The characteristic polynomials of AB and BA

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Let A and B be square matrices of the same order, with elements in any field F; it is well known that the characteristic polynomials of AB and BA are the same (see, *e.g.* C. C. Macduffee, Theory of Matrices, p. 23). The proof of this is easy when one at least of the matrices is non-singular; the object of the following remarks (which are not claimed as original) is to point out that the case $|A| = |B| = 0$ is just as easy. If one attempts to deduce the result in this case from the result in the non-singular case, unnecessary restrictions on the field F are apt to appear (see *e.g.*, W. V. Parker, American Mathematical Monthly, vol. 60 (1953) p. 316). If one proceeds directly to the general case, no difficulties are encountered.

Consider the elements of A as indeterminates over F; then

$$|A||BA - \lambda I| = |ABA - \lambda A| = |AB - \lambda I||A|,$$

the equality holding in the sense of an identity between polynomials in $a_{11}, a_{12}, \ldots, a_{nn}, \lambda$, with coefficients in F. Since $|A|$, as a polynomial in the elements of A, is not zero, we may divide by it to obtain $|BA - \lambda I| = |AB - \lambda I|$. This relation must still be true (as between polynomials in λ with coefficients in F) when we regard the elements of A as being fixed members of F; this is the required result.

We have used here only the results that the determinant of the product of two square matrices is equal to the product of their determinants (the usual proofs cover the present case without alteration), and that the product of two polynomials, in any indeterminates with coefficients in a field, cannot be zero unless one of the factors is zero. Both these results are quite elementary.

A variation of the above reasoning, which minimises the number of indeterminates, is to consider the matrix $(A - \mu I) B (A - \mu I) - \lambda (A - \mu I)$, where now the elements of A are fixed members of F, and λ and μ are indeterminates. The argument is as before, and we put finally $\mu = 0$.

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