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COUNTABLE PERIODIC *CC*-GROUPS AS AUTOMORPHISM GROUPS

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It is shown that if G is a group and $\operatorname{Aut} G$ is a countable periodic CC-group then $\operatorname{Aut} G$ is FC.

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1. Introduction

The thrust of several recent papers (see, for example, [**2, 7, 9** and **11**]) has been to show that certain kinds of infinite group cannot occur as the full automorphism group, $\operatorname{Aut} G$, of some group G. Indeed in [**2**, Theorem A], [**7**, Theorem] and [**9**, Theorem A] it is shown that if $\operatorname{Aut} G$ is periodic and satisfies some other appropriate hypothesis then the divisible radical of $\operatorname{Aut} G$ is trivial. (Here, the divisible radical of a group is the unique largest radicable abelian subgroup, if it exists.) In this short note we add one further result of this type and hence provide further examples of groups which cannot be automorphism groups. For example a countable direct product of Černikov groups, at least one of which is infinite, can never be an automorphism group.

In this paper we shall be concerned with the class of CC-groups. A group G is called a CC-group if $G/C_G(x^G)$ is Černikov for all $x \in G$; if $G/C_G(x^G)$ is finite for all $x \in G$ then we call G an FC-group. The class of CC-groups has been the subject of much recent work (see, for example, [**1, 4** and **6**].) Furthermore, Zimmerman [**11**] has considered countable, periodic FC-groups as automorphism groups and showed there that with certain extra hypotheses the structure of $\operatorname{Aut} G$ is very restricted. Our main result is as follows.

Theorem. *Suppose G is a group such that* $\operatorname{Aut} G$ *is a countable periodic CC-group. Then* $\operatorname{Aut} G$ *is an FC-group.*

The main part of the argument here is to show using the techniques of Lemma 2 of [**11**] that G must be a CC-group. We remark in passing that it seems reasonable to expect that the theorem holds in the absence of countability; however, this extra hypothesis is necessary to show that G is CC. In [**5**], Menegazzo and Stonehewer have given an example of a countable group G which is not CC but which has uncountable, elementary abelian automorphism group.

Our notation is standard and is that used in [**10**].

2. The proof of the theorem

Our first result is well known but we include it for completeness.

Lemma 1. *Suppose G is a CC-group. Then G has a periodic divisible radical D and G/D is an FC-group.*

Proof. It follows using arguments similar to those of Lemma 3 of [**1**] that the semiradicable radical, R, of G is nilpotent of class 2. Furthermore if M,L are periodic divisible subgroups of G then $M,L \leqq R$ so are subnormal in G. Hence by [**10**, vol. I, Lemma 4.46], G has a periodic divisible radical D, lying in R and D is the torsion subgroup of R, by [**10**, vol. II, Theorem 9.23]. If bars denote groups modulo R and if $x \in G$ then x^G is Černikov-by-cyclic (see [**10**, vol. I, Theorem 4.36]). Hence $\bar{x}^{\bar{G}} \cong x^G/x^G \cap R$ is finite-by-cyclic so $\bar{G}$ is FC by [**10**, vol. I, Corollary 3, p. 122]. Also if T is the torsion subgroup of G then [**10**, vol. I, Corollary, p. 129] implies G/T is abelian. Hence $G/(T \cap R) = G/D$ is an FC-group as required. □

In their recent paper [**4**], Franciosi, de Giovanni and Tomkinson remark that if G is a CC-group and M is a normal subgroup with G/M Černikov then one cannot necessarily find a normal Černikov subgroup K such that $G = MK$. Our next results are aimed at showing that this can be done in certain cases, which are of importance to our situation.

Lemma 2. *Suppose D is a periodic CC-group and M is an abelian normal subgroup such that D/M is a divisible abelian group. Then D is abelian.*

Proof. Suppose $x \in M$. Then x^D is Černikov by [**10**, vol. I, Theorem 4.36(ii)] and abelian. It follows from [**10**, vol. I, Corollary, p. 85] that $D/C_D(x^D)$ is finite and also divisible. Hence $M \leqq Z(D)$, and D is nilpotent of class at most 2. Furthermore $D/Z(D)$ is divisible so the same argument implies D is abelian, as required. □

Lemma 3. *Let G be a periodic CC-group and suppose M is a normal abelian subgroup of finite exponent such that G/M is Černikov. Then G has a normal Černikov subgroup K such that $G = MK$.*

Proof. Let D/M be the divisible radical of G/M. Then G/D is finite so there is a finite subset X of G such that $G = DX^G$. Note that X^G is Černikov.

By Lemma 2, D is abelian. Let B be a basic subgroup of D. Then BM/M is a subgroup of the Černikov group D/M and hence has finite rank. Hence $B/M \cap B$ is finite and B has finite exponent. Thus $D = B \oplus R$ for some (characteristic) divisible

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group R. Now R must be Černikov since M has finite exponent. Since $B/M\cap B$ is finite, D has a finite subset Y such that $B=Y(M\cap B)$. Hence $G=X^GD=X^GY^GRM$ and we set $K=X^GY^GR$, a Černikov normal subgroup of G. □

Our next result uses the argument of Lemma 2 of [**11**]. We let Z denote the centre of a group G.

Lemma 4. *Let G be a group and suppose* Aut G *is a countable periodic CC-group. Then G is a CC-group.*

Proof. Let $Q=G/Z$. If $x\in G$ let $R=x^G/x^G\cap Z, M=C_G(R)$ and $L=C_G(x^G)$. Since G/Z is a periodic CC-group, R is Černikov. By a theorem of Baer [**10**, vol. I, Theorem 3.29] it follows that G/M is Černikov. Let D/M be its divisible radical and let bars denote groups modulo L. Then $\bar{G}$ is a periodic CC-group. There is a monomorphism from $\bar{M}$ to $\mathrm{Hom}(x^G, x^G\cap Z)$ defined by $mL\mapsto f_m$, for $m\in M$, where $f_m(g)=g^{-1}g^m$ for $g\in x^G$. Since $Z\cap x^G\leqq \ker f_m$ there is an induced map $\bar{f}_m$ from $x^G/(x^G\cap Z)\ (x^G)'$ to $x^G\cap Z$. Note that f_m and $\bar{f}_m$ have the same order. Since $x^G/(x^G\cap Z)(x^G)'$ is Černikov it follows that $\mathrm{Hom}(x^G/(x^G\cap Z)(x^G)', x^G\cap Z)=K\oplus J$ where K is torsion-free and J is an abelian π-group of finite exponent, where π is the set of primes dividing the orders of the elements of Z. Since $\bar{M}$ is periodic it follows that $\bar{M}$ is an abelian π-group of finite exponent.

Now by Lemma 3, $\bar{G}=\bar{N}\bar{M}$ with $\bar{N}$ a Černikov normal subgroup of $\bar{G}$. Since $\bar{G}/\bar{N}$ is an abelian group of finite exponent and an image of Q_{ab}, it follows by [**11**, Lemma 1] that $\bar{G}/\bar{N}$ is finite. (It is here that we require the countability of Aut G). Hence $\bar{G}$ is Černikov and G is a CC-group, as required. □

Corollary. *If G is a group and* Aut G *is a periodic countable CC-group then G/Z is finite.*

Proof. By Lemma 1, G is a CC-group and G/Z is periodic. Since Aut G is countable, Theorem 4.5 of [**6**] shows that G/Z is a Černikov group. Let D/Z be the divisible radical of G/Z. Since a periodic group of automorphisms of an abelian Černikov group is finite, it follows that $G/C_G(D/Z)$ is finite. A result of Pettet [**8**, Proposition 5.5] now shows that G/Z has finite exponent. It follows that G/Z is finite. □

Proof of the Theorem.

We shall let D denote the divisible radical of Aut G, which exists by Lemma 1. We show that $D=1$.

Since D is divisible it must act trivially on G/Z which is finite, by the above corollary. Hence

$$[G,D]\leqq Z.$$

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Also if T is the torsion subgroup of G and if $A = C_{\mathrm{Aut}\,G}(G/Z)$ then Proposition 3.8 of [8] shows that $A/C_A(G/T)$ has exponent dividing 12. Since $D \leqq A$ it follows that

$$[G, D] \leqq T.$$

Hence $[G, D] \leqq T \cap Z = T(Z)$, the torsion subgroup of Z.

On the other hand, since G/Z is finite, G is finite-by-abelian and hence [3, Corollary 2] implies the Sylow p-subgroups of G are finite. Hence D acts trivially on $T(Z)$. Finally if $\alpha \in D$ consider the map $\alpha \mapsto [g, \alpha]$, for fixed $g \in G$. Since $[g, \alpha] \in T(Z) \leqq C_G(D)$ this map is a homomorphism. Hence $[g, D]$ is a periodic divisible subgroup of $T(Z)$ whence

$$[g, D] = 1.$$

Thus, D acts trivially on G and $D = 1$ as required. □

We remark finally that the proof shows that if $\mathrm{Aut}\,G$ is a countable periodic FC-group then the divisible radical of $\mathrm{Aut}\,G$ is trivial, which can of course be deduced from the work of Zimmerman.

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