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FREE FINITARY ALGEBRAS IN A COCOMPLETE CARTESIAN CLOSED CATEGORY

BY
C. HOWLETT AND D. SCHUMACHER

In [2] Volger proved that the underlying functor of a category of set-valued models of an r-ary theory has a left adjoint. We want to show that his proof remains valid if instead of set valued models of an r-ary theory models of a finitary theory with values in an arbitrary cocomplete cartesian closed category are considered. As Volger for sets we show for any cocomplete cartesian closed category $\mathbf{C}$ that for every finitary theory $\mathbf{S}^* \xrightarrow{T} \textcircled{H}$ ($\mathbf{S}$ being a skeleton of the full subcategory of finite sets) the restriction of the left adjoint of $\mathbf{C}^{\textcircled{H}} \xrightarrow{\mathbf{C}^T} \mathbf{C}^{\mathbf{S}}$ on $\mathbf{C}^{(\mathbf{S})}$ is a functor in $\mathbf{C}^{(\textcircled{H})}$; here brackets around the exponent indicate as usual a restriction to functors which preserve finite products. We are very much indebted to the referee for pointing out that our proof of the last statement is only based on the properties of $\mathbf{C}$ mentioned above and the fact that $\mathbf{S}$ has and T preserves finite products. With this in mind and retaining only that part of the cartesian closedness which is relevant for the following considerations we can state the following.

THEOREM. *Let C be a cocomplete category with finite products such that for all objects C of* $\mathbf{C}$ $C\times(\ )$ *preserves colimits. Then for every functor F from a category* $\mathbf{A}$ *with finite products to a category* $\mathbf{B}$ *which preserves finite products the restriction of the left adjoint of* $\mathbf{C}^{\mathbf{B}} \xrightarrow{\mathbf{C}^F} \mathbf{C}^{\mathbf{A}}$ *to* $\mathbf{C}^{(\mathbf{A})}$ *is a functor in* $\mathbf{C}^{(\mathbf{B})}$.

The proof of the theorem depends essentially on two observations:

(1) Suppose F and G are functors into $\mathbf{C}$ from small categories $\mathbf{D}$ and $\mathbf{T}$ respectively. Let $(i_D)_{D\in ob(\mathbf{D})}$ and $(j_T)_{T\in ob(\mathbf{T})}$ be the injections of colim F and colim G respectively. Then colim $F\times$colim G is a colimit of $\mathbf{D}\times\mathbf{T}\xrightarrow{F\times G}\mathbf{C}\times\mathbf{C}\xrightarrow{\times}\mathbf{C}$ with injections $(i_D\times j_T)_{(D,T)\in ob(\mathbf{D}\times\mathbf{T})}$. By induction we have a corresponding result for any finite number of functors to $\mathbf{C}$ from small categories.

(2) Let F be a finite product preserving functor from a small category $\mathbf{A}$ having finite products into a small category $\mathbf{B}$ and let for all objects B of $\mathbf{B}$ $V_{F,B}$ be the usual forgetful functor from the comma category (F, B) which takes a morphism of (F, B) i.e. a triple (g, f, g') with $g, g' \in \mathbf{B}, f\in\mathbf{A}$, and $g=g'F(f)$ to f.

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Then for every finite nonempty family $(B_k)_{k\in n}$ of objects of **B** with a product $\prod B_k$ the functor $\prod (F, B_k)\xrightarrow{P}(F, \prod B_k)$ which takes each $(g_k, f_k, g'_k)_{k\in n}$ of its domain to $(\prod g_k, F(\prod f_k), \prod g'_k)$ is a right adjoint (namely of the "splitting functor" from $(F, \prod B_k)$ to $\prod (F, B_k)$ which takes (g, f, g') of $(F, \prod B_k)$ to $(q_k g, f, q_k g)_{k\in n}$ with q_k being the k-th projection of $\prod B_k$) and is hence clearly cofinal [1].

From (1) and (2) the theorem is easily obtained. Let F be as under (2) and let G be a functor from **A** to **C** which preserves finite products and let G' be its left Kan extension along F.

$$\begin{array}{ccccc}
(F, \prod B_k) & \xrightarrow{V_{F,\Pi B_k}} & \mathbf{A} & \xrightarrow{G} & \mathbf{C} \\
{\scriptstyle P}\uparrow & & \uparrow{\scriptstyle \times_n} & & \\
\prod (F, B_k) & \xrightarrow[\Pi V_{F,B_k}]{} & \mathbf{A}^n & &
\end{array}$$

Then for all finite nonempty families $(B_k)_{k\in n}$ of objects of **B**, for which a product $\prod B_k$ exists,

$$\begin{aligned}
G'(\Pi B_k) &= \operatorname{colim} GV_{F,\Pi B_k} && \text{(Construction of left Kanextension)}\\
&= \operatorname{colim} GV_{F,\Pi B_k}P && \text{(since } P \text{ is cofinal)}\\
&= \operatorname{colim} G \underset{n}{\times} \prod V_{F,B_k} && \\
&= \operatorname{colim} \underset{n}{\times} \prod GV_{F,B_k} && \text{(because } G \text{ preserves finite products)}\\
&= \underset{n}{\times}\, G'(B_k) && \text{(because of (1))}
\end{aligned}$$

It should be remarked that even for sets the assumption of finiteness in (1) cannot be dropped. This can be seen by observing that an infinite product of connected categories might not be connected. That nevertheless at least for sets free infinitary algebras exist is due to the observation that for sets the above theorem remains true if the finiteness is dropped provided (a) the domain of the functor along which the left Kan extension is taken has pullbacks and (b) the functors to be mapped under the left Kan extension preserve not only products but also pullbacks (i.e. limits).

References

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McMaster University,
Hamilton, Ontario