the linear case obtained by Coifman, Rochberg, and Weiss in [2].

Theorem 1.3 *Suppose* $1 \leq l \leq m$, $1 < p_1, \ldots, p_m < \infty$, *and* $1 \leq p < \infty$ *with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

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and suppose that T is an m-linear Calderón–Zygmund operator that is mn-homogeneous. Then for every function $f \in H^1(\mathbb{R}^n)$, there exist sequences $\{\lambda_s^k\} \in \ell^1$ and functions $g_s^k, h_{s,1}^k, \dots, h_{s,m}^k \in L_c^\infty(\mathbb{R}^n)$, the space of bounded functions with compact support, such that

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) \tag{1.5}$$

in the sense of $H^1(\mathbb{R}^n)$. Moreover, we have that

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \inf\Big\{ \sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|g_s^k\|_{L^{p'}(\mathbb{R}^n)} \|h_{s,1}^k\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_{s,m}^k\|_{L^{p_m}(\mathbb{R}^n)} \Big\},$$

where the infimum above is taken over all possible representations of f that satisfy (1.5).

We then obtain the following characterization of $\mathrm{BMO}(\mathbb{R}^n)$ in terms of the commutators with the multilinear Riesz transforms, again in analogy with the main results in [2]. We point out that the necessity of the BMO condition was obtained in [1], while the sufficiency was obtained in [7, 10, 11]. A contribution of this work is to provide a new proof of these results.

Theorem 1.4 *Let $1 \le l \le m$. Suppose that T is an m-linear Calderón–Zygmund operator. If b is in $\mathrm{BMO}(\mathbb{R}^n)$, then the commutator $[b, T]_l(f_1, \dots, f_m)(x)$ is a bounded map from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$, with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}$$

and with the operator norm

$$\big\| [b, T]_l : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n) \big\| \le C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}.$$

Conversely, for $b \in \cup_{q>1} L^q_{\mathrm{loc}}(\mathbb{R}^n)$, if T is mn-homogeneous, and $[b, T]_l$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$, with

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

then b is in $\mathrm{BMO}(\mathbb{R}^n)$ and

$$\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \le C \big\| [b, T]_l : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n) \to L^p(\mathbb{R}^n) \big\|.$$

As a specific example of an operator T that is an m-linear Calderón–Zygmund operator and is mn-homogeneous, we now recall the multilinear Riesz transforms; see, for example, [5, p. 162].

Definition 1.5 Suppose $f_1, \dots, f_m$ are m functions on $\mathbb{R}^n$. For $j = 1, 2, \dots, m$, define

$$\vec{R}_j(f_1, \dots, f_m)(x) := \int_{\mathbb{R}^{mn}} \vec{K}_j(x, y_1, \dots, y_m) \prod_{s=1}^m f_s(y_s) \, dy_1 \cdots dy_m,$$

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where the kernel $\vec{K}_j(x, y_1, \dots, y_m)$ is given as

$$\vec{K}_j(x, y_1, \dots, y_m) := \frac{x - y_j}{|(x - y_1, \dots, x - y_m)|^{mn+1}}.$$

To be more specific,

$$\vec{R}_j = (R_j^{(1)}, \dots, R_j^{(n)}),$$

where for each $i = 1, 2, \dots, n$, $R_j^{(i)}$ is the multilinear operator with the kernel

$$K_j^{(i)}(x, y_1, \dots, y_m) := \frac{x^i - y_j^i}{|(x - y_1, \dots, x - y_m)|^{mn+1}}.$$

Here $x = (x^1, \dots, x^m)$ and $y_j = (y_j^1, \dots, y_j^m)$.

According to [5, Corollary 2], $\vec{R}_j$ is an m-linear Calderón–Zygmund operator for $j = 1, 2, \dots, m$. Moreover, we have that

$$|\vec{R}_j(\chi_{B_1}, \dots, \chi_{B_m})(x)| = \Big| \int_{B_1} \cdots \int_{B_m} \frac{x - y_j}{|(x - y_1, \dots, x - y_m)|^{mn+1}} dy_1 \cdots dy_m \Big| \geq \frac{C}{M^{mn}}$$

for $m + 1$ pairwise disjoint balls $B_0 = B_0(x_0, r), \dots, B_m = B_m(x_m, r)$ satisfying the condition $|y_0 - y_l| \approx Mr$ for all $y_0 \in B_0$ and $y_l \in B_l$, $l = 1, 2, \dots, m$, where $r > 0$, and $M > 10$. Thus, $\vec{R}_j$ is mn-homogeneous.

Remark 1.6 We remark that the necessity in Theorem 1.4 was obtained by Chaffee in [1]. His proof uses a technique applied by Janson [6], which is different than the one used here. One advantage of the approach taken in this paper is that it provides for a constructive algorithm to produce the weak factorization of $H^1(\mathbb{R}^n)$. As mentioned in [1] it would be interesting to show the equivalence between $\mathrm{BMO}(\mathbb{R}^n)$ and the commutators when $p < 1$. Both the methods used there and in this paper hinge upon duality, which will not be a viable strategy when $p < 1$. We also again point out that the sufficiency can be found in the works [7, 10, 11] under varying conditions on p.

2 Weak Factorization of the Hardy Space $H^1(\mathbb{R}^n)$

In this section we turn to proving Theorem 1.3. We collect some facts that will be useful in proving the main result.

To begin with, we recall a technical lemma about certain $H^1(\mathbb{R}^n)$ functions.

Lemma 2.1 *Suppose f is a function defined on $\mathbb{R}^n$ satisfying $\int_{\mathbb{R}^n} f(x)\,dx = 0$ and $|f(x)| \leq \chi_{B(x_0,1)}(x) + \chi_{B(y_0,1)}(x)$, where $|x_0 - y_0| := M > 10$. Then we have*

$$\|f\|_{H^1(\mathbb{R}^n)} \leq C_n \log M.$$

We can obtain this lemma using the maximal function characterization of $H^1(\mathbb{R}^n)$, as well as the atomic decomposition characterization of $H^1(\mathbb{R}^n)$. For details of the proof, we refer the reader to similar versions of this lemma in [3, Lemma 3.1], where we use the atomic decomposition as the main tool, and in [8, Lemma 4.3], where we use the maximal function characterization of $H^1(\mathbb{R}^n)$.

Suppose $1 \le l \le m$. Ideally, given an $H^1(\mathbb{R}^n)$-atom a, we would like to find functions $g \in L^{p'}(\mathbb{R}^n)$, $h_1 \in L^{p_1}(\mathbb{R}^n), \dots, h_m \in L^{p_m}(\mathbb{R}^n)$ such that $\Pi_l(g, h_1, \dots, h_m) = a$ pointwise. While this cannot be accomplished in general, the theorem below shows that it is "almost" true.

Theorem 2.2 *Suppose $1 \le l \le m$. Suppose that T is an m-linear Calderón–Zygmund operator that is mn-homogeneous. For every $H^1(\mathbb{R}^n)$-atom $a(x)$ and for all $\varepsilon > 0$ and for all $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$, with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p},$$

there exist $g, h_1, \dots, h_m \in L_c^\infty(\mathbb{R}^n)$ and a large positive number $M = M(\varepsilon)$ such that

$$\left\| a - \Pi_l(g, h_1, \dots, h_m) \right\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and $\|g\|_{L^{p'}(\mathbb{R}^n)} \|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)} \le CM^{mn}$, where C is an absolute positive constant.

Proof The proof here follows the same lines as in [12, Theorem 2]. Let $a(x)$ be an $H^1(\mathbb{R}^n)$-atom, supported in $B(x_0, r) \subset \mathbb{R}^n$, satisfying that

$$\int_{\mathbb{R}^n} a(x)dx = 0 \quad \text{and} \quad \|a\|_{L^\infty(\mathbb{R}^n)} \le r^{-n}.$$

Fix $1 \le l \le m$ and fix $\varepsilon > 0$. Choose M sufficiently large so that

$$\frac{\log M}{M^\eta} < \varepsilon,$$

where the constant η appearing in the power of M is from the regularity condition (1.1) of the multilinear Calderón–Zygmund kernel K.

Next, we denote $x_0 = (x_{0,1}, \dots, x_{0,n})$. Now select $y_l = (y_{l,1}, \dots, y_{l,n}) \in \mathbb{R}^n$ so that $y_{l,i} - x_{0,i} = \frac{Mr}{\sqrt{n}}$. Note that for this y_l, we have $|x_0 - y_l| = Mr$. Similar to the choice of y_l, we choose y_1 such that y_l and y_1 satisfy the same relationship as x_0 and y_l do, *i.e,* $y_1 = (y_{1,1}, \dots, y_{1,n}) \in \mathbb{R}^n$ with $y_{1,i} - y_{l,i} = \frac{Mr}{\sqrt{n}}$. Then we have $|y_l - y_1| = Mr$. In the same way as above, we choose $y_2, \dots, y_{l-1}, y_{l+1}, \dots, y_m$ so that we have a collection of disjoint balls so that we can apply the homogeneity of the kernel K.

We then set

$$\begin{aligned} g(x) &:= \chi_{B(y_l, r)}(x), \\ h_j(x) &:= \chi_{B(y_j, r)}(x), \quad j \ne l, \\ h_l(x) &:= \frac{a(x)}{T^{*l}(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0)}, \end{aligned}$$

where T^{*l} is the l-th transpose of T as defined in (1.2). It is essentially clear that these functions are in $L_c^\infty(\mathbb{R}^n)$.

More precisely though, we observe that, since T is mn-homogeneous, and so is T^{*l}, for the specific choice of the functions $h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m$ as above, we have that there exists a positive constant C such that

$$\left| T^{*l}(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)(x_0) \right| \ge CM^{-mn} \qquad \text{for } 1 \le l \le m. \tag{2.1}$$

From the definitions of the functions g and h_j, we obtain that $\operatorname{supp} g = B(y_l, r)$ and $\operatorname{supp} h_j = B(y_j, r)$. Moreover,

$$\|g\|_{L^{p'}(\mathbb{R}^n)} \approx r^{\frac{n}{p'}} \quad \text{and} \quad \|h_j\|_{L^{p_j}(\mathbb{R}^n)} \approx r^{\frac{n}{p_j}}$$

for $j = 1, \ldots, l-1, l+1, \ldots, m$. Also, we have $\operatorname{supp} h_l = B(x_0, r)$ and

$$\|h_l\|_{L^{p_l}(\mathbb{R}^n)} = \frac{1}{|T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|} \|a\|_{L^{p_l}(\mathbb{R}^n)} \leq CM^{mn} r^{-n} r^{\frac{n}{p_l}},$$

where the last inequality follows from (2.1). Hence, we obtain that

$$\|g\|_{L^{p'}(\mathbb{R}^n)} \|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)} \leq CM^{mn} r^{-n} r^{n(\frac{1}{p'} + \frac{1}{p_1} + \cdots + \frac{1}{p_m})} = CM^{mn}.$$

Next, we have

$$\begin{aligned}
&a(x) - \Pi_l(g, h_1, \ldots, h_m)(x) \\
&\quad = a(x) - \Big(h_l T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) - gT(h_1, \ldots, h_m)(x) \Big) \\
&\quad = a(x) \, \frac{\begin{array}{l} T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0) \\ \quad - T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) \end{array}}{T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)} \\
&\qquad + g(x) T(h_1, \ldots, h_m)(x) \\
&\quad =: W_1(x) + W_2(x).
\end{aligned}$$

By definition, it is obvious that $W_1(x)$ is supported on $B(x_0, r)$ and $W_2(x)$ is supported on $B(y_l, r)$. We first estimate W_1. For $x \in B(x_0, r)$, we have

$$\begin{aligned}
&|W_1(x)| \\
&\quad = |a(x)| \, \frac{\left| \begin{array}{l} T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0) \\ \quad - T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x) \end{array} \right|}{|T^{*l}(h_1, \ldots, h_{l-1}, g, h_{l+1}, \ldots, h_m)(x_0)|} \\
&\quad \leq C \frac{\|a\|_{L^\infty(\mathbb{R}^n)}}{M^{-mn}} \int_{\prod_{j=1}^m B(y_j, r)} \Big| K(z_l, z_1, \ldots, z_{l-1}, x_0, z_{l+1}, \ldots, z_m) \\
&\qquad\qquad - K(z_l, z_1, \ldots, z_{l-1}, x, z_{l+1}, \ldots, z_m) \Big| \, dz_1 \cdots dz_m \\
&\quad \leq CM^{mn} r^{-n} \int_{\prod_{j=1}^m B(y_j, r)} \frac{|x_0 - x|^\eta}{\Big(\sum_{i=1,\, i \neq l}^m |z_l - z_i| + |z_l - x_0| \Big)^{mn+\eta}} \, dz_1 \cdots dz_m \\
&\quad \leq CM^{mn} r^{-n} r^{mn} \frac{r^\eta}{(Mr)^{mn+\eta}} \\
&\quad \leq C \frac{1}{M^\eta r^n},
\end{aligned}$$

where in the second inequality we use the regularity condition (1.1) of the multilinear kernel K. Hence, we obtain that

$$|W_1(x)| \leq C \frac{1}{M^\eta r^n} \chi_{B(x_0, r)}(x).$$

Next we estimate $W_2(x)$. From the definition of $g(x)$ and $h_l(x)$, we have

$$\begin{aligned}
&|W_2(x)|\\
&\quad = \chi_{B(y_l,r)}(x)|T(h_1,\ldots,h_m)(x)|\\
&\quad = \chi_{B(y_l,r)}(x)\frac{1}{|T^{*l}(h_1,\ldots,h_{l-1},g,h_{l+1},\ldots,h_m)(x_0)|}\\
&\qquad \times \bigg| \int_{\prod_{j=1,j\neq l}^m B(y_j,r)\times B(x_0,r)} \Big(K(z_1,\ldots,z_{l-1},x_0,z_{l+1},\ldots,z_m)\\
&\qquad\quad - K(z_1,\ldots,z_{l-1},x,z_{l+1},\ldots,z_m)\Big)\, a(z_l)\, dz_1\cdots dz_m \bigg|\\
&\quad \le C\chi_{B(y_l,r)}(x) M^{mn} \int_{\prod_{j=1,j\neq l}^m B(y_j,r)\times B(x_0,r)} \|a\|_{L^\infty(\mathbb{R}^n)}\\
&\qquad \times \frac{|x_0-x|^\eta}{\big(\sum_{s=1}^m |x_0-z_s|\big)^{mn+\eta}}\, dz_1\cdots dz_m\\
&\quad \le C\chi_{B(y_l,r)}(x) M^{mn} r^{-n}\frac{r^\eta\cdot r^{mn}}{(Mr)^{mn+\eta}}\\
&\quad = \frac{C}{M^\eta r^n},
\end{aligned}$$

where in the second equality we use the cancellation property of the atom $a(y_l)$. Hence, we have

$$|W_2(x)| \le \frac{C}{M^\eta r^n}\chi_{B(y_l,r)}(x).$$

Combining the estimates of W_1 and W_2, we obtain that

$$\big|a(x)-\Pi_l(g,h_1,\ldots,h_m)(x)\big| \le \frac{C}{M^\eta r^n}\big(\chi_{B(x_0,r)}(x)+\chi_{B(y_l,r)}(x)\big). \tag{2.2}$$

Next we point out that

$$\int_{\mathbb{R}^n}\big[a(x)-\Pi_l(g,h_1,\ldots,h_m)(x)\big]\,dx = 0 \tag{2.3}$$

since the atom $a(x)$ has cancellation and the second integral equals 0 just by the definitions of Π_l.

Then the size estimate (2.2) and the cancellation (2.3), together with Lemma 2.1, imply that

$$\big\|a(x)-\Pi_l(g,h_1,\ldots,h_m)(x)\big\|_{H^1(\mathbb{R}^n)} \le C\frac{\log M}{M^\eta} < C\varepsilon.$$

This proves the result. ∎

To prove the main Theorem 1.3, we also need the following estimate of the multilinear operator Π_l, which is defined in Definition 1.2. The reader can compare this proposition to recent work in [9] where similar estimates are obtained.

Proposition 2.3 *Suppose T is an m-linear Calderón–Zygmund operator. Assume $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Then for any fixed $g, h_1, \dots, h_m \in L_c^\infty(\mathbb{R}^n)$, we obtain that $\Pi_l(g, h_1, \dots, h_m)$ is in $H^1(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} \le C\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)}. \tag{2.4}$$

Proof For any fixed $g, h_1, \dots, h_m \in L_c^\infty(\mathbb{R}^n)$, to show that $\Pi_l(g, h_1, \dots, h_m)$ is in $H^1(\mathbb{R}^n)$ with the required norm (2.4), we now consider the properties of $\Pi_l(g, h_1, \dots, h_m)$.

To begin with, since $g, h_1, \dots, h_m$ are in $L_c^\infty(\mathbb{R}^n)$, we have that $g \in L^{p'}(\mathbb{R}^n)$ and $h_i \in L^{p_i}(\mathbb{R}^n)$, $i = 1, \dots, m$, for any $p_1, \dots, p_m \in (1, \infty)$, $p \in [1, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. Then, from the definition of Π_l as in (1.4), the boundedness of the m-linear Calderón–Zygmund operator T and Hölder's inequality, we have that

$$\Pi_l(g, h_1, \dots, h_m)(x) \in L^1(\mathbb{R}^n).$$

Moreover, note that from the definition of Π_l as in (1.4), we have

$$\int_{\mathbb{R}^n} \Pi_l(g, h_1, \dots, h_m)(x)\, dx = 0.$$

Next, since $g, h_1, \dots, h_m$ are all in $L_c^\infty(\mathbb{R}^n)$, from the definition of Π_l as in (1.4) and the boundedness of the m-linear Calderón–Zygmund operator T, it is direct to see that $\Pi_l(g, h_1, \dots, h_m)$ is in $L^2(\mathbb{R}^n)$ with compact support. Hence, we immediately have that $\Pi_l(g, h_1, \dots, h_m)$ is a multiple of an $H^1(\mathbb{R}^n)$ atom; *i.e,* we get that $\Pi_l(g, h_1, \dots, h_m)$ is in $H^1(\mathbb{R}^n)$. Then it suffices to verify that the $H^1(\mathbb{R}^n)$ norm of $\Pi_l(g, h_1, \dots, h_m)$ satisfies (2.4).

To see this, for $b \in \mathrm{BMO}(\mathbb{R}^n)$, we now consider the inner product

$$\big\langle b, \Pi_l(g, h_1, \dots, h_m)\big\rangle := \int_{\mathbb{R}^n} b(x)\Pi_l(g, h_1, \dots, h_m)(x)\, dx. \tag{2.5}$$

We first show that $\langle b, \Pi_l(g, h_1, \dots, h_m)\rangle$ is well defined.

Without loss of generality we assume that $\Pi_l(g, h_1, \dots, h_m)$ is supported in a cube Q_Π. We also note that for $b \in \mathrm{BMO}(\mathbb{R}^n)$, b is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$. As a consequence, we get that

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} b(x)\Pi_l(g, h_1, \dots, h_m)(x)dx\Big| \\
&\quad = |Q_\Pi|\Big|\frac{1}{|Q_\Pi|}\int_{Q_\Pi}(b(x) - b_{Q_\Pi})\Pi_l(g, h_1, \dots, h_m)(x)dx\Big| \\
&\quad \le C_{Q_\Pi}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|\Pi_l(g, h_1, \dots, h_m)\|_{L^2(\mathbb{R}^n)} < \infty,
\end{aligned}$$

where C_{Q_Π} is a constant related to the cube Q_Π, the equality above follows from the cancellation condition of $\Pi_l(g, h_1, \dots, h_m)$, and the first inequality above follows from Hölder's inequality. This implies that the inner product $\langle b, \Pi_l(g, h_1, \dots, h_m)\rangle$ in (2.5) is well defined.

We now further claim that for any fixed $g, h_1, \dots, h_m \in L_c^\infty(\mathbb{R}^n)$,

$$\left\langle b, \Pi_l(g, h_1, \dots, h_m)\right\rangle = \left\langle [b, T]_l(h_1, \dots, h_m), g\right\rangle \tag{2.6}$$

To see this, we first note that since $g, h_1, \dots, h_m$ are in $L_c^\infty(\mathbb{R}^n)$ and b is in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$\left\langle b, g\, T(h_1, \dots, h_m)\right\rangle = \left\langle g, b\, T(h_1, \dots, h_m)\right\rangle.$$

Moreover, from the definition of T^{*l} as in (1.2), we also have

$$\left\langle b, h_l\, T^{*l}(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m)\right\rangle = \\ \left\langle g,\ T(h_1, \dots, h_{l-1}, b \cdot h_l, h_{l+1}, \dots, h_m)\right\rangle.$$

Combining the above two equalities and the definition of Π_l as in (1.4), we have

$$\begin{aligned}
&\left\langle b, \Pi_l(g, h_1, \dots, h_m)\right\rangle \\
&\quad = \left\langle b, h_l\, T^{*l}(h_1, \dots, h_{l-1}, g, h_{l+1}, \dots, h_m) - g\, T(h_1, \dots, h_m)\right\rangle \\
&\quad = \left\langle g,\ T(h_1, \dots, h_{l-1}, b \cdot h_l, h_{l+1}, \dots, h_m)\right\rangle - \left\langle g, b\, T(h_1, \dots, h_m)\right\rangle \\
&\quad = \left\langle [b, T]_l(h_1, \dots, h_m), g\right\rangle,
\end{aligned}$$

which implies (2.6).

Now, from equality (2.6) and the boundedness of the multilinear commutator in terms of BMO as proved in [7, Theorem 3.18], we obtain that

$$\begin{aligned}
\left|\langle b, \Pi_l(g, h_1, \dots, h_m)\rangle\right| &= \left|\langle [b, T]_l(h_1, \dots, h_m), g\rangle\right| \\
&\le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots\cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)}.
\end{aligned} \tag{2.7}$$

We then verify (2.4). To see this, we point out that from the fundamental fact as in [4, Exercise 1.4.12 (b)], we have

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} \approx \sup_{b:\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\le 1} \left|\left\langle b, \Pi_l(g, h_1, \dots, h_m)\right\rangle\right|,$$

which, together with (2.7), immediately implies that (2.4) holds.

The proof of Proposition 2.3 is completed. ∎

We can now prove the main Theorem 1.3.

Proof of Theorem 1.3 By Proposition 2.3, we have that

$$\|\Pi_l(g, h_1, \dots, h_m)\|_{H^1(\mathbb{R}^n)} \le C\|g\|_{L^{p'}(\mathbb{R}^n)}\|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)}.$$

It is immediate that for any representation of f as in (1.5), *i.e.,*

$$f = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k\, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k),$$

we have that $\|f\|_{H^1(\mathbb{R}^n)}$ is bounded by

$$C \inf\left\{\sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|h_1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_m\|_{L^{p_m}(\mathbb{R}^n)}\|g\|_{L^{p'}(\mathbb{R}^n)} : f \text{ satisfies (1.5)}\right\}.$$

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We turn to showing that the other inequality holds and that it is possible to obtain such a decomposition for any $f \in H^1(\mathbb{R}^n)$. Utilizing the atomic decomposition, for any $f \in H^1(\mathbb{R}^n)$ we can find a sequence $\{\lambda_s^1\} \in \ell^1$ and sequence of $H^1(\mathbb{R}^n)$-atoms $\{a_s^1\}$ so that

$$f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 \quad \text{and} \quad \sum_{s=1}^{\infty} |\lambda_s^1| \le C \|f\|_{H^1(\mathbb{R}^n)}.$$

We explicitly track the implied absolute constant C appearing from the atomic decomposition, since it will play a role in the convergence of the algorithm. Fix $\varepsilon > 0$ so that $\varepsilon C < 1$. We apply Theorem 2.2 to each atom a_s^1. So there exist $g_s^1, h_{s,1}^1, \dots, h_{s,m}^1 \in L_c^\infty(\mathbb{R}^n)$, satisfying

$$\left\| a_s^1 - \Pi_{j,l}(g_s^1, h_{s,1}^1, \dots, h_{s,m}^1) \right\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and $\|g_s^1\|_{L^{p'}(\mathbb{R}^n)} \|h_{s,1}^1\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_{s,m}^1\|_{L^{p_m}(\mathbb{R}^n)} \le C(\varepsilon)$ for every $s = 1, 2, \dots$, where

$$C(\varepsilon) = CM^{nm}$$

is a constant depending on ε that we can track from Theorem 2.2. Now note that we have

$$\begin{aligned} f = \sum_{s=1}^{\infty} \lambda_s^1 a_s^1 &= \sum_{s=1}^{\infty} \lambda_s^1 \, \Pi_l(g_s^1, h_{s,1}^1, \dots, h_{s,m}^1) + \sum_{s=1}^{\infty} \lambda_s^1 \big(a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \dots, h_{s,m}^1) \big) \\ &=: M_1 + E_1. \end{aligned}$$

Observe that we have

$$\|E_1\|_{H^1(\mathbb{R}^n)} \le \sum_{s=1}^{\infty} |\lambda_s^1| \|a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \dots, h_{s,m}^1)\|_{H^1(\mathbb{R}^n)} \le \varepsilon \sum_{s=1}^{\infty} |\lambda_s^1| \le \varepsilon C \|f\|_{H^1(\mathbb{R}^n)}.$$

We now iterate the construction on the function E_1. Since $E_1 \in H^1(\mathbb{R}^n)$, we can apply the atomic decomposition in $H^1(\mathbb{R}^n)$ to find a sequence $\{\lambda_s^2\} \in \ell^1$ and a sequence of $H^1(\mathbb{R}^n)$-atoms $\{a_s^2\}$ so that $E_1 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2$ and

$$\sum_{s=1}^{\infty} |\lambda_s^2| \le C \|E_1\|_{H^1(\mathbb{R}^n)} \le \varepsilon C^2 \|f\|_{H^1(\mathbb{R}^n)}.$$

Again, we will apply Theorem 2.2 to each atom a_s^2. So there exists $g_s^2, h_{s,1}^2, \dots, h_{s,m}^2 \in L_c^\infty(\mathbb{R}^n)$, satisfying

$$\left\| a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \dots, h_{s,m}^2) \right\|_{H^1(\mathbb{R}^n)} < \varepsilon$$

and $\|g_s^2\|_{L^{p'}(\mathbb{R}^n)} \|h_{s,1}^2\|_{L^{p_1}(\mathbb{R}^n)} \cdots \|h_{s,m}^2\|_{L^{p_m}(\mathbb{R}^n)} \le C(\varepsilon)$ for every $s = 1, 2, \dots$, where $C(\varepsilon) = CM^{nm}$.

We then have that

$$\begin{aligned} E_1 = \sum_{s=1}^{\infty} \lambda_s^2 a_s^2 &= \sum_{s=1}^{\infty} \lambda_s^2 \, \Pi_l(g_s^2, h_{s,1}^2, \dots, h_{s,m}^2) + \sum_{s=1}^{\infty} \lambda_s^2 \big(a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \dots, h_{s,m}^2) \big) \\ &:= M_2 + E_2. \end{aligned}$$

But, as before, observe that

$$\begin{aligned}\|E_2\|_{H^1(\mathbb{R}^n)} &\le \sum_{s=1}^{\infty}|\lambda_s^2|\big\|a_s^2 - \Pi_l(g_s^2, h_{s,1}^2, \ldots, h_{s,m}^2)\big\|_{H^1(\mathbb{R}^n)} \le \varepsilon\sum_{s=1}^{\infty}|\lambda_s^2|\\ &\le (\varepsilon C)^2\|f\|_{H^1(\mathbb{R}^n)}.\end{aligned}$$

This implies that for f we have

$$\begin{aligned}f &= \sum_{s=1}^{\infty}\lambda_s^1 a_s^1 = \sum_{s=1}^{\infty}\lambda_s^1\,\Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1) + \sum_{s=1}^{\infty}\lambda_s^1\big(a_s^1 - \Pi_l(g_s^1, h_{s,1}^1, \ldots, h_{s,m}^1)\big)\\ &= M_1 + E_1 = M_1 + M_2 + E_2\\ &= \sum_{k=1}^{2}\sum_{s=1}^{\infty}\lambda_s^k\,\Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_2.\end{aligned}$$

Repeating this construction for each $1 \le k \le K$ produces functions

$$g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k \in L_c^\infty(\mathbb{R}^n),$$
$$\|g_s^k\|_{L^{p'}(\mathbb{R}^n)}\|h_{s,1}^k\|_{L^{p_1}(\mathbb{R}^n)}\cdots\|h_{s,m}^k\|_{L^{p_m}(\mathbb{R}^n)} \le C(\varepsilon)$$

for all s, sequences $\{\lambda_s^k\} \in \ell^1$ with $\|\{\lambda_s^k\}\|_{\ell^1} \le \varepsilon^{k-1}C^k\|f\|_{H^1(\mathbb{R}^n)}$, and a function $E_K \in H^1(\mathbb{R}^n)$ with $\|E_K\|_{H^1(\mathbb{R}^n)} \le (\varepsilon C)^K\|f\|_{H^1(\mathbb{R}^n)}$ so that

$$f = \sum_{k=1}^{K}\sum_{s=1}^{\infty}\lambda_s^k\,\Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k) + E_K.$$

Letting $K \to \infty$ gives the desired decomposition of

$$f = \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}\lambda_s^k\,\Pi_l(g_s^k, h_{s,1}^k, \ldots, h_{s,m}^k).$$

We also have that

$$\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k| \le \sum_{k=1}^{\infty}\varepsilon^{-1}(\varepsilon C)^k\|f\|_{H^1(\mathbb{R}^n)} = \frac{C}{1-\varepsilon C}\|f\|_{H^1(\mathbb{R}^n)}. \qquad \blacksquare$$

Finally, we deal with the proof of Theorem 1.4.

Proof of Theorem 1.4 The upper bound in this theorem is contained in [7, Theorem 3.18]. It suffices to consider only the lower bound.

From the definition of $H^1(\mathbb{R}^n)$, given $f \in H^1(\mathbb{R}^n)$, there exists a number sequence $\{\lambda_j\}_{j=1}^\infty$ and atoms $\{a_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$, where the series converges in the $H^1(\mathbb{R}^n)$ norm and

$$\|f\|_{H^1(\mathbb{R}^n)} \approx \sum_{j=1}^{\infty}|\lambda_j|.$$

Hence, we have that $f_N := \sum_{j=1}^N \lambda_j a_j$ tends to f as $N \to +\infty$ in the $H^1(\mathbb{R}^n)$ norm, which implies that $H^1(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n)$, where recall that $L_c^\infty(\mathbb{R}^n)$ is the subspace of $L^\infty(\mathbb{R}^n)$ consisting of functions with compact support in $\mathbb{R}^n$.

Suppose that T is an m-linear Calderón–Zygmund operator, and T is mn-homogeneous. Consider now a function $b \in \bigcup_{q>1} L^q_{\mathrm{loc}}(\mathbb{R}^n)$ such that $[b, T]_l$ is bounded from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for some $1 < p_1, \dots, p_m < \infty$ and $1 \le p < \infty$, with

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.$$

Since $b \in \bigcup_{q>1} L^q_{\mathrm{loc}}(\mathbb{R}^n)$, without lost of generality, we can assume that $b \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ for some $q > 1$. We now use q' to denote the conjugate index of q, *i.e.*,

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Then for $f \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, by using the weak factorization in Theorem 1.3, we choose a weak factorization of f such that

$$f(x) = \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k)(x) \tag{2.8}$$

in the sense of $H^1(\mathbb{R}^n)$, where $\{\lambda_s^k\} \in \ell^1$ and $g_s^k, h_{s,1}^k, \dots, h_{s,m}^k \in L^\infty_c(\mathbb{R}^n)$, and that

$$\sum_{k=1}^\infty \sum_{s=1}^\infty |\lambda_s^k| \|g_s^k\|_{L^{p'}(\mathbb{R}^n)} \prod_{j=1}^m \|h_{s,j}^k\|_{L^{p_j}(\mathbb{R}^n)} \le C \|f\|_{H^1(\mathbb{R}^n)}.$$

Moreover, since T is an m-linear Calderón–Zygmund operator and $g_s^k, h_{s,1}^k, \dots, h_{s,m}^k$ are in $L^\infty_c(\mathbb{R}^n)$, from the definition of Π_l as in (1.4), we get that

$$\Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) \in L^{q'}(\mathbb{R}^n).$$

Since $f \in H^1(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, we see that f is in $L^{q'}(U)$, where we use the set U to denote the support of f. Hence,

$$\int_{\mathbb{R}^n} b(x) f(x) \, dx$$

is well defined, since $b \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ and hence in $L^q(U)$.

We now define

$$b_i(x) = b(x) \chi_{\{x \in \mathbb{R}^n : |b(x)| \le i\}}(x), \quad i = 1, 2, \dots.$$

It is clear that $b_i(x) \to b(x)$ as $i \to \infty$ in the sense of $L^q(U)$. And then we have

$$\int_{\mathbb{R}^n} b(x) f(x) \, dx = \lim_{i \to \infty} \int_{\mathbb{R}^n} b_i(x) f(x) \, dx$$

Next, for each $i = 1, 2, \dots$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^n} b_i(x) f(x) \, dx &= \int_{\mathbb{R}^n} b_i(x) \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k)(x) \, dx \\
&= \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \int_{\mathbb{R}^n} b_i(x) \, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k)(x) \, dx \\
&= \sum_{k=1}^\infty \sum_{s=1}^\infty \lambda_s^k \langle b_i, \Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k) \rangle
\end{aligned}$$

since b_i is in $L^\infty(U)$ and hence is in $\mathrm{BMO}(\mathbb{R}^n)$, (2.8) holds in $H^1(\mathbb{R}^n)$ and each $\Pi_l(g_s^k, h_{s,1}^k, \dots, h_{s,m}^k)(x)$ is in $H^1(\mathbb{R}^n)$ as shown in Proposition 2.3.

As a consequence, we obtain that

$$
\begin{aligned}
(2.9)\qquad |\langle b, f\rangle| &\le \lim_{i\to\infty}\Big|\int_{\mathbb{R}^n} b_i(x)f(x)\,dx\Big| \\
&\le \lim_{i\to\infty}\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k||\langle b_i, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\rangle| \\
&= \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\ \lim_{i\to\infty}|\langle b_i, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\rangle|,
\end{aligned}
$$

where the equality above holds, since all the terms are non-negative. Next, since $b_i(x)\to b(x)$ as $i\to\infty$ in the sense of $L^q(V)$ and $\Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)$ is in $L^{q'}(V)$ with V the support of $\Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)$, we have that

$$
\lim_{i\to\infty}\big\langle b_i, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\big\rangle = \big\langle b, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\big\rangle,
$$

which implies that

$$
\lim_{i\to\infty}\big|\big\langle b_i, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\big\rangle\big| = \big|\big\langle b, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\big\rangle\big|
$$

This, together with (2.9), yields that

$$
|\langle b, f\rangle| \le \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\,\big|\big\langle b, \Pi_l(g_s^k, h_{s,1}^k,\dots,h_{s,m}^k)\big\rangle\big|.
$$

Now, from (2.6), we obtain that

$$
|\langle b, f\rangle| \le \sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\,|\langle g_s^k, [b,T]_l(h_{s,1}^k,\dots,h_{s,m}^k)\rangle|,
$$

which is further controlled by

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\big\|[b,T]_l(h_{s,1}^k,\dots,h_{s,m}^k)\big\|_{L^p(\mathbb{R}^n)}\|g_s^k\|_{L^{p'}(\mathbb{R}^n)} \\
&\quad\le \big\|[b,T]_l\colon L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n) \\
&\qquad\longrightarrow L^p(\mathbb{R}^n)\big\|\sum_{k=1}^{\infty}\sum_{s=1}^{\infty}|\lambda_s^k|\|g_s^k\|_{L^{p'}(\mathbb{R}^n)}\prod_{j=1}^{m}\|h_{s,j}^k\|_{L^{p_j}(\mathbb{R}^n)} \\
&\quad\le C\big\|[b,T]_l\colon L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\longrightarrow L^p(\mathbb{R}^n)\big\|\|f\|_{H^1(\mathbb{R}^n)}.
\end{aligned}
$$

By the duality between $\mathrm{BMO}(\mathbb{R}^n)$ and $H^1(\mathbb{R}^n)$ and the density argument, we have that

$$
\begin{aligned}
\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} &\approx \sup_{f\in H^1(\mathbb{R}^n)\cap L_c^\infty(\mathbb{R}^n):\|f\|_{H^1(\mathbb{R}^n)}\le 1}\Big|\int_{\mathbb{R}^n} b(x)f(x)dx\Big| \\
&\le C\big\|[b,T]_l\colon L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_m}(\mathbb{R}^n)\to L^p(\mathbb{R}^n)\big\|. \qquad\blacksquare
\end{aligned}
$$

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