# SOME MOMENT PROBLEMS IN A FINITE INTERVAL

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**1. Introduction.** Let the sequence $\{\lambda_i\}$ $(i \geqq 0)$ satisfy the following conditions.

1. $0 < \lambda_0 < \lambda_1 < \ldots < \lambda_n < \ldots,$

2. $\lim_{n\to\infty} \lambda_n = \infty,$

3. $\sum_{i=1}^{\infty} (1/\lambda_i) = \infty.$

We shall deal with the following moment problems: what are the conditions, necessary and sufficient, on a sequence $\{\mu_n\}$ $(n \geqq 0)$ in order that it should possess the representation

$$(1.1) \qquad \mu_n = \int_0^1 t^{\lambda_n} f(t)\, dt, \qquad n = 0, 1, 2, \ldots,$$

where $f(t)$ belongs to a given class of functions integrable over $[0, 1]$.

For a sequence $\{\mu_n\}$ $(n \geqq 0)$ we write, for each $0 \leqq m \leqq n = 0, 1, 2, \ldots,$

$$(1.2) \quad [\mu_m, \ldots, \mu_n] = \sum_{i=m}^{n} \frac{\mu_i}{(\lambda_i - \lambda_m)\cdot \ldots \cdot(\lambda_i - \lambda_{i-1})\cdot(\lambda_i - \lambda_{i+1})\cdot \ldots \cdot(\lambda_i - \lambda_n)}.$$

**2. Main results.** Let $M(u)$ be an even, convex, continuous function satisfying (1) $M(u)/u \to 0$ as $u \to 0$ and (2) $M(u)/u \to \infty$ as $u \to \infty$. Denote by $L_M[0, 1]$ the class of all functions integrable over $[0, 1]$ satisfying

$$\int_0^1 M[f(t)]\, dt < \infty.$$

$L_M[0, 1]$ is known as the Orlicz class related to $M(u)$ (see **2**).

THEOREM 1. *The sequence* $\{\mu_n\}$ $(n \geqq 0)$ *possesses the representation* (1.1), *where* $f \in L_M[0, 1]$, *if and only if*

$$(2.1) \quad \sup_{n \geqq 0} \sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1}\cdot \ldots \cdot\lambda_n \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt \times M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right) \equiv H < \infty$$

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($[t^{\lambda_m}, \ldots, t^{\lambda_n}]$ *is defined by* (1.2) *for* $\mu_n = t^{\lambda_n}$); *if* $m = n$, *then*

$$\lambda_{m+1} \cdot \ldots \cdot \lambda_n = 1.$$

If $\lambda_0 = 0$ instead of $\lambda_0 > 0$, then the proof of Theorem 1 is as shown in (**3**), Theorem 2.3 (i).

Denote by $M[0, 1]$ the space of all functions bounded in $[0, 1]$.

THEOREM 2. *The sequence* $\{\mu_n\}$ $(n \geqq 0)$ *possesses the representation* (1.1), *where* $f \in M[0, 1]$, *if and only if*

$$\sup_{\substack{0 \leqq m \leqq n, \\ n \geqq 0}} \frac{|[\mu_m, \ldots, \mu_n]|}{\left| \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt \right|} \equiv H < \infty. \tag{2.2}$$

If $\lambda_0 = 0$ instead of $\lambda_0 > 0$, then the proof of Theorem 2 is as shown in (**3**, Theorem 2.3 (ii)).

**3. Proofs.** We need the following two lemmas.

LEMMA 1. *For each* $0 \leqq t \leqq 1$ *and* $n \geqq 0$,

$$\sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n [t^{\lambda_m}, \ldots, t^{\lambda_n}] \leqq 1 \quad (\textit{if } m = n, \textit{ then } \lambda_{m+1} \cdot \ldots \cdot \lambda_n = 1).$$

*Proof.* Define the sequence $\{\tilde{\lambda}_i\}$ $(i \geqq 0)$ by

$$\tilde{\lambda}_0 = 0, \quad \tilde{\lambda}_i = \lambda_{i-1}, \quad i \geqq 1. \tag{3.1}$$

By (**4**, p. 46 (11)) we have, for $n \geqq 0$,

$$\sum_{m=0}^{n} (-1)^{n-m} \tilde{\lambda}_{m+1} \cdot \ldots \cdot \tilde{\lambda}_n [t^{\tilde{\lambda}_m}, \ldots, t^{\tilde{\lambda}_n}] = 1.$$

By an easy calculation we get, for $n \geqq 1$,

$$[t^{\tilde{\lambda}_m}, \ldots, t^{\tilde{\lambda}_n}] = [t^{\lambda_{m-1}}, \ldots, t^{\lambda_{n-1}}]$$

and by (**4**, p. 46 (10)),

$$(-1)^n [t^{\tilde{\lambda}_0}, \ldots, t^{\tilde{\lambda}_n}] \geqq 0.$$

Hence

$$\sum_{m=0}^{n-1} (-1)^{n-1-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_{n-1} [t^{\lambda_m}, \ldots, t^{\lambda_{n-1}}] \leqq 1.$$

LEMMA 2. *For a sequence* $\{\tilde{\lambda}_i\}$ $(i \geqq 0)$ *satisfying* (3.1) *we have*

$$\lim_{n \to \infty} (-1)^n \tilde{\lambda}_1 \cdot \ldots \cdot \tilde{\lambda}_n \int_0^1 [t^{\tilde{\lambda}_0}, \ldots, t^{\tilde{\lambda}_n}]\, dt = 0.$$

*Proof.* If

$$\mu_n = \int_0^1 t^{\tilde{\lambda}_n}\, d\alpha(t), \qquad n = 0, 1, 2, \ldots,$$

where $\alpha(t)$ is of bounded variation in [0, 1], then, by (**1**, p. 287 (25)), we have

$$\lim_{n\to\infty} (-1)^n \tilde{\lambda}_1 \cdot \ldots \cdot \tilde{\lambda}_n [\mu_0, \ldots, \mu_n] = \alpha(0+) - \alpha(0).$$

In this case we have $\alpha(t) = t$ for $0 \leqq t \leqq 1$, hence $\alpha(0+) - \alpha(0) = 0$.

*Proof of Theorem* 1. First we prove necessity. Suppose that $\{\mu_n\}$ $(n \geqq 0)$ satisfies (1.1), where $f \in L_M[0, 1]$, then we have

$$[\mu_m, \ldots, \mu_n] = \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}] f(t)\, dt,$$

and by the Jensen inequality (see **8**, pp. 23–24)

$$M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right) \leqq \frac{\int_0^1 (-1)^{n-m} [t^{\lambda_m}, \ldots, t^{\lambda_n}] M[f(t)]\, dt}{\int_0^1 (-1)^{n-m} [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt},$$

since $(-1)^{n-m}[t^{\lambda_m}, \ldots, t^{\lambda_m}] \geqq 0$ for each $0 \leqq t \leqq 1$ and $0 \leqq m \leqq n = 0, 1, 2, \ldots$ (see **4**, p. 46). Hence, by Lemma 1,

$$\sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt\, M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right) \leqq$$

$$\int_0^1 \left[\sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n [t^{\lambda_m}, \ldots, t^{\lambda_n}]\right] M[f(t)]\, dt \leqq \int_0^1 M[f(t)]\, dt < \infty.$$

In order to prove sufficiency let us assume that condition (2.1) is fulfilled. Denote by $N(v)$ the complement function of $M(u)$ (see **2**, p. 11), then by the Young inequality (see **2**, p. 12)

$$\left|\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right| \leqq N(1) + M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right).$$

Hence, by Lemma 1 and (2.1),

$$\sum_{m=0}^{n} \lambda_{m+1} \cdot \ldots \cdot \lambda_n |[\mu_m, \ldots, \mu_n]| \leqq N(1) \int_0^1 \left[\sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \right.$$

$$\left. \times [t^{\lambda_m}, \ldots, t^{\lambda_n}]\right] dt + \sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt$$

$$\times M\left(\frac{[\mu_m, \ldots, \mu_n]}{\int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}]\, dt}\right) \leqq K < \infty.$$

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Let

$$\tilde{t}_{nm} = [(1 - \tilde{\lambda}_1/\tilde{\lambda}_{m+1})\cdot\ldots\cdot(1 - \tilde{\lambda}_1/\tilde{\lambda}_n)]^{1/\tilde{\lambda}_1},$$

where $\{\tilde{\lambda}_i\}$ $(i \geqq 0)$ is defined by (3.1). Define the functions $\alpha_n(x)$ by:

$$\text{(3.2)} \qquad \begin{aligned} \alpha_n(0) &= 0, \\ \alpha_n(x) &= \sum_{\tilde{t}_{n+1,m+1} \leqq x} (-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n[\mu_m, \ldots, \mu_n]. \end{aligned}$$

The functions $\alpha_n(x)$ are of variations uniformly bounded in [0, 1]. Define the sequence $\{\tilde{\mu}_n\}$ $(n \geqq 0)$ by

$$\text{(3.3)} \qquad \tilde{\mu}_0 \text{ arbitrary}, \qquad \tilde{\mu}_n = \tilde{\mu}_{n-1}, \quad n > 0.$$

We have that

$$\begin{aligned} \int_0^1 t^{\lambda_k} d\alpha_n(t) &= \sum_{m=0}^{n} \tilde{t}_{n+1,m+1}^{\lambda_k}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n[\mu_m, \ldots, \mu_n] \\ &= \sum_{m=0}^{n+1} \tilde{t}_{n+1,m}^{\tilde{\lambda}_{k+1}}(-1)^{n+1-m}\tilde{\lambda}_{m+1}\cdot\ldots\cdot\tilde{\lambda}_{n+1}[\tilde{\mu}_m, \ldots, \tilde{\mu}_{n+1}] \end{aligned}$$

since $[\tilde{\mu}_m, \ldots, \tilde{\mu}_{n+1}] = [\mu_{m-1}, \ldots, \mu_n]$ for $1 \leqq m \leqq n = 1, 2, 3, \ldots$ and $\tilde{t}_{n+1,0} = 0$. By (**6**) we have, for every $k \geqq 0$,

$$\lim_{n\to\infty} \sum_{m=0}^{n+1} \tilde{t}_{n+1,m}^{\tilde{\lambda}_{k+1}}(-1)^{n+1-m}\tilde{\lambda}_{m+1}\cdot\ldots\cdot\tilde{\lambda}_{n+1}[\tilde{\mu}_m, \ldots, \tilde{\mu}_{n+1}] = \tilde{\mu}_{k+1} = \mu_k.$$

Hence

$$\mu_k = \lim_{n\to\infty} \int_0^1 t^{\lambda_k} d\alpha_n(t), \qquad k = 0, 1, 2, \ldots .$$

By the Helly Theorem (see **7**, p. 29) there exists a subsequence $\{n_i\}$ $(i \geqq 0)$ such that $\lim_{i\to\infty} \alpha_{n_i}(t) = \alpha(t), 0 \leqq t \leqq 1$, and by the Helly-Bray Theorem (see **7**, p. 31),

$$\text{(3.4)} \qquad \mu_k = \int_0^1 t^{\lambda_k} d\alpha(t), \qquad k = 0, 1, 2, \ldots ,$$

where $\alpha(t)$ is of bounded variation in [0, 1].

Let $x_0 = 0 < x_1 < \ldots < x_k = 1$ be a fixed division of [0, 1] and let $0 \leqq p < k$. There exist $r, s$ (depending on $n$) satisfying

$$\begin{aligned} \tilde{t}_{n+1,r+1} &\leqq x_p < \tilde{t}_{n+1,r+2}, \\ \tilde{t}_{n+1,s+1} &\leqq x_{p+1} < \tilde{t}_{n+1,s+2}. \end{aligned}$$

Now

$$\alpha_n(x_{p+1}) - \alpha_n(x_p) = \sum_{m=r+1}^{s} (-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n[\mu_m, \ldots, \mu_n].$$

Hence by the Jensen inequality (see **8**, pp. 23–24)

$$M\left(\frac{\alpha_n(x_{p+1}) - \alpha_n(x_p)}{\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt}\right) \leqq$$

$$\frac{\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt\, M\left(\frac{[\mu_m,\ldots,\mu_n]}{\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt}\right)}{\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt}.$$

Hence we have, by (2.1),

$$\sum_{p=0}^{k-1}\left[\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt\right]$$

$$\times M\left(\frac{\alpha_n(x_{p+1}) - \alpha_n(x_p)}{\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt}\right) \leqq$$

$$\sum_{m=0}^{n}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt\, M\left(\frac{[\mu_m,\ldots,\mu_n]}{\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt}\right) \leqq H.$$

We have that $\lim_{i\to\infty}[\alpha_{n_i}(x_{p+1}) - \alpha_{n_i}(x_p)] = \alpha(x_{p+1}) - \alpha(x_p)$ and since the sequence $\{t^{\lambda_n}\}$ $(n \geqq 0)$ spans the space of all functions $f(x)$ continuous in $[0, 1]$ for which $f(0) = 0$, we get, in the supremum norm,

$$\lim_{n\to\infty}\sum_{m=r+1}^{s}(-1)^{n-m}\lambda_{m+1}\cdot\ldots\cdot\lambda_n\int_0^1[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt = x_{p+1} - x_p.$$

Hence

$$\sum_{p=0}^{k-1}(x_{p+1} - x_p)\cdot M\left(\frac{\alpha(x_{p+1}) - \alpha(x_p)}{x_{p+1} - x_p}\right) \leqq H. \tag{3.5}$$

Since (3.5) holds for every fixed division of $[0, 1]$ we get, by a theorem of Medvedev (**5**), that

$$\alpha(x) = c + \int_0^x f(t)\,dt, \quad \text{where } f \in L_M[0, 1].$$

Hence by (3.4)

$$\mu_n = \int_0^1 t^{\lambda_n}f(t)\,dt, \qquad n = 0, 1, 2, \ldots .$$

*Proof of Theorem* 2. First we prove necessity. By (1.1),

$$|[\mu_m,\ldots,\mu_n]| \leqq \int_0^1(-1)^{n-m}[t^{\lambda_m},\ldots,t^{\lambda_n}]\,|f(t)|\,dt$$

$$\leqq H\int_0^1(-1)^{n-m}[t^{\lambda_m},\ldots,t^{\lambda_n}]\,dt,$$

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where

$$H \equiv \operatorname*{ess\,sup}_{0 \leqq t \leqq 1} |f(t)|.$$

Thus we prove necessity. We now prove the sufficiency, By Lemma 1 and (2.2)

$$\sum_{m=0}^{n} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \, |[\mu_m, \ldots, \mu_n]| \leqq$$

$$H \int_0^1 \left[ \sum_{m=0}^{n} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n [t^{\lambda_m}, \ldots, t^{\lambda_n}] \right] dt \leqq H < \infty.$$

As in the proof of Theorem 1, we get that $\{\mu_n\}$ $(n \geqq 0)$ possesses the representation (3.4). Moreover, if we define the functions $\alpha_n(x)$ by (2.3), we have $\lim_{x\to\infty} \alpha_{n_i}(x) = \alpha(x)$, $0 \leqq x \leqq 1$. Let $x$ and $y$, $0 \leqq x < y \leqq 1$, be two points in $[0, 1]$. There exist $r, s$ (depending on $n$) such that

$$\tilde{t}_{n+1,r+1} \leqq x < \tilde{t}_{n+1,r+2},$$

$$\tilde{t}_{n+1,s+1} \leqq y < \tilde{t}_{n+1,s+2}.$$

By (2.2),

$$\frac{|\alpha_n(y) - \alpha_n(x)|}{\sum_{m=r+1}^{s} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \int_0^1 [t^{\lambda_m}, \ldots, t^{\lambda_n}] \, dt} =$$

$$\frac{\left| \sum_{m=r+1}^{s} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n [\mu_m, \ldots, \mu_n] \right|}{\sum_{m=r+1}^{s} (-1)^{n-m} \lambda_{m+1} \cdot \ldots \cdot \lambda_n \int_0^1 [t^{\lambda_m}, \ldots, . \, t^{\lambda_n}] \, dt} \leqq H.$$

As in the proof of Theorem 1, we get that for every $x$ and $y$, $0 \leqq x < y \leqq 1$,

$$\left| \frac{\alpha(y) - \alpha(x)}{y - x} \right| \leqq H.$$

Hence

$$\alpha(x) = c + \int_0^x f(t) \, dt,$$

where $f \in M[0, 1]$ and by (3.4)

$$\mu_n = \int_0^1 t^{\lambda_n} f(t) \, dt, \qquad m = 0, 1, 2, \ldots,$$

where $f \in M[0, 1]$.

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