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RANKS OF CHAIN COMPLEXES OVER THE COMPLEX POLYNOMIAL RING

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ABSTRACT. Using LePotier's vanishing theorem, we establish a lower bound on the rank of nontrivial free differential complex in terms of the dimension of the support for its cohomology. Our bound specializes to the one predicted by the syzygy theorem of Evans and Griffith.

Introduction. Let A denote the graded polynomial ring $\mathbb{C}[x_1,\dots,x_n]$ with $|x_i| = 1$ for all i. Let $\mathcal{M}$ be the category of finitely generated, graded A-modules and let $A(k)$ denote the free module with generator of degree k. Pair (M,d) where $M = \oplus_{i=0}^{s} M_i$, $M_i = A(i_1) \oplus \cdots \oplus A(i_k)$ and $d \in \operatorname{End}_{\mathcal{M}}(M)$ will be called a *complex* if $d^2 = 0$ and $d(M_k) \subset \oplus_{i=0}^{k-1} M_i$. A complex is *regular* if $d(M_k) \subset M_{k-1}$.

In this paper we shall establish a lower bound on rank(M) in terms of the dimension of the support of homology module $\mathcal{H}(M) = \operatorname{Ker} d / \operatorname{Im} d$, providing that M is a regular complex. We use the Kodaira-LePotier vanishing theorem. In the case when M is a free resolution of an artinian module, we recover the bounds predicted by the syzygy theorem of Evans and Griffith ([3]).

Our work is motivated by the abundance of A-complexes in studies of equivariant cohomology of torus actions on topological spaces ([1],[2],[4]). Theorems tying up complexity of M with the support of $\mathcal{H}(M)$ should eventually shed some light on the nature of obstructions to the existence of fixed-point free actions. So far, we have not been able to remove, in a convincing way, the assumption of regularity of M. Our bounds are perhaps far from being sharp—this is directly related to a very unsatisfactory state of the Horrocks Problem. The main result of this note is stated as Theorem 2.3.

1. **Preliminaries.** For $M \in \mathcal{M}$ we shall write $\operatorname{supp} M = \{p \in \operatorname{Spec} A \mid M_p \neq 0\}$, where M_p denotes localization at p. The set $\operatorname{supp} M$ is $\mathbb{C}$-invariant and Zariski closed in $\operatorname{Spec} A$. Let $\operatorname{Supp} M = \operatorname{supp} M \cap \operatorname{Proj} A$. If we interpret $\operatorname{supp} M$ as $\mathbb{C}^*$-invariant, affine variety in $\mathbb{C}^n$, then $\operatorname{Supp} M$ is the space of one dimensional subspaces contained in $\operatorname{supp} M$. If $\operatorname{supp} M$ has codimension k in $\mathbb{C}^n$, then there exists a codimension-one subspace H of $\mathbb{C}^n$ such that $\operatorname{supp} M \cap H$ has codimension k in H. By induction, there exists a vector space $E \subset \mathbb{C}^n$ with $\dim E = k$, such that $E \cap \operatorname{supp} M = \{0\}$.

Let $\mathcal{S}$ denote the category of coherent sheaves on $\operatorname{Proj} A \approx \mathbb{P}^{n-1}$. There is the standard and exact functor $\mathbb{F}: \mathcal{M} \longrightarrow \mathcal{S}$ ([5], p. 116) such that $\mathbb{F}\big(A(k)\big) = \mathcal{O}(-k)$ and

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$\operatorname{supp}\mathbb{F}(M) = \operatorname{Supp} M$, where the left side denotes the sheave-theoretical support. We shall need the following additional property of $\mathbb{F}$ whose verification we leave to the reader as an exercise.

PROPOSITION 1.1. *Let (M, d) be a regular complex and let X be a subvariety in $\mathbb{P}^{n-1}$. If $X \cap \operatorname{Supp}\mathcal{H}(M) = \emptyset$, then the restriction $\mathbb{F}(M)|_X$ is an extension of vector bundles, i.e., it is locally acyclic on X.*

The proof of Theorem 2.3 uses (1.1) with $X = \mathbb{P}(E)$.

2. Let (M, d) be a complex and $m = \min(|x|; 0 \neq x \in M)$. We shall say that M is *reduced* if $d(M^m) = 0$, where M^m denotes the vector space of elements of degree m in M. Notice that if M is reduced then $\mathcal{H}(M) \neq 0$ (we adopt the convention that the trivial complex is not reduced).

LEMMA 2.1. *If M is regular and $\mathcal{H}(M) \neq 0$, then there exists a reduced regular complex M' such that* $\operatorname{rank} M_i \geq \operatorname{rank} M'_i$ *and $\mathcal{H}(M) = \mathcal{H}(M')$.*

PROOF. M^m is a complex of vector spaces, hence there exists a splitting $M^m = N \oplus R$ where $d(N) = 0$, $N \approx \mathcal{H}(M^m)$ and R is an acyclic subcomplex of M^m. Thus, $A \cdot R$ is acyclic, and hence $\mathcal{H}(M) = \mathcal{H}(M/A \cdot R)$. Define: $M' = M/A \cdot R$.

DEFINITION 2.2. Suppose that M is reduced, regular and m is as above, then $m' = \max\{k \mid M^m \cap M_k \neq 0\}$ is the *minimal number* for M.

THEOREM 2.3. *Let (M, d) be reduced, regular complex and* $\dim \operatorname{supp}\mathcal{H}(M) = n - k$. *Let m be its minimal number. Write $s_i = \dim_Q d(M_{m+i+1})$, $0 < i < k$, where Q denotes the field of fractions of A. Then* $\operatorname{rank} M_m \geq 1$ *and $s_i \geq k - i$.*

COROLLARY 2.4. *If $1 < i \leq k$ then* $\operatorname{rank}(M_{m+i}) \geq s_{i-1} + s_i \geq 2k - 2i + 1$ *and* $\operatorname{rank}(M_{m+1}) \geq k$.

PROOF OF (2.3). We can assume without losing generality that $M^0 \neq 0$ and $M^k = 0$ for $k < 0$. Let E be a k-dimensional subspace of $\mathbb{C}^n$ chosen so that $\operatorname{supp}\mathcal{H}(M) \cap E = \{0\}$. Set $V_i = \mathbb{F}(M_i)|_{\mathbb{P}(E)}$. By (1.1), the complex of line bundles $V_s \rightarrowtail V_{s-1} \mapsto \cdots \to \to V_0$ is an extension, hence sheaves $d(V_s), d(V_{s-1}), \ldots, d(V_1)$ are locally free—they are holomorphic vector bundles on $\mathbb{P}(E) \approx \mathbb{P}^{k-1}$.

CLAIM 2.5. *$H^i\big(d(V_{m+i+1})\big) \neq 0$ for $0 \leq i < k$ and in particular: $d(V_{m+k}), d(V_{m+k-1}), \ldots, d(V_{m+1})$ are not zero. Here $H^i(V) \approx H^i\big(\mathbb{P}(E); V\big)$.*

PROOF. Let x be a nonzero element in $(M_m)^0$. Since $d(x) = 0$, we have $\mathbb{F}(Ax) \approx \mathcal{O} \subset d(V_{m+1})$ and hence $H^0\big(d(V_{m+1})\big) \neq 0$. By minimality of m, V_{m+i} does not contain summands $\mathcal{O}(k)$ with positive k for $i \geq 0$, and therefore, $H^{i-1}(V_{m+i}) = 0$ for $1 \leq i < k$. The argument is completed inductively using the exact sequences:

$$H^{i-1}(V_{m+i}) \to H^{i-1}\big(d(V_{m+i})\big) \to H^i\big(d(V_{m+i+1})\big),$$

obtained from short exact sequences of vector bundles splitting the extension $V_s \rightarrowtail V_{s+1} \mapsto \cdots \twoheadrightarrow d(V_{m+1})$.

CLAIM 2.6. *Vector bundles* $d(V_{m+k}), \cdots, d(V_{m+2})$ *are negative in the sense of* [6, p. 83].

PROOF. V_{m+i} is negative for $i \geq 1$ since it is a direct sum of negative line bundles. Since $d(V_{m+i+1})$ is a sub-bundle of V_{m+i} and negativity is inherited by sub-bundles (see [6], p. 87), the claim follows.

The sought inequalities follow now from the LePotier vanishing theorem ([6], p. 84).

REMARK 2.5. Our proof shows that $\dim \operatorname{supp} \mathcal{H}(M) \geq \dim \operatorname{supp} \mathcal{H}(M^*)$ where $M^* = \operatorname{Hom}(M, A)$. If one defines *maximal number* for M as the minimal one for M^* then the statement of (2.3) can be dualized in the obvious way. This process considerably strengthens our theorem.

EXAMPLE 2.6. Suppose that M is the minimal resolution of an artinian module V; $M_n \rightarrowtail M_{n-1} \longrightarrow \cdots M_0 \twoheadrightarrow V$. Then $\mathcal{H}(M) = V$ and $\operatorname{supp} V = \{0\}$, minimal and maximal numbers are zero and n respectively. The dual version of (2.3) gives estimates coinciding with those implied by the syzygy theorem ([3]). If $n \leq 4$, then (2.3) together with its dual yield that $\operatorname{rank} M_i \geq \binom{n}{i}$—they solve the Horrocks Problem.

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