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BRAIDED MIXED DATUMS AND THEIR APPLICATIONS ON HOM-QUANTUM GROUPS

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Abstract. In this paper, we mainly provide a categorical view on the braided structures appearing in the Hom-quantum groups. Let $\mathcal{C}$ be a monoidal category on which F is a bimonad, G is a bicomonad, and φ is a distributive law, we discuss the necessary and sufficient conditions for $\mathcal{C}_F^G(\varphi)$, the category of mixed bimodules to be monoidal and braided. As applications, we discuss the Hom-type (co)quasitriangular structures, the Hom–Yetter–Drinfeld modules, and the Hom–Long dimodules.

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1. Introduction. In 2006, Hartwig, Larsson, and Silvestrov introduced the Hom–Lie algebras when they concerned about the q-deformations of Witt and Virasoro algebras (see [**8**]). Hom-associative algebras, the corresponding structure of associative algebras, were introduced by Makhlouf and Silvestrov in [**14**]. The associativity of a Hom-algebra is twisted by an endomorphism (here we call it the Hom-structure map). The generalized notions, Hom-bialgebras, Hom–Hopf algebras were developed in [**13, 15, 16**]. Further research on various Hom–Lie structures and Hom-type algebras by many scholars could be found in [**10, 11**]. Quasitriangular Hom-bialgebras were considered by Yau [**21**], which provided a solution of the quantum Hom–Yang–Baxter euqation, a twisted version of the quantum Yang–Baxter equation [**22, 23**].

An interesting question is to explain Hom-type algebras use the theory of monoidal categories. In 2011, in order to provide a categorical approach to Hom-type algebras, Caenepeel and Goyvaerts [**6**] introduced the notions of Hom-categories and monoidal Hom–Hopf algebras. In a Hom-category $\widetilde{\mathscr{H}}(\mathscr{M}_k)$, the associativity and unit constraints are twisted by the Hom-structure maps. A (co)monoid in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is a Hom-(co)algebra, and a bimonoid in $\widetilde{\mathscr{H}}(\mathscr{M}_k)$ is a monoidal Hom-bialgebra (see Section 2, [**6**]). Note that a monoidal Hom-bialgebra is a Hom-bialgebra if and only if the Hom-structure map α satisfies $\alpha^2 = id$. Further, there is no monoidal category such that the Hom-bialgebra is a bimonoid in it. That is the main difference between Hom-bialgebra and monoidal Hom-bialgebra.

The aim of this paper is to provide a categorical view on the braided structures appearing in the Hom-quantum groups.

Let B be a bialgebra, ${}_B\mathcal{M}$ the category of left B-modules. Obviously, the monoidal structure on ${}_B\mathcal{M}$ is determined by the bialgebra structure on B. Furthermore, if ${}_B\mathcal{M}$ is a braided category with the braiding t, then there is an R-matrix $R = t_{B,B}(1_B \otimes 1_B)$ on B such that (B, R) is quasitriangular. But in the Hom case, recall from Remark 2.7 [**6**], if (H, α) is a Hom-bialgebra (the monoidal Hom-bialgebra case can be discussed in the same way), H is not a generator in its representation category. That means, if ${}_H\mathcal{M}$ is the category of left H-Hom-modules, and if we define $f_m : H \to M$ by $f_m(h) = h \cdot m$ for any $M \in {}_H\mathcal{M}$, $m \in M$, then f_m is not H-linear. Thus, we cannot prove that $t_{H,H}(1_H \otimes 1_H)$ is a quasitriangular structure in H as the same way in the usual bialgebras.

The natural question is to ask how we describe the braided structure on ${}_H\mathcal{M}$? If ${}_H\mathcal{M}$ is braided, is there any relation between the braiding in ${}_H\mathcal{M}$ and the Hom-bialgebra structure on H? This is the motivation of the present paper.

In 2015, Zhang and Wang (see [**24**]) showed that the tensor functor of a Hom-bialgebra H is a bi(co)monad on a special monoidal category. Hence, we can use the theory of monoidal (co)monads to interpret the braided structures obtained from Hom-quantum groups.

In 2002, Moerdijk [**17**] used a comonoidal monad to define a bimonad. Although Moerdijk called his bimonad "Hopf monad", the antipode was not involved in his definition. In 2007, Bruguières and Virelizier [**4**] introduced the notion of Hopf monad with antipode in another direction, which is different from Moderijk. Because of their close connections with the monoidal structures, the theory of Bruguières and Virelizier had developed rapidly and got many fundamental achievements (see [**3, 5**]).

Note that Beck [**2**] gave the notion of mixed distributive law which was the compatible condition for monads and comonads to be an entwining structure. Hobst and Pareigis [**9**] showed that the category of entwined modules over a field k could be made into a braided monoidal category if and only if there exists a k-linear morphism $\gamma : C \otimes C \to A \otimes A$ which satisfies some axioms. Since the entwined module can be seen as a mixed bimodule over a monad and a comonad, the braided structure over the mixed structure also could be summarized. Inspired by this conclusion, we introduce the notion of the braided mixed datum, which generalizes both quasitriangular bimonads (Section 8, [**4**]) and double quantum groups (Section 5, [**9**]), and give the examples and applications in Hom-quantum groups.

Further, one is prompted to answer several questions:

- Could a mixed sturcture admit the monoidal structure and the braided structure?
- Is it possible to characterize Hom-type braidings by mixed distributive laws?
- Does the mixed bimoduless can be view as the generalization of some Hom-type modules such as Hom-(co)modules, Hom–Yetter–Drinfeld modules, Hom–Long dimodules?
- What is the necessary and sufficient condition for the category of the Hom-(co)modules becomes a braided category?

The propose of this paper is to investigate these questions. Indeed, we find equivalent conditions to describe the braidings in the category of mixed bimodules. And finally, we use the Hom-type (co)quasitriangular structures, the braided structures in Hom–Yetter–Drinfeld modules and in the Hom–Long dimodules to verify our theory.

The paper is organized as follows. In Section 2, we first review some basic definitions such as bi(co)monads,distributive laws, and Hom-type algebras. In Section 3, we discuss the monoidal structure on $\mathcal{C}_F^G(\varphi)$, the category of mixed bimodules, and give some necessary and sufficient conditions of the property that $\mathcal{C}_F^G(\varphi)$ is a monoidal category. In Section 4, we find equivalent conditions to describe the braidings in $\mathcal{C}_F^G(\varphi)$.

As applications, in section 5, we discuss when the (co)representations category of a Hom-bialgebra is a braided monoidal category, and discuss the Hom–Yetter–Drinfeld modules and Hom–Long dimodules to verify our theory.

2. Preliminaries. Let $\mathcal{C}$ be a category, $F, G: \mathcal{C} \to \mathcal{C}$ two functors. Recall from [**20**] that if there exist natural transformations $m: FF \to F$, and $\eta: id_{\mathcal{C}} \to F$, satisfying

$$m \circ mF = m \circ Fm, \quad \text{and} \quad id_F = m \circ \eta F = m \circ F\eta,$$

then we call the triple (F, m, η) a *monad* on $\mathcal{C}$. If there exist natural transformations δ: $G \to GG$, and $\varepsilon: G \to id_{\mathcal{C}}$, such that the following identities hold:

$$G\delta \circ \delta = \delta G \circ \delta, \quad \text{and} \quad id_G = G\varepsilon \circ \delta = \varepsilon G \circ \delta,$$

then we call the triple (G, δ, ε) a *comonad* on $\mathcal{C}$.

Let $\mathcal{C}$ be a category, $A \in \mathcal{C}$, and (F, m, η) a monad on $\mathcal{C}$. If there exists a morphism $\theta_A: FA \to A$, such that

$$\theta_A \circ m_A = \theta_A \circ F(\theta_A), \quad \text{and} \quad \theta_A \circ \eta_A = id_A,$$

then we call the couple (A, θ_A) an *F-module* in $\mathcal{C}$.

A morphism between F-modules $f: A \to A'$ is called F-linear in $\mathcal{C}$, if f satisfies: $\theta_{A'} \circ Ff = f \circ \theta_A$. The category of F-modules is denoted by $\mathcal{C}_F$.

Let $\mathcal{C}$ be a category, $B \in \mathcal{C}$, and (G, δ, ε) a comonad on $\mathcal{C}$. If there exists a morphism $\rho^B: B \to GB$, satisfying

$$G\rho^B \circ \rho^B = \delta_B \circ \rho^B, \quad \text{and} \quad \varepsilon_B \circ \rho^B = id_B,$$

then we call the couple (B, ρ^B) a *G-comodule*.

A morphism between G-comodules $g: B \to B'$ is called G-colinear in $\mathcal{C}$, if g satisfies $Gg \circ \rho^B = \rho^{B'} \circ g$. The category of G-comodules is denoted by $\mathcal{C}^G$.

Let $\mathcal{C}$ be a category on which (F, m, η) is a monad and (G, δ, ε) is a comonad. A natural transformation $\varphi: FG \to GF$ is called a *mixed distributive law* or an *entwining map*, if φ induces the following commutative diagrams:

$$\begin{tikzcd}
FFG \arrow[rr, "mG"] \arrow[d, "F\varphi"'] & & FG \arrow[d, "\varphi"] \\
FGF \arrow[r, "\varphi F"] & GFF \arrow[r, "Gm"] & GF,
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \arrow[r, "F\delta"] \arrow[d, "\varphi"'] & FGG \arrow[r, "\varphi G"] & GFG \arrow[d, "G\varphi"] \\
GF \arrow[rr, "\delta F"] & & GGF,
\end{tikzcd}$$

$$\begin{tikzcd}
G \arrow[r, "\eta G"] \arrow[dr, "G\eta"'] & FG \arrow[d, "\varphi"] \\
& GF,
\end{tikzcd}
\qquad
\begin{tikzcd}
FG \arrow[r, "F\varepsilon"] \arrow[d, "\varphi"'] & F \\
GF. \arrow[ur, "\varepsilon F"'] &
\end{tikzcd}$$

For simplicity, we call (F, G, φ) a *mixed structure* on $\mathcal{C}$.

EXAMPLE 2.1. Let A be an algebra, C a coalgebra over a commutative ring k. Then it is easy to check that $F = \_ \otimes A$ is a monad, $G = \_ \otimes C$ is a comonad on ${}_k\mathcal{M}$. If we

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define $\varphi : FG \to GF$ by

$$\varphi_X : \ X \otimes C \otimes A \to X \otimes A \otimes C, \quad x \otimes c \otimes a \mapsto x \otimes \phi(c \otimes a),$$

where $\phi : C \otimes A \to A \otimes C$ is a k-linear map, then (F, G, φ) is a mixed structure if and only if (A, C, ϕ) is a right–right entwining structure over k.

Let $\mathcal{C}$ be a category, (F, G, φ) a mixed structure, $M \in \mathcal{C}$, (M, θ_M) an F-module, and (M, ρ^M) a G-comodule. If the diagram

$$\begin{tikzcd} FM \arrow[r, "\theta_M"] \arrow[d, "F\rho^M"'] & M \arrow[r, "\rho^M"] & GM \\ FGM \arrow[rr, "\varphi_M"] & & GFM \arrow[u, "G\theta_M"'] \end{tikzcd}$$

is commutative, then we call the triple (M, θ_M, ρ^M) a *mixed bimodule* or an *entwined module*.

A morphism between two mixed bimodules is called a *bimodule morphism* if it is both F-linear and G-colinear. The category of mixed bimodules is denoted by $\mathcal{C}_F^G(\varphi)$.

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category, (F, m, η) a monad on $\mathcal{C}$, and F also an opmonoidal functor, which means that there exists a natural transformation F_2: $F\otimes \to F \otimes F$ (here, $F \otimes F$ denotes $\otimes \circ (F \times F)$) and a morphism F_0: $F(I) \to I$ in $\mathcal{C}$, such that for any $X, Y, Z \in \mathcal{C}$, the following equalities hold:

$$\begin{aligned} &(id_{F(X)} \otimes F_2(Y, Z)) \circ F_2(X, Y \otimes Z) \circ F(a_{X,Y,Z}) \\ &\qquad = a_{FX,FY,FZ} \circ (F_2(X, Y) \otimes id_{F(Z)}) \circ F_2(X \otimes Y, Z), \\ &r_{FX} \circ (id_{F(X)} \otimes F_0) \circ F_2(X, I) \circ F(r_X^{-1}) \\ &\qquad = id_{F(X)} = l_{FX}(F_0 \otimes id_{F(X)})F_2(I, X) \circ F(l_X^{-1}). \end{aligned}$$

Then recall from **[4]** (or "Hopf monad" in **[17]**) that F is called a *bimonad* (or an *opmonoidal monad*) on $\mathcal{C}$ if the following identities hold:

$$\begin{cases} (M1) \ \ (m_X \otimes m_Y) \circ F_2(FX, FY) \circ F(F_2(X, Y)) = F_2(X, Y) \circ m_{X\otimes Y}; \\ (M2) \ \ F_2(X, Y) \circ \eta_{X\otimes Y} = \eta_X \otimes \eta_Y; \\ (M3) \ \ F_0 \circ F(F_0) = F_0 \circ m_I; \\ (M4) \ \ F_0 \circ \eta_I = id_I. \end{cases}$$

Note that if F is a bimonad on $\mathcal{C}$, then $\mathcal{C}_F$ is a monoidal category with the monoidal structure

$$\theta_{M\otimes N} : \ F(M \otimes N) \xrightarrow{F_2(M,N)} FM \otimes FN \xrightarrow{\theta_M \otimes \theta_N} M \otimes N,$$

for any $(M, \theta_M), (N, \theta_N) \in \mathcal{C}_F$, and with monoidal unit $(I, F_0) \in \mathcal{C}_F$.

Let $(\mathcal{C}, \otimes, I, a, l, r)$ be a monoidal category, (G, δ, ε) a comonad on $\mathcal{C}$, and G also a monoidal functor, i.e., there exists a natural transformation G_2: $G \otimes G \to G\otimes$ and a morphism G_0: $I \to G(I)$ in $\mathcal{C}$, such that for any $X, Y, Z \in \mathcal{C}$, the following equations

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hold:

$$G_2(X, Y\otimes Z)\circ(id_{G(X)}\otimes G_2(Y,Z))\circ a_{GX,GY,GZ} = G(a_{X,Y,Z})\circ G_2(X\otimes Y, Z)\circ(G_2(X, Y)\otimes id_{G(Z)}),$$

$$G(r_X)\circ G_2(X,I)\circ(id_{G(X)}\otimes G_0)\circ r_{GX}^{-1} = id_{G(X)} = G(l_X)\circ G_2(I,X)\circ(G_0\otimes id_{G(X)})\circ l_{GX}^{-1}.$$

Then recall from [**4**] that G is called a *bicomonad* (or a *monoidal comonad*) on $\mathcal{C}$ if the following identities hold:

$$\begin{cases} (C1)\ \ G(G_2(X, Y))\circ G_2(GX, GY)\circ(\delta_X\otimes\delta_Y) = \delta_{X\otimes Y}\circ G_2(X, Y);\\ (C2)\ \ \varepsilon_{X\otimes Y}\circ G_2(X, Y) = \varepsilon_X\otimes\varepsilon_Y;\\ (C3)\ \ G(G_0)\circ G_0 = \delta_I\circ G_0;\\ (C4)\ \ \varepsilon_I\circ G_0 = id_I.\end{cases}$$

Note that if G is a bicomonad on $\mathcal{C}$, then $\mathcal{C}^G$ is a monoidal category with the monoidal structure

$$\rho^{M\otimes N}:\ M\otimes N\xrightarrow{\rho^M\otimes\rho^N} GM\otimes GN\xrightarrow{G_2(M,N)} G(M\otimes N),$$

for any $(M,\rho^M), (N,\rho^N)\in\mathcal{C}^G$, and with monoidal unit $(I, G_0)\in\mathcal{C}^G$.

3. The monoidal structure in $\mathcal{C}_F^G(\varphi)$. Throughout this section, assume that $(\mathcal{C},\otimes, I, a, l, r)$ is a monoidal category on which (F, m, η) is a bimonad and (G,δ,ε) is a bicomonad such that (F, G,φ) is a mixed structure.

Notice that for any $X\in\mathcal{C}$, if we define

$$\theta_{FGX}:\ FFGX\xrightarrow{m_{GX}} FGX$$

and

$$\rho^{FGX}:\ FGX\xrightarrow{F(\delta_X)} FGGX\xrightarrow{\varphi_{GX}} GFGX\ ,$$

then it is easy to check that $(FGX,\theta_{FGX},\rho^{FGX})\in\mathcal{C}_F^G(\varphi)$.

LEMMA 3.1. *Let (M,θ_M,ρ^M) and (N,θ_N,ρ^N) be objects in $\mathcal{C}_F^G(\varphi)$. If the F-action $\theta_{M\otimes N}$ and G-coaction $\rho^{M\otimes N}$ on $M\otimes N$ are given by*

$$\theta_{M\otimes N}:\ F(M\otimes N)\xrightarrow{F_2(M,N)} FM\otimes FN\xrightarrow{\theta_M\otimes\theta_N} M\otimes N$$

and

$$\rho^{M\otimes N}:\ M\otimes N\xrightarrow{\rho^M\otimes\rho^N} GM\otimes GN\xrightarrow{G_2(M,N)} G(M\otimes N)\ ,$$

then $(\mathcal{C}_F^G(\varphi),\otimes, I, a, l, r)$ is a monoidal category if and only if (F, G,φ) satisfies the following equations for any $X, Y\in\mathcal{C}$:

(a) $GF_2(X, Y)\circ\varphi_{X\otimes Y}\circ FG_2(X, Y) = G_2(FX, FY)\circ(\varphi_X\otimes\varphi_Y)\circ F_2(GX, GY)$;

(b) $G(F_0)\circ\varphi_I\circ F(G_0) = G_0\circ F_0$.

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Proof. $\Rightarrow$): By the assumption, we have $(FGX \otimes FGY, \theta_{FGX\otimes FGY}, \rho^{FGX\otimes FGY})$ is a mixed bimodule for any $X, Y \in \mathcal{C}$, i.e.

$$\begin{aligned} &G_2(FGX, FGY) \circ (\varphi_{GX} \otimes \varphi_{GY}) \circ (F\delta_X \otimes F\delta_Y) \\ &\quad \circ (m_{GX} \otimes m_{GY}) \circ F_2(FGM, FGY) \\ &= G(m_{GX} \otimes m_{GY}) \circ GF_2(FGX, FGY) \circ \varphi_{FGX\otimes FGY} \circ FG_2(FGX, FGY) \\ &\quad \circ F(\varphi_{GX} \otimes \varphi_{GY}) \circ F(F\delta_X \otimes F\delta_Y). \end{aligned}$$

Multiplied by $G(F\varepsilon_X \otimes F\varepsilon_Y)$ left and by $F(\eta_{GX}) \otimes F(\eta_{GY})$ right on both sides of the above identity, we immediately get the conclusion (a). Since $(I, F_0, G_0) \in \mathcal{C}_F^G(\varphi)$, one can see that (b) holds.

$\Leftarrow$): First, assume that $(M, \theta_M, \rho^M), (N, \theta_N, \rho^N) \in \mathcal{C}_F^G(\varphi)$, it is easy to show that $(M \otimes N, \theta_{M\otimes N}) \in \mathcal{C}_F$ and $(M \otimes N, \rho^{M\otimes N}) \in \mathcal{C}^G$. Then from the following commutative diagram

$$\begin{tikzcd}
F(M\otimes N) \arrow[r, "F_2(M,N)"] \arrow[d, "F(\rho^M\otimes\rho^N)"'] & FM\otimes FN \arrow[r, "\theta_M\otimes\theta_N"] \arrow[d, "F\rho^M\otimes F\rho^N"] & M\otimes N \arrow[r, "\rho^M\otimes\rho^N"] & GM\otimes GN \arrow[dd, "G_2(M,N)"] \\
F(GM\otimes GN) \arrow[r, "F_2(GM,GN)"'] \arrow[d, "F(G_2(M,N))"'] & FGM\otimes FGN \arrow[r, "\varphi_M\otimes\varphi_N"] & GFM\otimes GFN \arrow[ur, "G\theta_M\otimes G\theta_N"] \arrow[d, "G_2(FM,FN)"] & \\
FG(M\otimes N) \arrow[r, "\varphi_{M\otimes N}"] & GF(M\otimes N) \arrow[r, "G(F_2(M,N))"'] & G(FM\otimes FN) \arrow[r, "G(\theta_M\otimes\theta_N)"'] & G(M\otimes N),
\end{tikzcd}$$

we get that $(M \otimes N, \theta_{M\otimes N}, \rho^{M\otimes N}) \in \mathcal{C}_F^G(\varphi)$ is also a mixed bimodule.

Second, from the assumption (b), one can easily get $(I, F_0, G_0) \in \mathcal{C}_F^G(\varphi)$.

Third, since F is opmonoidal and G is monoidal, we immediately get that the coherence morphisms a, l, r lift to morphisms in $\mathcal{C}_F^G(\varphi)$. Then, $(\mathcal{C}_F^G(\varphi), \otimes, I, a, l, r)$ is a monoidal category. $\square$

Recall from [**19**] and [**20**], if $\mathbb{C}$ denotes any 2-category, then the following data forms the 2-category of monads, which is denoted by **Mnd**$(\mathbb{C})$:

• The 0-cell contains an object X, a 1-cell $S : X \to X$ in $\mathbb{C}$, together with the multiplication $m : SS \to S$, and the unit $\eta : 1_X \to S$, which satisfy the associative law and the unit law, respectively.

• The 1-cell in **Mnd**$(\mathbb{C})$ from (X, S, m, η) to (X', S', m', η') is a 1-cell $J : X \to X'$ in $\mathbb{C}$ together with a 2-cell $j : S'J \Rightarrow JS$ in $\mathbb{C}$, satisfying the following commutative diagrams:

$$\begin{tikzcd}
S'S'J \arrow[r, "S'j"] \arrow[d, "m'J"'] & S'JS \arrow[r, "jS"] & JSS \arrow[d, "Jm"] \\
S'J \arrow[rr, "j"] & & JS,
\end{tikzcd}
\qquad
\begin{tikzcd}
J \arrow[r, "\eta' J"] \arrow[dr, "J\eta'"'] & S'J \arrow[d, "j"] \\
& JS.
\end{tikzcd}$$

• The 2-cell in **Mnd**$(\mathbb{C})$ from (J, j) to (K, k) is a 2-cell $\varrho : J \Rightarrow K$ in $\mathbb{C}$ which satisfies the equation

$$\varrho S \circ j = k \circ S'\varrho.$$

Let $\mathbf{S} = (X, S, m, \eta)$ and $\mathbf{S}' = (X', S', m', \eta')$ be 0-cells in $\mathbf{Mnd}(\mathbb{C})$. We say a 1-cell $J : X \to X'$ *lifts to a 1-cell* $\overline{J} : X_{\mathbf{S}} \to X'_{\mathbf{S}'}$ if the following diagram commutes:

$$\begin{tikzcd} X_{\mathbf{S}} \arrow[r, "\overline{J}"] \arrow[d, "U^{\mathbf{S}}"'] & X'_{\mathbf{S}'} \arrow[d, "U^{\mathbf{S}'}"] \\ X \arrow[r, "J"] & X', \end{tikzcd}$$

where U means the underlying functor.

Suppose both 1-cells $J, K : X \to X'$ lifts to J', K', respectively. We say a 2-cell $\varrho : J \Rightarrow K$ *lifts to a 2-cell* $\overline{\varrho}$ if the equation $U^{\mathbf{S}'}\overline{\varrho} = \varrho U^{\mathbf{S}}$ holds.

Dually, we have the following 2-category $\mathbf{Cmd}(\mathbb{C})$ of comonads:

• The 0-cell contains an object Y, a 1-cell $T : Y \to Y$ in $\mathbb{C}$, together with the comultiplication $\delta : T \to TT$, and the counit $\epsilon : T \to 1_Y$, which satisfies the coassociative law and the counit law, respectively.

• The 1-cell in $\mathbf{Cmd}(\mathbb{C})$ from (Y, T, δ, ϵ) to $(Y', T', \delta', \epsilon')$ is a 1-cell $W : Y \to Y'$ in $\mathbb{C}$ together with a 2-cell $w : WT \Rightarrow T'W$ in $\mathbb{C}$, satisfying

$$\delta' W \circ w = T'w \circ wT \circ W\delta, \quad \text{and} \quad \epsilon' W \circ w = W\epsilon.$$

• The 2-cell in $\mathbf{Cmd}(\mathbb{C})$ from (W, w) to (V, v) is a 2-cell $\chi : W \Rightarrow V$ in $\mathbb{C}$ which satisfies

$$v \circ \chi T = T'\chi \circ w.$$

Let $\mathbf{T} = (Y, T, \delta, \epsilon)$ and $\mathbf{T}' = (Y', T', \delta', \epsilon')$ be 0-cells in $\mathbf{Cmd}(\mathbb{C})$. We say a 1-cell $W : Y \to Y'$ *lifts to a 1-cell* $\overline{W} : Y^{\mathbf{T}} \to Y'^{\mathbf{T}'}$ if the following diagram commutes:

$$\begin{tikzcd} Y^{\mathbf{T}} \arrow[r, "\overline{W}"] \arrow[d, "U^{\mathbf{T}}"'] & Y'^{\mathbf{T}'} \arrow[d, "U^{\mathbf{T}'}"] \\ Y \arrow[r, "W"] & Y', \end{tikzcd}$$

where U means the underlying functor.

Suppose both 1-cells $W, V : Y \to Y'$ lifts to W', V', respectively. We say a 2-cell $\chi : W \Rightarrow V$ *lifts to a 2-cell* $\overline{\chi}$ if the equation $U^{\mathbf{T}'}\overline{\chi} = \chi U^{\mathbf{T}}$ holds.

Similarly, the following data forms a 2-category $\mathbf{Dist}(\mathbb{C})$ of the distributive laws:

• The 0-cell (X, T, D, ν) consists of an object X of $\mathbb{C}$, a monad T on X, a comonad D on X, and a 2-cell $\nu : TD \Rightarrow DT$ in $\mathbb{C}$ which is a distributive law.

• The 1-cell $(J, j_t, j_d) : (X, T, D, \nu) \to (X', T', D', \nu')$ consists of a 1-cell $J : X \to X'$ in $\mathbb{C}$, together with 2-cells $j_t : T'J \Rightarrow JT$ and $j_d : JD \Rightarrow D'J$, where j_t is a monad law and j_d is a comonad law in $\mathbb{C}$, and satisfies the following diagram:

$$\begin{tikzcd} T'JD \arrow[r, "j_t D"] \arrow[d, "T'j_d"'] & JTD \arrow[r, "J\nu"] & JDT \arrow[d, "j_d T"] \\ T'D'J \arrow[r, "\nu' J"] & D'T'J \arrow[r, "D'j_t"] & D'JT. \end{tikzcd}$$

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• The 2-cell $\varpi : (J, j_t, j_d) \Rightarrow (H, h_t, h_d)$, where $\varpi : J \Rightarrow H$ is a 2-cell in $\mathbb{C}$, and satisfies

$$\begin{tikzcd} T'J \arrow[r, "T'\varpi"] \arrow[d, "j_t"'] & T'H \arrow[d, "h_t"] \\ JT \arrow[r, "\varpi T"] & HT, \end{tikzcd} \quad \begin{tikzcd} JD \arrow[r, "\varpi D"] \arrow[d, "j_d"'] & HD \arrow[d, "h_d"] \\ D'J \arrow[r, "D'\varpi"] & D'H. \end{tikzcd}$$

Use the definition of $\mathbf{Mnd}(\mathbb{C})$, $\mathbf{Cmd}(\mathbb{C})$, and $\mathbf{Dist}(\mathbb{C})$, we get the following theorem.

THEOREM 3.2. *The following statements are equivalent:*

(1) $(\mathcal{C}_F^G(\varphi), \otimes, I, a, l, r)$ *is a monoidal category.*
(2) *The equations (a) and (b) in Lemma 3.1 hold.*
(3) $G_2 : (G \otimes G, (\varphi \otimes \varphi) \circ F_2(G, G)) \Rightarrow (G\otimes, GF_2 \circ (\varphi\otimes))$ *and* $G_0 : (I, F_0) \Rightarrow (GI, GF_0 \circ \varphi_I)$ *are 2-cells in the 2-category* $\mathbf{Mnd}(\mathbb{C})$.
(4) $G_2 : G \otimes G \Rightarrow G\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *lifts to a 2-cell* $\overline{G_2} : \overline{G \otimes G} \Rightarrow \overline{G\otimes}$ *such that* $U_{F\times F} \circ \overline{G_2} = G_2 \circ U_F$ *and* $G_0 : I \Rightarrow GI : \mathfrak{I} \to \mathcal{C}$ *lifts to a 2-cell* $\overline{G_0} : \overline{I} \Rightarrow \overline{GI}$ *such that* $U_{id_{\mathfrak{I}}} \circ \overline{G_0} = G_0 \circ U_F$*, where* U *is the forgetful functor.*
(5) $F_2 : (F\otimes, (\varphi\otimes) \circ (FG_2)) \Rightarrow (F \otimes F, G_2(F, F) \circ (\varphi \otimes \varphi))$ *and* $F_0 : (FI, \varphi_I \circ (FG_0)) \Rightarrow (I, G_0)$ *are 2-cells in the 2-category* $\mathbf{Cmd}(\mathbb{C})$.
(6) $F_2 : F\otimes \Rightarrow F \otimes F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *lifts to a 2-cell* $\overline{F_2} : \overline{F\otimes} \Rightarrow \overline{F \otimes F}$ *such that* $U^{G\times G} \circ \overline{F_2} = F_2 \circ U^G$ *and* $F_0 : FI \Rightarrow I : \mathfrak{I} \to \mathcal{C}$ *lifts to a 2-cell* $\overline{F_0} : \overline{FI} \Rightarrow \overline{I}$ *such that* $U^{id_{\mathfrak{I}}} \circ \overline{F_0} = F_0 \circ U^G$.
(7) $(\otimes, F_2, G_2)$ *and* (I, F_0, G_0) *are 1-cells in* $\mathbf{Dist}(\mathbb{C})$.

Proof. From Lemma 3.1, (1) and (2) are equivalent. Further, it is a direct computation to check that the conditions (3) (resp. (5), resp. (7)) hold if and only if (2) holds. Finally, by Corollary 3.11, [**19**], (3) is equivalent to (4). Similarly, by Corollary 5.11, [**19**], (5) is equivalent to (6). □

DEFINITION 3.3. We call (F, G, φ) a *monoidal mixed datum* if (F, G, φ) is a mixed structure and the properties in Theorem 3.2 hold.

EXAMPLE 3.4. In the setting of Example 2.1, if A and C are both bialgebras over k, then (F, G, φ) is a monoidal mixed structure if and only if (A, C, ϕ) is a monoidal entwining structure (see Section 4, [**9**]).

4. The braided structure in $\mathcal{C}_F^G(\varphi)$.

4.1. Convolution product. Given a category $\mathcal{C}$ and a positive integer n, we denote $\mathcal{C}^n = \mathcal{C} \times \mathcal{C} \times \cdots \times \mathcal{C}$ the n-tuple cartesian product of $\mathcal{C}$. If F is a monad, G is a comonad on $\mathcal{C}$, then $F^{\times n}$ (the n-tuple cartesian product of F) is a monad, and $G^{\times n}$ is a comonad on $\mathcal{C}^n$, and we have $\mathcal{C}^n_{F^{\times n}} = (\mathcal{C}_F)^n$, $\mathcal{C}^{nG^{\times n}} = (\mathcal{C}^G)^n$. Furthermore, if $\varphi : FG \to GF$ is a mixed distributive law, then $\mathcal{C}^{nG^{\times n}}_{F^{\times n}}(\varphi^{\times n}) = \mathcal{C}_F^G(\varphi)^n$.

Assume that (F, m, η) is a monad, (G, δ, ε) is a comonad on $\mathcal{C}$, (F, G, φ) is a mixed structure, and $U : \mathcal{C}_F^G(\varphi) \to \mathcal{C}$ is the forgetful functor. Let $P, Q : \mathcal{C}^n \to \mathcal{D}$ be functors. Then we have the following result which generalizes Lemma 1.3 [**4**].

PROPOSITION 4.1. *There is a canonical bijection:*

$$Nat(PU^{\times n}, QU^{\times n}) \cong Nat(PG^{\times n}, QF^{\times n}).$$

Proof. Define $?^\flat : Nat(PU^{\times n}, QU^{\times n}) \to Nat(PG^{\times n}, QF^{\times n}), f \mapsto f^\flat$ by

$$f^\flat_{(X_1,\ldots,X_n)} := Q(F\varepsilon_{X_1}, \ldots, F\varepsilon_{X_n}) \circ f_{(FGX_1,\ldots,FGX_n)} \circ P(\eta_{GX_1}, \ldots, \eta_{GX_n})$$

and $?^\sharp : Nat(PG^{\times n}, QF^{\times n}) \to Nat(PU^{\times n}, QU^{\times n}), \alpha \mapsto \alpha^\sharp$ by

$$\alpha^\sharp_{(M_1,\ldots,M_n)} := Q(\theta_{M_1}, \cdots, \theta_{M_n}) \circ \alpha_{(M_1,\ldots,M_n)} \circ P(\rho^{M_1}, \ldots, \rho^{M_n})$$

for any $f \in Nat(PU^{\times n}, QU^{\times n})$, $\alpha \in Nat(PG^{\times n}, QF^{\times n})$, and $X_i \in \mathcal{C}$, $(M_i, \theta_{M_i}, \rho^{M_i}) \in \mathcal{C}_F^G(\varphi)$. It is easy to check that $?^\flat$ and $?^\sharp$ are well defined.

Then from the following diagram

$$\begin{tikzcd}
P(FGX_i) \arrow[r, "P(F\delta_{X_i})"] & P(FGGX_i) \arrow[r, "P(\varphi_{GX_i})"] & P(GGX_i) \arrow[d, "\alpha_{FGX_i}"] \\
P(GX_i) \arrow[u, "P(\eta_{X_i})"] \arrow[r, "P(\delta_{X_i})"'] \arrow[d, "\alpha_{X_i}"'] & P(GGX_i) \arrow[ur, "P(G\eta_{GX_i})"] \arrow[dr, "\alpha_{GX_i}"] & Q(FFGX_i) \arrow[d, "Q(m_{GX_i})"] \\
Q(FX_i) & & Q(FGX_i), \arrow[ll, "Q(F\varepsilon_{X_i})"]
\end{tikzcd}$$

we obtain $\alpha^{\sharp\flat}_{(X_1,\ldots,X_n)} = \alpha_{(X_1,\ldots,X_n)}$. Similarly, we also have $f^{\flat\sharp}_{(M_1,\ldots,M_n)} = f_{(M_1,\ldots,M_n)}$. Hence, $?^\sharp$ and $?^\flat$ are inverse to each other. □

Let $P, Q, R : \mathcal{C}^n \to \mathcal{D}$ be functors. For any $\alpha \in Nat(PG^{\times n}, QF^{\times n})$ and $\beta \in Nat(QG^{\times n}, RF^{\times n})$, define their *convolution product* $\beta * \alpha \in Nat(PG^{\times n}, RF^{\times n})$ by setting, for any objects $X_1, \ldots, X_n$ in $\mathcal{C}$,

$$\begin{aligned}
&(\beta * \alpha)_{(X_1,\ldots,X_n)} \\
&= R(m_{X_1}, \ldots, m_{X_n}) \circ \beta_{FX_1,\ldots,FX_n} \circ Q(\varphi_{X_1}, \ldots, \varphi_{X_n}) \circ \alpha_{GX_1,\ldots,GX_n} \circ P(\delta_{X_1}, \ldots, \delta_{X_n}).
\end{aligned}$$

We say that $\alpha \in Nat(PG^{\times n}, QF^{\times n})$ is $*$-*invertible* if there exists $\beta \in Nat(QG^{\times n}, PF^{\times n})$ such that $\beta * \alpha = P\eta \circ P\varepsilon$ and $\alpha * \beta = Q\eta \circ Q\varepsilon$. We denote β by α^{*-1}.

PROPOSITION 4.2. *The $*$-invertible elements in $Nat(PG^{\times n}, QF^{\times n})$ are in corresponding with the natural isomorphisms in $Nat(PU^{\times n}, GU^{\times n})$.*

Proof. Suppose that $f \in Nat(PU^{\times n}, GU^{\times n})$ is a natural isomorphism. Then we immediately get that $f^\flat$ has a $*$-inverse $(f^{-1})^\flat$.

Conversely, if $\alpha \in Nat(PG^{\times n}, QF^{\times n})$ is $*$-invertible, then $(\alpha^{*-1})^\sharp$ is the inverse element of $\alpha^\sharp$. □

4.2. The braidings. Throughout this section, assume that $(\mathcal{C}, \otimes, I, a, l, r)$ is a monoidal category in which (F, G, φ) is a monoidal mixed datum.

Recall that a *braiding* in $\mathcal{C}$ is a natural isomorphism $t\colon \otimes \Rightarrow \otimes^{op} : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ such that the following diagrams

$$\begin{tikzcd}
(U\otimes V)\otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "t_{U,V}\otimes id_W"'] & U\otimes(V\otimes W) \arrow[r, "t_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[d, "a_{V,W,U}"] \\
(V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[r, "id_V\otimes t_{U,W}"] & V\otimes(W\otimes U),
\end{tikzcd} \tag{B1}$$

$$\begin{tikzcd}
U\otimes(V\otimes W) \arrow[r, "a^{-1}_{U,V,W}"] \arrow[d, "id_U\otimes t_{V,W}"'] & (U\otimes V)\otimes W \arrow[r, "t_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[d, "a^{-1}_{W,U,V}"] \\
U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[r, "t_{U,W}\otimes id_V"] & (W\otimes U)\otimes V
\end{tikzcd} \tag{B2}$$

are commutative for any $U, V, W \in \mathcal{C}$.

We suppose that there is a natural transformation $\sigma\colon G\otimes G \Rightarrow F\otimes^{op} F : \mathcal{C}^{\times 2}\to\mathcal{C}$. From Proposition 4.1, for any objects M, N in $\mathcal{C}_F^G(\varphi)$, σ can induce a natural transformation

$$t_{M,N} = \sigma^{\sharp}_{M,N}:\ M\otimes N \xrightarrow{\rho^M\otimes\rho^N} GM\otimes GN \xrightarrow{\sigma_{M,N}} FN\otimes FM \xrightarrow{\theta_N\otimes\theta_M} N\otimes M\ . \tag{4.1}$$

Conversely, if there is a natural transformation $t : \otimes \Rightarrow \otimes^{op} : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$, then from Proposition 4.1, for any $X, Y \in \mathcal{C}$, t can induce a natural transformation

$$\sigma_{X,Y} = t^{\flat}_{X,Y}:\ GX\otimes GY \xrightarrow{\eta_{GX}\otimes\eta_{GY}} FGX\otimes FGY \xrightarrow{t_{FGX,FGY}} FGY\otimes FGX \xrightarrow{F\varepsilon_Y\otimes F\varepsilon_X} FY\otimes FX\ . \tag{4.2}$$

Next, we will discuss when t is a braiding in $\mathcal{C}_F^G(\varphi)$.

DEFINITION 4.3. Let (F, m, η) be a bimonad, (G, δ, ε) a bicomonad on a monoidal category $\mathcal{C}$, and (F, G, φ) a monoidal mixed datum. If there is a $*$-invertible natural transformation $\sigma \in Nat(G\otimes G, F\otimes^{op} F)$, satisfying the following identities for any $X, Y, Z \in \mathcal{C}$

$$\begin{cases}
(m_Y\otimes m_X)\circ\sigma_{FX,FY}\circ(\varphi_X\otimes\varphi_Y)\circ F_2(GX,GY) = (m_Y\otimes m_X)\circ F_2(FY,FX) \\
\quad\circ F(\sigma_{X,Y}); & (4.3)\\
G(\sigma_{X,Y})\circ G_2(GX,GY)\circ(\delta_X\otimes\delta_Y) = G_2(FY,FX)\circ(\varphi_Y\otimes\varphi_X)\circ\sigma_{GX,GY} \\
\quad\circ(\delta_X\otimes\delta_Y); & (4.4)\\
(id_{FY}\otimes id_{FZ}\otimes m_X)\circ(id_{FY}\otimes\sigma_{FX,Z})\circ a_{FY,GFX,GZ}\circ(id_{FY}\otimes\varphi_X\otimes id_{GZ}) \\
\quad\circ(\sigma_{GX,Y}\otimes id_{GZ})\circ(\delta_X\otimes id_{GY}\otimes id_{GZ}) \\
\quad = a_{FY,FZ,FX}\circ(F_2(Y,Z)\otimes id_{FX})\circ\sigma_{X,Y\otimes Z}\circ(id_{GX}\otimes G_2(Y,Z))\circ a_{GX,GY,GZ}; & (4.5)\\
(m_Z\otimes id_{FX}\otimes id_{FY})\circ(\sigma_{X,FZ}\otimes id_{FY})\circ(id_{GX}\otimes\varphi_Z\otimes id_{FY})\circ a^{-1}_{GX,FGZ,FY} \\
\quad\circ(id_{GX}\otimes\sigma_{Y,GZ})\circ(id_{GX}\otimes id_{GY}\otimes\delta_Z) \\
\quad = a^{-1}_{FZ,FX,FY}\circ(id_{FX}\otimes F_2(X,Y))\circ\sigma_{X\otimes Y,Z}\circ(G_2(X,Y)\otimes id_{GZ})\circ a^{-1}_{GX,GY,GZ}, & (4.6)
\end{cases}$$

then the quadruple (F, G, φ, σ) is called a *braided mixed datum.*

THEOREM 4.4. *Let (F, m, η) be a bimonad, (G, δ, ε) a bicomonad on a monoidal category $\mathcal{C}$, and (F, G, φ) a monoidal mixed datum. Then, $\mathcal{C}_F^G(\varphi)$ is a braided monoidal category if and only if there exists a natural transformation $\sigma : G\otimes G\to F\otimes^{op} F$ such that (F, G, φ, σ) is a braided mixed datum. Moreover, the braiding in $\mathcal{C}_F^G(\varphi)$ is $t = \sigma^{\sharp}$.*

To prove Theorem 4.4, we need the following lemmas.

LEMMA 4.5. *t is F-linear if and only if σ satisfies equation (4.3) for any X, $Y \in \mathcal{C}$.*

Proof. $\Leftarrow$): Since the following diagram

$$
\begin{tikzcd}
F(M\otimes N) \arrow[r, "F_2(M,N)"] \arrow[d, "F(\rho^M\otimes\rho^N)"'] & FM\otimes FN \arrow[rr, "\theta_M\otimes\theta_N"] \arrow[d, "F\rho^M\otimes F\rho^N"] & & M\otimes N \arrow[d, "\rho^M\otimes\rho^N"] \\
F(GM\otimes GN) \arrow[r, "F_2(GM,GN)"] \arrow[d, "F(\sigma_{M,N})"'] & FGM\otimes FGN \arrow[r, "\varphi_M\otimes\varphi_N"] & GFM\otimes GFN \arrow[r, "G\rho^M\otimes G\rho^N"'] \arrow[dl, "\sigma_{FM,FN}"'] & GM\otimes GN \arrow[d, "\sigma_{M,N}"] \\
F(FN\otimes FM) \arrow[r, "F_2(FN,FM)"] \arrow[dd, "F(\theta_N\otimes\theta_M)"'] & FFN\otimes FFM \arrow[rr, "F\theta_N\otimes F\theta_M"'] \arrow[dr, "m_N\otimes m_M"] \arrow[dd, "F\theta_N\otimes F\theta_M"] & & FN\otimes FM \arrow[dd, "\theta_N\otimes\theta_M"] \\
 & & FN\otimes FM \arrow[dr, "\theta_N\otimes\theta_M"] & \\
F(N\otimes M) \arrow[r, "F_2(N,M)"] & FN\otimes FM \arrow[rr, "\theta_N\otimes\theta_M"] & & N\otimes M
\end{tikzcd}
$$

is commutative for any $M, N \in \mathcal{C}_F^G(\varphi)$, $t_{M,N}$ is F-linear.

$\Rightarrow$): Notice that $t_{FGX,FGY}$ is F-linear for any X, $Y \in \mathcal{C}$, then it follows

$$
\begin{aligned}
&(m_{GY}\otimes m_{GX})\circ F_2(FGY, FGX)\circ F(m_{GY}\otimes m_{GX})\circ F(\sigma_{FGX,FGY}) \\
&\quad \circ F(\varphi_{GX}\otimes\varphi_{GY})\circ F(F\delta_X\otimes F\delta_Y) \\
&= (m_{GY}\otimes m_{GX})\circ\sigma_{FGX,FGY}\circ(\varphi_{GX}\otimes\varphi_{GY})\circ(F\delta_X\otimes F\delta_Y) \\
&\quad \circ(m_{GX}\otimes m_{GY})\circ F_2(FGX, FGY).
\end{aligned}
$$

On the one hand, by constructing the suitable commutative diagram, we have

$$
\begin{tikzcd}
F(FGX\otimes FGY) \arrow[r, "F(F\delta_X\otimes F\delta_Y)"] & F(FGGX\otimes FGGY) \arrow[r, "F(\varphi_{GX}\otimes\varphi_{GY})"] & F(GFGX\otimes GFGY) \arrow[d, "F(\sigma_{FGX,FGY})"] \\
F(GX\otimes GY) \arrow[u, "F(\eta_{GX}\otimes\eta_{GY})"] \arrow[r, "F(\delta_X\otimes\delta_Y)"] \arrow[d, "F(\sigma_{X,Y})"'] & F(GGX\otimes GGY) \arrow[ur, "F(G\eta_{GX}\otimes G\eta_{GY})"] \arrow[d, "F(G\varepsilon_Y\otimes G\varepsilon_X)"'] \arrow[dr, "F(\sigma_{GX,GY})"'] & F(FFGY\otimes FFGX) \arrow[d, "F(m_{GY}\otimes m_{GX})"] \\
F(FY\otimes FX) \arrow[d, "F_2(FY,FX)"'] & F(GX\otimes GY) \arrow[l, "F\sigma_{X,Y}"'] & F(FGY\otimes FGX) \arrow[ll, bend left, "F(F\varepsilon_Y\otimes F\varepsilon_X)"] \arrow[d, "F_2(FGY,FGX)"] \\
FFY\otimes FFX \arrow[r, "m_Y\otimes m_X"] & FY\otimes FX & FFGY\otimes FFGX. \arrow[l, "m\varepsilon_Y\otimes m\varepsilon_X"']
\end{tikzcd}
$$

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On the other hand, we compute

$$\begin{tikzcd}
F(FGX \otimes FGY) \arrow[r, "F_2(FGX \otimes FGY)"] & FFGX \otimes FFGY \arrow[r, "m_{GX} \otimes m_{GY}"] & FGX \otimes FGY \arrow[d, "F\delta_X \otimes F\delta_Y"] \\
F(GX \otimes GY) \arrow[u, "F(\eta_{GX} \otimes \eta_{GY})"] \arrow[r, "F_2(GX \otimes GY)"] & FGX \otimes FGY \arrow[u, "F\eta_{GX} \otimes F\eta_{GY}"'] \arrow[r, "F\delta_X \otimes F\delta_Y"] \arrow[d, "\varphi_X \otimes \varphi_Y"'] & FGGY \otimes FGGX \arrow[d, "\varphi_{GX} \otimes \varphi_{GY}"] \\
FFY \otimes FFX \arrow[d, "m_Y \otimes m_X"'] & GFX \otimes GFY \arrow[l, "\sigma_{FX,FY}"'] & GFGX \otimes GFGY \arrow[l, "GF\varepsilon_X \otimes GF\varepsilon_Y"'] \arrow[d, "\sigma_{FGX,FGY}"] \\
FY \otimes FX & FGY \otimes FGX \arrow[l, "\varepsilon_Y \otimes \varepsilon_X"'] & FFGY \otimes FFGX. \arrow[l, "m_{GY} \otimes m_{GX}"']
\end{tikzcd}$$

Comparing the two diagrams, we get the conclusion. □

LEMMA 4.6. *t is G-colinear if and only if σ satisfies equation (4.4) for any $X, Y \in \mathcal{C}$.*

Proof. The proof is similar to Lemma 4.5. □

LEMMA 4.7. *With the above notations, Diagram (B1) is commutative in $\mathcal{C}_F^G(\varphi)$ if and only if σ satisfies equation (4.5) for any $X, Y, Z \in \mathcal{C}$.*

Proof. $\Leftarrow$): Take $X = M$, $Y = N$, $Z = K$ for any mixed bimodules M, N, K. Multiplied by $\theta_K \otimes \theta_M \otimes \theta_N$ left and by $\rho^K \otimes \rho^M \otimes \rho^N$ right on both sides of equation (4.5), we immediately get that Diagram (B1) is commutative.

$\Rightarrow$): Obviously, FGX, FGY, FGZ satisfy

$$\begin{aligned} & a_{FGY,FGZ,FGX} \circ t_{FGX,FGY \otimes FGZ} \circ a_{FGX,FGY,FGZ} \\ & \quad = (id_{FGY} \otimes t_{FGX,FGZ}) \circ a_{FGY,GX,FGZ} \circ (t_{FGX,FGY} \otimes id_{FGZ}) \end{aligned}$$

for any $X, Y, Z \in \mathcal{C}$. Multiplied by $F\varepsilon_Y \otimes F\varepsilon_Z \otimes F\varepsilon_X$ left and by $\eta_{GX} \otimes \eta_{GY} \otimes \eta_{GZ}$ right on both sides of the above equation, we get equation (4.5). □

LEMMA 4.8. *With the above notations, Diagram (B2) holds if and only if σ satisfies equation (4.6) for any $X, Y, Z \in \mathcal{C}$.*

Proof. The proof is similar to Lemma 4.7. □

LEMMA 4.9. *t is a natural isomorphism if and only if σ is $*$-invertible.*

Proof. Straightforward from Proposition 4.2. □

By Lemmas 4.5–4.9, we immediately get Theorem 4.4.

EXAMPLE 4.10. If $G = id_{\mathcal{C}}$, $\varphi = id_F$, then a braided mixed datum (F, G, φ, σ) is exactly a quasitriangular bimonad defined in Section 8.2 [**4**], and σ is an R-matrix for F.

EXAMPLE 4.11. In the setting of Example 2.1, if A and C are both bialgebras over k, then (F, G, φ) is a braided mixed datum in ${}_k\mathcal{M}$ if and only if (A, C, ϕ) is a double quantum group (see Section 5, [**9**]).

DEFINITION 4.12. If $F = id_{\mathcal{C}}$, $\varphi = id_G$, then a braided mixed datum (F, G, φ, σ) on $\mathcal{C}$ is called a *coquasitriangular bicomonad* (G, σ).

5. Applications in Hom-quantum groups. In this section, we will give some applications on Hom-type algebras to verify our theories. First, let us review several definitions and notations related to Hom-bialgebras. Note that when we say a "Hom-algebra" or a "Hom-coalgebra", we mean the unital Hom-algebra and counital Hom-coalgebra.

Let k be a commutative ring. Recall from [**1**] that a *Hom-algebra* over k is a quadruple $(A, \mu, 1_A, \alpha)$, in which A is a k-module, $\alpha : A \to A$, $\mu : A \otimes A \to A$ are k-linear maps, with notation $ab = \mu(a \otimes b)$, and $1_A \in A$, satisfying the following conditions, for all $a, b, c \in A$:

$$\alpha(a)(bc) = (ab)\alpha(c), \quad \alpha(1_A) = 1_A, \quad 1_A a = a1_A = \alpha(a).$$

Let $(A, \alpha, \mu, 1_A)$ and $(A', \alpha', \mu', 1_{A'})$ be two Hom-algebras. A linear map $f : A \to A'$ is said to be a *morphism of Hom-algebras* if

$$f \circ \mu = \mu' \circ (f \otimes f), \quad f(1_A) = 1_{A'}, \text{ and } f \circ \alpha = \alpha' \circ f.$$

Recall from [**1**] that a *Hom-coalgebra* over k is a quadruple $(C, \alpha, \Delta, \epsilon)$, in which C is a k-module, $\alpha : C \to C$, $\Delta : C \to C \otimes C$ and $\epsilon : C \to k$ are linear maps, with notation $\Delta(c) = c_1 \otimes c_2$, satisfying the following conditions for all $c \in C$:

$$\epsilon \circ \alpha = \epsilon, \quad \alpha(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \alpha(c_2), \quad \epsilon(c_1)c_2 = c_1\epsilon(c_2) = \alpha(c).$$

Let $(C, \alpha, \Delta, \epsilon)$ and $(C', \alpha', \Delta', \epsilon')$ be two Hom-coalgebras. A linear map $f : C \to C'$ is said to be a *morphism of Hom-coalgebras* if

$$(f \otimes f) \circ \Delta = \Delta' \circ f, \quad \epsilon' \circ f = \epsilon, \text{ and } f \circ \alpha = \alpha' \circ f.$$

Note that in the earlier definition of Hom-(co)algebras by Makhlouf and Silvestrov (see [**13**] or [**14**]), an axiom was redundant as shown in [**1**]. The reader will easily check that the definition above is equivalent to the one in those papers.

Recall from [**14**] that a *Hom-bialgebra* H over k is a sextuple $H = (H, \alpha, \mu, 1_H, \Delta, \epsilon)$, in which $(H, \alpha, \mu, 1_H)$ is a Hom-algebra, $(H, \alpha, \Delta, \epsilon)$ is a Hom-coalgebra, and Δ, ϵ are morphisms of Hom-algebras preserving unit.

Example 5.1. Let k be a commutative ring. Suppose $(B, m, \eta, \Delta, \epsilon)$ is a k-bialgebra endowed with a bialgebra isomorphism $\alpha : B \to B$. Then, $(B, \alpha, \alpha \circ m, \eta, \Delta \circ \alpha, \epsilon)$ is a Hom-bialgebra over k. We denote this Hom-bialgebra by B^α.

Conversely, if $(H, \alpha, m, \eta, \Delta, \epsilon)$ is a Hom-bialgebra and α is invertible, then $(H, \alpha^{-1} \circ m, \eta, \Delta \circ \alpha^{-1}, \epsilon)$ is a bialgebra over k. We denote this bialgebra by H_α.

Thus, we immediately get a bijective map $B \to B^\alpha$ between the collection of all bialgebras over k endowed with an invertible endomorphism on it, and the collection of all Hom-bialgebras with invertible Hom-structure maps.

Let (H, α) be a Hom-algebra. A left (H, α)-*Hom-module* is a triple (M, α_M, θ_M), where M is a k-module, $\theta_M : H \otimes M \to M$ is a k-linear map with notation $\theta_M(h \otimes m) = h \cdot m$, and $\alpha_M : M \to M$ is also a k-linear map defined by $1_H \cdot m = \alpha_M(m)$, satisfying the following condition:

$$\alpha(h) \cdot (h' \cdot m) = (hh') \cdot \alpha_M(m), \quad \text{for all } h, h' \in H, \ m \in M.$$

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A *morphism* $f : M \to N$ *of H-Hom-modules* is a k-linear map such that $\theta_N \circ (id_H \otimes f) = f \circ \theta_M$.

Let C be a Hom-coalgebra. Recall that a *right C-comodule* is a triple (M, α_M, ρ^M), where M is a k-module, $\rho^M : M \to M \otimes C$ is a k-linear map with notation $\rho^M(m) = m_0 \otimes m_1$, and $\alpha_M : M \to M$ is also a k-linear map defined by $\epsilon(m_1)m_0 = \alpha_M(m)$, satisfying the following conditions:

$$\alpha_M(m_0) \otimes \Delta(m_1) = \rho^M(m_0) \otimes \alpha(m_1), \quad \text{for all } m \in M.$$

A morphism $f : M \to N$ *of C-Hom-comodules* is a k-linear map such that $\rho^N \circ f = (id_C \otimes f) \circ \rho^M$.

Recall that in the earlier definition of Hom-(co)modules by Makhlouf and Silvestrov, there is also a redundant axiom (see **[1]** for details).

Let (H, α) be a Hom-bialgebra over k. Recall from **[24]** that if there exists an invertible element $R \in H \otimes H$, satisfying

$$\begin{cases} (q1)\ \ (\alpha \otimes \alpha)R = R; \\ (q2)\ \ R\Delta(x) = \Delta^{op}(x)R; \\ (q3)\ \ \sum R_1^{(1)} \otimes R_2^{(1)} \otimes \alpha(R^{(2)}) = \alpha(r^{(1)}) \otimes \alpha(R^{(1)}) \otimes r^{(2)}R^{(2)}; \\ (q4)\ \ \sum \alpha(R^{(1)}) \otimes R_1^{(2)} \otimes R_2^{(2)} = r^{(1)}R^{(1)} \otimes \alpha(R^{(2)}) \otimes \alpha(r^{(2)}), \end{cases}$$

for any $x \in H$, where $R = \sum R^{(1)} \otimes R^{(2)} = \sum r^{(1)} \otimes r^{(2)}$, then R is called an *R-matrix* of H, (H, α, R) is called a *quasitriangular Hom-bialgebra*.

Under the condition of Example 5.1, the following theorem can be seen as the corollary of Proposition 1.14 **[6]** and Example 2.3 **[21]**.

THEOREM 5.2. *Suppose that* $(B, m, \eta, \Delta, \varepsilon)$ *is a k-bialgebra endowed with a bialgebra isomorphism* $\alpha : B \to B$. *Then there exists an element* $R \in B \otimes B$, *such that* (B^α, α, R) *is a quasitriangular Hom-bialgebra if and only if* $R \in B \otimes B$ *is an R-matrix of B and satisfies* $(\alpha \otimes \alpha)R = R$.

Proof. Straightforward. □

5.1. Quasitriangular Hom-bialgebras. Let k be a commutative ring, ${}_k\mathcal{M} = ({}_k\mathcal{M}, \otimes, k)$ be the category of k-modules. Now from this category, we can construct a new monoidal category $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ for any $i, j \in \mathbb{Z}$ as follows:

- The objects of $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ are pairs (U, α_U), where $U \in {}_k\mathcal{M}$ and $\alpha_U \in Aut_k(U)$.
- The morphism $f : (U, \alpha_U) \to (V, \alpha_V)$ in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ is a k-linear map from U to V such that $\alpha_V \circ f = f \circ \alpha_U$.
- The monoidal structure is given by

$$(U, \alpha_U) \otimes (V, \alpha_V) = (U \otimes V, \alpha_U \otimes \alpha_V),$$

and the unit is (k, id_k).

- The associativity constraint a is given by

$$a_{U,V,P} : (U \otimes V) \otimes W \to U \otimes (V \otimes W), \ \ (u \otimes v) \otimes w \mapsto \alpha_U^{-i-1}(u) \otimes (v \otimes \alpha_W^{j+1}(w)).$$

- For any $M \in {}_k\mathcal{M}$, $m \in M$ and $\lambda \in k$, the unit constraints l and r are given by

$$l_U(\lambda \otimes u) = \lambda\alpha_U^{-j-1}(u), \quad r_U(u \otimes \lambda) = \lambda\alpha_U^{-i-1}(u).$$

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It is a direct computation to check that $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M}) = (\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M}), \otimes, k, a, l, r)$ is a monoidal category.

PROPOSITION 5.3 [**24**, Corollary 4.2]. *If $(H, \mu, 1_H, \Delta, \epsilon, \alpha)$ is a Hom-bialgebra over k, then $F = (H \otimes \_, m, \eta, F_2, F_0)$ is a bimonad on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ with the following structures:*

- *$m : FF \to F$ is given by*

$$m_X : H \otimes (H \otimes X) \to H \otimes X, \quad h \otimes (g \otimes x) \mapsto \alpha^{-1}(h)g \otimes \alpha_X(x).$$

- *$\eta : id_{\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})} \to F$ is given by $\eta_X : X \to H \otimes X$, $x \mapsto 1_H \otimes \alpha_X^{-1}(x)$.*
- *$F_2 : F\otimes \to F \otimes F$ is given by*

$$\begin{aligned} F_2(X, Y) : H \otimes (X \otimes Y) &\to (H \otimes X) \otimes (H \otimes Y), \\ h \otimes (x \otimes y) &\mapsto (\alpha^i(h_1) \otimes x) \otimes (\alpha^j(h_2) \otimes y), \end{aligned}$$

for any $X, Y \in \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$.

- *$F_0 : F(k) \to k$ is given by $F_0 : H \otimes k \to k$, $h \otimes \lambda \mapsto \varepsilon(h)\lambda$.*

Note that ${}_H\mathcal{M} = \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})_F$ as monoidal categories, where $F = H \otimes \_$. Thus, monoidal structure in ${}_H\mathcal{M}$ is given by

$$h \cdot (u \otimes v) = \alpha^i(h_1) \cdot u \otimes \alpha^j(h_2) \cdot v, \quad \forall u \in U, \quad v \in V, \quad h \in H,$$

where (U, α_U) and (V, α_V) are all H-Hom-modules.

THEOREM 5.4. *If (H, α) is a Hom-bialgebra, then the category of H-Hom-modules ${}_H\mathcal{M}$ is a braided monoidal category if and only if F is a quasitriangular bimonad on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$.*

Proof. Directly induced by Theorem 4.4. □

PROPOSITION 5.5. *For the fixed elements $R, R' \in H \otimes H$, define $\sigma : \otimes \Rightarrow F \otimes^{op} F : \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^{\times 2} \to \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ by*

$$\sigma_{X,Y}(x \otimes y) = (\alpha^i(R^{(2)}) \otimes \alpha_Y^{i-j-1}(y)) \otimes (\alpha^j(R^{(1)}) \otimes \alpha_X^{j-i-1}(x)),$$

and define $\sigma' : \otimes^{op} \Rightarrow F \otimes F : \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^{\times 2} \to \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ by

$$\sigma'_{X,Y}(y \otimes x) = (\alpha^i(R'^{(2)}) \otimes \alpha_X^{i-j-1}(x)) \otimes (\alpha^j(R'^{(1)}) \otimes \alpha_Y^{j-i-1}(y)),$$

for any $(X, \alpha_X), (Y, \alpha_Y) \in \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, $x \in X$, $y \in Y$. Then, (F, σ) is a quasitriangular bimonad with the $*$-inverse σ' if and only if R is the R-matrix of H with the inverse R' such that (H, α, R) is a quasitriangular Hom-bialgebra. Moreover, the braiding in ${}_H\mathcal{M}$ is given by $t_{U,V}(u \otimes v) = \alpha^i(R^{(2)}) \cdot \alpha_V^{i-j-1}(v) \otimes \alpha^j(R^{(1)}) \cdot \alpha_U^{j-i-1}(u)$, for any $U, V \in {}_H\mathcal{M}$.

Proof. $\Rightarrow$: Suppose (F, σ) is a quasitriangular bimonad. First, since $\sigma_{k,k}$ is a morphism in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, we have $\sigma_{k,k} \circ (id_k \otimes id_k) = ((\alpha \otimes id_k) \otimes (\alpha \otimes id_k)) \circ \sigma_{k,k}$, which implies $(\alpha \otimes \alpha)R = R$.

Second, since σ satisfies equation (4.3), we have

$$(m_k \otimes m_k) \circ \sigma_{Fk,Fk} \circ F_2(k, k) = (m_k \otimes m_k) \circ F_2(Fk, Fk) \circ F(\sigma_{k,k}).$$

For one thing, we compute

$$
\begin{aligned}
&((m_k \otimes m_k) \circ \sigma_{Fk,Fk} \circ F_2(k,k))(x \otimes (1_k \otimes 1_k))\\
&= (m_k \otimes m_k)((\alpha^i(R^{(2)}) \otimes (\alpha^{j+i-j-1}(x_2) \otimes 1_k)) \otimes (\alpha^j(R^{(1)}) \otimes (\alpha^{i+j-i-1}(x_1) \otimes 1_k)))\\
&= (\alpha^{i-1}(R^{(2)})\alpha^{i-1}(x_2) \otimes 1_k) \otimes (\alpha^{j-1}(R^{(1)})\alpha^{j-1}(x_1) \otimes 1_k).
\end{aligned}
$$

For another thing, we have

$$
\begin{aligned}
&((m_k \otimes m_k) \circ F_2(Fk, Fk) \circ F(\sigma_{k,k}))(x \otimes (1_k \otimes 1_k))\\
&= (m_k \otimes m_k)((\alpha^i(x_1) \otimes (\alpha^i(R^{(2)}) \otimes 1_k)) \otimes (\alpha^j(x_2) \otimes (\alpha^j(R^{(1)}) \otimes 1_k)))\\
&= (\alpha^{i-1}(x_1)\alpha^i(R^{(2)}) \otimes 1_k) \otimes (\alpha^{j-1}(x_2)\alpha^j(R^{(2)}) \otimes 1_k).
\end{aligned}
$$

Comparing the above two equations, since $(\alpha \otimes \alpha)R = R$, we immediately get equation (q2).

Third, take $X = Y = Z = k$ in equations (4.5) and (4.6), it is a direct computation to prove equations (q3) and (q4).

At last, since σ' is the $*$-inverse of σ, we have $\sigma_{k,k} * \sigma'_{k,k} = \eta_k \otimes^{op} \eta_k$ and $\sigma'_{k,k} * \sigma_{k,k} = \eta_k \otimes \eta_k$, which implies R and R' are inverse to each other.

$\Rightarrow$: Straightforward. □

EXAMPLE 5.6 (the Sweedler's 4-dimensional Hom-bialgebra). Let k be a field and H_4 the Sweedler's 4-dimensional bialgebra $H_4 = k\{1_H, g, x, y | g^2 = 1_H, x^2 = 0, y = gx = -xg\}$ with the following structures:

$$
\Delta(g) = g \otimes g,\ \ \Delta(x) = x \otimes 1_H + g \otimes x,\ \ \Delta(y) = y \otimes g + 1_H \otimes y,
$$
$$
\epsilon(g) = 1,\ \ \epsilon(x) = \epsilon(y) = 0.
$$

Note that H_4 is a quasitriangular Hopf algebra with the R-matrix

$$
R_\lambda = \frac{1}{2}(1_H \otimes 1_H + 1_H \otimes g + g \otimes 1_H - g \otimes g) + \frac{\lambda}{2}(x \otimes x - x \otimes y + y \otimes x + y \otimes y),
$$

where $\lambda \in k$ (see Example 10.1.17 [**18**]).

By (Example 3.5 [**7**]), any bialgebra isomorphism $\alpha : H_4 \to H_4$ takes the form

$$
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & d & c \end{pmatrix},
$$

where $c, d \in k$ satisfying $c^2 \neq d^2$. Thus, we immediately get a Hom-bialgebra $H_4{}^\alpha = (H_4, \alpha, \alpha \circ \mu, 1_H, \Delta \circ \alpha, \epsilon)$ (usually called *Sweedler's 4-dimensional Hom-bialgebra*).

Moreover, from Theorem 5.2, through a direct computation, we obtain that $H_4{}^\alpha$ is a quasitriangular Hom-bialgebra, and the R-matrix of $H_4{}^\alpha$ is given by

$$
R = \begin{cases} \frac{1}{2}(1_H \otimes 1_H + 1_H \otimes g + g \otimes 1_H - g \otimes g) & \\ \quad +\frac{\lambda}{2}(x \otimes x - x \otimes y + y \otimes x + y \otimes y), & \text{when } \ c^2 = 1,\ d = 0,\ \lambda \neq 0, \\ & \quad \text{or } c = 0,\ d^2 = 1,\ \lambda \neq 0; \\ \frac{1}{2}(1_H \otimes 1_H + 1_H \otimes g + g \otimes 1_H - g \otimes g), & \text{otherwise,} \end{cases}
$$

where $\lambda \in k$.

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5.2. Coquasitriangular Hom-bialgebras. Dual to the above property, we have the following results.

Assume that k is a commutative ring. For any $i', j' \in \mathbb{Z}$, a monoidal category $\overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$ is defined as follows:

- The objects, morphisms, and tensor products are the same as in $\widetilde{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$.
- The associativity constraint a is given by

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W), \quad (u \otimes v) \otimes w \mapsto \alpha_U^{i'+1}(u) \otimes (v \otimes \alpha_W^{-j'-1}(w)).$$

- For any $U \in {}_k\mathcal{M}$, $u \in U$, and $\lambda \in k$, the unit constraints l and r are given by

$$l_U(\lambda \otimes u) = \lambda\alpha_U^{j'+1}(u), \quad r_U(u \otimes \lambda) = \lambda\alpha_U^{i'+1}(u).$$

Note that if $i' = j' = 0$, then $\overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$ is the monoidal Hom-category defined in [**6**].

PROPOSITION 5.7 [**24**, Theorem 4.3]. *Let i', j' be two integers. If $(H, \mu, 1_H, \Delta, \epsilon, \alpha)$ is a Hom-bialgebra over k, then $G = ({}_- \otimes H, \delta, \epsilon, G_2, G_0)$ is a bicomonad on $\overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$ with the following structures:*

- *$\delta : G \to GG$ is given by*

$$\delta_X : X \otimes H \to (X \otimes H) \otimes H, \quad x \otimes h \mapsto (\alpha_X(x) \otimes h_1) \otimes \alpha^{-1}(h_2).$$

- *$\epsilon : G \to id_{\overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})}$ is given by $\epsilon_X : X \otimes H \to X$, $x \otimes h \mapsto \epsilon(h)\alpha_X^{-1}(x)$.*
- *$G_2 : G \otimes G \to G\otimes$ is given by*

$$G_2(X, Y) : (X \otimes H) \otimes (Y \otimes H) \to (X \otimes Y) \otimes H,$$
$$(x \otimes a) \otimes (y \otimes b) \mapsto (x \otimes y) \otimes \alpha^{i'}(a)\alpha^{j'}(b),$$

for any $X, Y \in \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$.

- *$G_0 : k \to G(k)$ is given by $G_0 : k \to k \otimes H$, $\lambda \mapsto \lambda \otimes 1_H$.*

Notice that $\mathcal{M}^H = \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})^G$ as monoidal categories, where $G = {}_- \otimes H$. Thus, monoidal structure in $\mathcal{M}^H$ is given by

$$(u \otimes v)_{(0)} \otimes (u \otimes v)_{(1)} = u_{(0)} \otimes v_{(0)} \otimes \alpha^{i'}(u_{(1)})\alpha^{j'}(v_{(1)}), \quad \forall u \in U, \quad v \in V,$$

where (U, α_U) and (V, α_V) are all H-Hom-comodules.

THEOREM 5.8. *The category of Hom-comodules of a Hom-bialgebra (H, α) is a braided monoidal category if and only if ${}_- \otimes H$ is a coquasitriangular bicomonad on $\overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$.*

Recall from Definition 6.5 [**24**] that a Hom-bialgebra (H, α) is called *coquasitriangular* if there exists a convolution invertible bilinear form $\xi : H \otimes H \to k$, such that the following conditions hold:

$$\begin{cases} (cq1) & \xi(\alpha(a), \alpha(b)) = \xi(a, b); \\ (cq2) & \xi(a_1, b_1)a_2b_2 = b_1a_1\xi(a_2, b_2); \\ (cq3) & \xi(\alpha(a), bc) = \xi(a_1, \alpha(c))\xi(a_2, \alpha(b)); \\ (cq4) & \xi(ab, \alpha(c)) = \xi(\alpha(a), c_1)\xi(\alpha(b), c_2), \end{cases}$$

for any $a, b, c \in H$.

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PROPOSITION 5.9. *For the fixed linear forms* $\xi, \xi' \in (H \otimes H)^*$, *define* $\sigma : G \otimes G \Rightarrow \otimes^{op} : \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})^{\times 2} \to \overline{\mathcal{H}}^{i,j}({}_k\mathcal{M})$ *by*

$$\sigma_{X,Y}((x \otimes a) \otimes (y \otimes b)) = \alpha_Y^{j'-i'-1}(y) \otimes \alpha_X^{i'-j'-1}(x)\xi(\alpha^{i'}(a), \alpha^{j'}(b)),$$

and $\sigma' : G \otimes^{op} G \Rightarrow \otimes : \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})^{\times 2} \to \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$ *by*

$$\sigma'_{X,Y}((y \otimes b) \otimes (x \otimes a)) = \alpha_X^{j'-i'-1}(x) \otimes \alpha_Y^{i'-j'-1}(y)\xi'(\alpha^{i'}(b), \alpha^{j'}(a)),$$

for any (X, α_X), $(Y, \alpha_Y) \in \overline{\mathcal{H}}^{i',j'}({}_k\mathcal{M})$, $x \in X$, $y \in Y$, $a, b \in H$. *Then,* (G, σ) *is a coquasitriangular bicomonad with the* $*$*-inverse* σ' *if and only if* (H, α, ξ) *is a coquasitriangular Hom-bialgebra and* ξ' *is the convolution inverse of* ξ. *Moreover, the braiding in* $\mathcal{M}^H$ *is given by* $t_{U,V}(u \otimes v) = \alpha_V^{j'-i'-1}(v_{(0)}) \otimes \alpha_U^{i'-j'-1}(u_{(0)})\xi(\alpha^{i'}(u_{(1)})\alpha^{j'}(v_{(1)}))$, *where* $U, V \in \mathcal{M}^H$, $u \in U$, $v \in V$.

5.3. Hom–Yetter–Drinfeld modules. Note that for any $i, j \in \mathbb{Z}$, we immediately get $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M}) = \overline{\mathcal{H}}^{-i-2,-j-2}({}_k\mathcal{M})$. Suppose that $H = (H, \alpha, \mu, 1_H, \Delta, \epsilon, S)$ is a Hom–Hopf algebra over k.

Let $F = H \otimes \_$ be the bimonad in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, and $G = \_ \otimes H$ be the bicomonad in $\overline{\mathcal{H}}^{-i-2,-j-2}({}_k\mathcal{M})$. For any $p \in \mathbb{Z}$ and $(X, \alpha_X) \in \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, define $\varphi : FG \to GF$ by

$$\begin{aligned}\varphi_X : FGX = H \otimes (X \otimes H) &\to (H \otimes X) \otimes H = GFX,\\ h \otimes (x \otimes g) &\mapsto (\alpha^{-1}(h_{21}) \otimes x) \otimes (\alpha^{p-4}(h_{22})\alpha^{-1}(g))S^{-1}(\alpha^{p-2}(h_1)),\end{aligned}$$

it is a direct computation to check that (F, G, φ) is a monoidal mixed datum on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$. Moreover, $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$, the category of mixed bimodules is a monoidal category satisfying

• the tensor product, the associativity constraint, and the unity constraints are the same as in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$;

• the objects in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$ are pairs (U, α_U), where (U, α_U) is both a left H-Hom-module and a right H-Hom-comodule, satisfying

$$\rho(h \cdot u) = \alpha^{-1}(h_{21}) \cdot u_{(0)} \otimes (\alpha^{p-4}(h_{22})\alpha^{-1}(u_{(1)}))S^{-1}(\alpha^{p-2}(h_1)), \quad u \in U, \quad h \in H.$$

We call such a mixed bimodule a *pth Hom–Yetter–Drinfeld module*, and we write ${}_H\mathcal{HYD}^H(p)$ for $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$.

For example, if we take $i = j = 0$ and $p = 2$, then the mixed bimodule becomes the Makhlouf's left–right Yetter–Drinfeld module which is defined in [**12**] (see Remark 5.4, [**12**]).

Furthermore, ${}_H\mathcal{HYD}^H(p)$ is a braided category with the following braiding:

$$\tau_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto \alpha_V^{i-j-1}(v_{(0)}) \otimes \alpha^{-p}(v_{(1)}) \cdot \alpha_U^{j-i-1}(u).$$

Thus, from Theorem 4.4, there is a natural transformation $\sigma : G \otimes G \to F \otimes^{op} F$ such that (F, G, φ, σ) is a braided mixed datum on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$. Actually, σ is defined as follows:

$$\begin{aligned}\sigma_{U,V} : (U \otimes H) \otimes (V \otimes H) &\longrightarrow (H \otimes V) \otimes (H \otimes U)\\ (u \otimes h) \otimes (v \otimes g) &\longmapsto (1_H \otimes \alpha_V^{i-j-2}(v_{(0)})) \otimes (\alpha^{-p}(g) \otimes \alpha_U^{j-i-2}(u)\epsilon(h)).\end{aligned}$$

5.4. Generalized Hom–Long dimodules. Suppose that $H = (H, \alpha_H, \mu_H, 1_H, \Delta_H, \epsilon_H)$ and $B = (B, \alpha_B, \mu_B, 1_B, \Delta_B, \epsilon_B)$ are two Hom-bialgebras over k. Since $F = H \otimes \_$ is a bimonad in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, and $G = \_ \otimes B$ a bicomonad in $\overline{\mathcal{H}}^{-i-2,-j-2}({}_k\mathcal{M})$, for any $(X, \alpha_X) \in \widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$, one can define $\varphi : FG \to GF$ by

$$\begin{aligned}\varphi_X : FGX = H \otimes (X \otimes B) &\to (H \otimes X) \otimes B = GFX,\\ h \otimes (x \otimes a) &\mapsto (\alpha_H(h) \otimes x) \otimes \alpha_B(a).\end{aligned}$$

It is a direct computation to check that (F, G, φ) is a monoidal mixed datum on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$. Moreover, $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$, the category of mixed bimodules is a monoidal category satisfying

• the tensor product, the associativity constraint, and the unity constraints are the same as in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$;

• the objects in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$ are pairs (U, α_U), where (U, α_U) is both a left H-Hom-module and a right B-Hom-comodule, satisfying

$$\rho(h \cdot u) = \alpha_H(h) \cdot u_{(0)} \otimes \alpha_B(u_{(1)}), \quad u \in U, \quad h \in H, \quad a \in B.$$

We call such a mixed bimodule a *generalized Hom–Long dimodule*, and we write ${}_H\mathcal{HL}^B$ for $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$.

Now suppose that (H, R) is a quasitriangular Hom-bialgebra where $R = \sum R^{(1)} \otimes R^{(2)}$ is the R-matrix, and (B, ξ) is a coquasitriangular Hom-bialgebra, ${}_H\mathcal{HL}^B$ denotes the category of generalized Hom–Long dimodules. Define the following maps τ by

$$\begin{aligned}\tau_{U,V} : U \otimes V &\longrightarrow V \otimes U\\ u \otimes v &\longmapsto \sum \beta(\alpha^i_B(u_{(1)}), \alpha^j_B(v_{(1)}))\alpha^i_H(R^{(2)}) \cdot \alpha_V^{i-j-2}(v_{(0)}) \otimes \alpha^j_H(R^{(1)}) \cdot \alpha_U^{j-i-2}(u_{(0)}),\end{aligned}$$

then it is straightforward to show that τ is a braiding in ${}_H\mathcal{HL}^B$. Indeed, τ is induced by the following natural transformation in $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})^F_G(\varphi)$ through Theorem 4.4:

$$\begin{aligned}&\sigma_{U,V} : (U \otimes B) \otimes (V \otimes B) \longrightarrow (H \otimes V) \otimes (H \otimes U)\\ &(u \otimes a) \otimes (v \otimes b) \longmapsto \beta(\alpha^i_B(a), \alpha^j_B(b))(\alpha^i_H(R^{(2)}) \otimes \alpha_V^{i-j-2}(v)) \otimes (\alpha^j_H(R^{(1)})\\ &\otimes \alpha_U^{j-i-2}(u)).\end{aligned}$$

It is easy to check that (F, G, φ, σ) is a braided mixed datum on $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$.

DEFINITION 5.10. Let U be a vector space over k and $R \in End_k(U \otimes U)$. We say that R is a *solution of the $\mathcal{D}$-equation* if

$$R^{12}R^{23} = R^{23}R^{12}$$

in $End_{\Bbbk}(U \otimes U \otimes U)$.

If we set $B = H$, then we have the following property.

PROPOSITION 5.11. *Let (H, α_H) be a Hom-bialgebra over k, ${}_H\mathcal{HL}^H$ denote the category of Hom-Long dimodules of H. For any integer $n \in \mathbb{Z}$, if we define the following k-linear map*

$$\begin{aligned}\beta_{U,V} : U \otimes V &\longrightarrow U \otimes V\\ u \otimes v &\longmapsto \alpha^n_H(v_{(1)}) \cdot \alpha_U^{-1}(u) \otimes \alpha_V^{-1}(v_{(0)}),\end{aligned}$$

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where $(U, \alpha_U), (V, \alpha_V) \in {}_H\mathcal{HL}^H$*, then* β *satisfies the following generalized Hom-type* $\mathcal{D}$*-equation in* $\widetilde{\mathcal{H}}^{i,j}({}_k\mathcal{M})$*:*

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "\beta_{U,V} \otimes id_W"] & U \otimes (V \otimes W) \arrow[r, "id_U \otimes \beta_{V,W}"] & U \otimes (V \otimes W) \arrow[r, "a^{-1}_{U,V,W}"] & (U \otimes V) \otimes W \arrow[d, "\beta_{U,V} \otimes id_W"] \\
(U \otimes V) \otimes W \arrow[r, "a_{U,W,V}"'] & U \otimes (V \otimes W) \arrow[r, "id_U \otimes \beta_{V,W}"'] & U \otimes (V \otimes W) \arrow[r, "a^{-1}_{U,V,W}"'] & (U \otimes V) \otimes W.
\end{tikzcd}$$

Proof. For any $u \in U$, $v \in V$, $w \in W$, since the following identities

$$\begin{aligned}
&((\beta_{U,V} \otimes id_W) \circ a^{-1}_{U,W,V} \circ (id_U \otimes \beta_{V,W}) \circ a_{U,V,W})((u \otimes u) \otimes w) \\
&\quad = ((\beta_{U,V} \otimes id_W) \circ a^{-1}_{U,W,V})(\alpha_U^{-i-1}(u) \otimes (\alpha_H^{n+j+1}(w_{(1)}) \cdot \alpha_V^{-1}(v) \otimes \alpha_W^{j}(w_{(0)}))) \\
&\quad = (\alpha_H^{n}(v_{(1)}) \cdot \alpha_U^{-1}(u) \otimes \alpha_H^{n+j+1}(w_{(1)}) \cdot \alpha_V^{-2}(v_{(0)})) \otimes \alpha_W^{-1}(w_{(0)}) \\
&\quad = (a^{-1}_{U,V,W} \circ (id_U \otimes \beta_{V,W}))(\alpha_H^{n-i-1}(v_{(1)}) \cdot \alpha_U^{-i-2}(u) \otimes (\alpha_V^{-1}(v_{(0)}) \otimes \alpha_W^{j+1}(w))) \\
&\quad = (a^{-1}_{U,V,W} \circ (id_U \otimes \beta_{V,W}) \circ a_{U,W,V} \circ (\beta_{U,V} \otimes id_W))((u \otimes u) \otimes w),
\end{aligned}$$

the conclusion holds. □

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