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RAY SEQUENCES OF BEST RATIONAL APPROXIMANTS FOR $|x|^\alpha$

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ABSTRACT. The convergence behavior of best uniform rational approximations r^*_{mn} with numerator degree m and denominator degree n to the function $|x|^\alpha$, $\alpha > 0$, on $[-1, 1]$ is investigated. It is assumed that the indices (m, n) progress along a ray sequence in the lower triangle of the Walsh table, *i.e.* the sequence of indices $\{(m, n)\}$ satisfies

$$\frac{m}{n} \longrightarrow c \in [1, \infty) \quad \text{as } m + n \longrightarrow \infty.$$

In addition to the convergence behavior, the asymptotic distribution of poles and zeros of the approximants and the distribution of the extreme points of the error function $|x|^\alpha - r^*_{mn}(x)$ on $[-1, 1]$ will be studied. The results will be compared with those for paradiagonal sequences ($m = n + 2[\alpha/2]$) and for sequences of best polynomial approximants.

1. **Introduction and statements of main results.** Our aim is to investigate the convergence of ray sequences in the Walsh table of the function

$$f(x) = |x|^\alpha, \quad x \in [-1, 1],\ 0 < \alpha. \tag{1.1}$$

Let Π_n denote the collection of all *real* polynomials p of degree at most n and R_{mn} the set of rational functions

$$R_{mn} := \{p/q \mid p \in \Pi_m, q \in \Pi_n, q \not\equiv 0\}, \quad m, n \in \mathbb{N},$$

where $\mathbb{N} := \{0, 1, 2, \ldots\}$. By $r^*_{mn} = r^*_{mn}(f, [-1, 1]; \cdot) \in R_{mn}$ we denote the *best uniform rational approximant* to f on the interval $[-1, 1]$, *i.e.*

$$E_{mn}(f, [-1, 1]) := \|f - r^*_{mn}\|_{[-1,1]} = \inf_{r \in R_{mn}} \|f - r\|_{[-1,1]}, \tag{1.2}$$

where $\|\cdot\|_{[-1,1]}$ denotes the sup norm on $[-1, 1]$.

We know that for each pair $m, n \in \mathbb{N}$ the best rational approximant r^*_{mn} exists and is unique (see [Me], §9.1 and §9.2, or [Ri], §5.1). The doubly infinite array of all rational

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functions $r^*_{mn}, m, n \in \mathbb{N}$, is called the *Walsh table* of the function f approximated on $[-1, 1]$.

An infinite sequence $N = N_c \subseteq \mathbb{N}^2$ of indices (m, n) as well as the corresponding sequence $\{r^*_{mn}\}_{(m,n)\in N_c}$ of approximants is called a *ray sequence* with associated *asymptotic numerator-denominator ratio c* if

$$\frac{m}{n} \longrightarrow c \in [0, \infty] \text{ as } m + n \longrightarrow \infty, \quad (m, n) \in N_c. \tag{1.3}$$

Since the sequence $\{(n, n)\}_{n=0}^{\infty}$ corresponds to the diagonal of the Walsh table, it is called *diagonal*, and any sequence N_c with $c = 1$ is called *near diagonal*, while sequences $\{(n + \lambda, n)\}_{n=M}^{\infty}$ with λ a constant, carry the name *paradiagonal*. The sequence of best polynomial approximants $\{r^*_{m0}\}_{m=0}^{\infty}$ corresponds to the sequence $\{(m, 0)\}_{m=0}^{\infty}$ of indices, *i.e.* they constitute the first column of the Walsh table and best reciprocal polynomial approximants r^*_{0n} correspond to the first row of the Walsh table. We shall investigate sequences in the lower triangle of the Walsh table, *i.e.* $c \in [1, \infty]$. It will turn out that asymptotic behavior of the approximants $\{r^*_{mn}\}_{(m,n)\in N_c}$ essentially depends on the parameter c.

Since f is an even function, it is an immediate consequence of the uniqueness of best real rational approximants that r^*_{mn} is also an even function, and we have

(1.4)

$$r^*_{2m+i,2n+j}(|x|^\alpha, [-1, 1]; \cdot) = r^*_{2m,2n}(|x|^\alpha, [-1, 1]; \cdot) \quad \text{for } m, n \in \mathbb{N} \text{ and } i, j \in \{0, 1\}.$$

Substituting x for x^2 leads to the identity

$$r^*_{2m,2n}(|x|^{2\alpha}, [-1, 1]; t) = r^*_{mn}(x^\alpha, [0, 1]; t^2) \tag{1.5}$$

for all $m, n \in \mathbb{N}$ and $\alpha > 0$. Identity (1.5) shows that an investigation of the Walsh table of the function $|x|^{2\alpha}$ with respect to approximation on the interval $[-1, 1]$ is equivalent to an investigation of the Walsh table of x^α with respect to approximation on $[0, 1]$. For $\alpha \in \mathbb{N}$ both *Walsh tables* are trivial, for in these cases each entry is identical with the function f if $m \geq \alpha$.

The approximation of $|x|$ on $[-1, 1]$ can be seen as a prototype of the somewhat more general problem of approximating $|x|^\alpha$ on $[-1, 1]$. Much attention has been given to both problems in the literature. After the pioneering result by Newman [Ne], who showed in 1964 that

$$\frac{1}{2}e^{-9\sqrt{n}} \leq E_{n,n}(|x|, [-1, 1]) \leq 3e^{-\sqrt{n}} \quad \text{for } n = 4, 5, \ldots, \tag{1.6}$$

a series of results has been published. A rather complete list of contributions can be found in [Vj2] and [SaSt1]. We will mention here only some results that are related to our special interest. In [Vj3] it is shown that there exist constants $0 < M_1 \leq M_2 < \infty$ such that

$$M_1 e^{-\pi\sqrt{n}} \leq E_{n,n}(|x|, [-1, 1]) \leq M_2 e^{-\pi\sqrt{n}} \quad \text{for } n \in \mathbb{N}. \tag{1.7}$$

From (1.7) we learn that $-\pi\sqrt{n}$ is the correct exponent in the error formula. However, nothing is said in [Vj3] about the constants M_1 and M_2 except that, from a result of Bulanow [Bu], it follows that $M_1 \geq 1/3$. Based on high precision calculations it has been conjectured by Varga, Ruttan and Carpenter [VRC] that

$$\lim_{n\to\infty} e^{\pi\sqrt{n}} E_{n,n}(|x|,[-1,1]) = 8. \tag{1.8}$$

This conjecture has been recently proved in [St1].

For the approximation of $|x|^\alpha$ with $\alpha \neq 1$, T. Ganelius [Ga] and N. S. Vjacheslavov [Vj2] have independently proved error estimates that are comparable with (1.7) in their precision. They have shown that there exists a constant $M_1(\alpha) > 0$ for each $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and a constant $M_2(\alpha) < \infty$ for each rational number α such that

$$M_1(\alpha)e^{-\pi\sqrt{\alpha n}} \leq E_{n,n}(|x|^\alpha,[-1,1]) \leq M_2(\alpha)e^{-\pi\sqrt{\alpha n}} \quad \text{for } n \in \mathbb{N}. \tag{1.9}$$

However, it could not be proved that the constant $M_2(\alpha)$ depends continuously on α, so that the upper estimate in (1.9) remained open for all irrational α. In [Ga] a slightly weaker result is proved which is not restricted to rational α. There it is shown that there exist three constants $0 < M_1(\alpha) < M_2(\alpha) < \infty$ and $c(\alpha) < \infty$ for all $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ such that

$$M_1(\alpha)e^{-\pi\sqrt{\alpha n}} \leq E_{n,n}(|x|^\alpha,[-1,1]) \leq M_2(\alpha)e^{-\pi\sqrt{\alpha n}}e^{c(\alpha)\sqrt[4]{n}} \quad \text{for } n \in \mathbb{N}. \tag{1.10}$$

The strong error estimate (1.8) can be extended to the problem of approximating $|x|^\alpha$ on $[-1,1]$. In [St2] a proof of the limit

$$\lim_{n\to\infty} e^{\pi\sqrt{\alpha n}} E_{n,n}(|x|^\alpha,[-1,1]) = 4^{1+\alpha/2}|\sin(\alpha\pi/2)| \tag{1.11}$$

has been announced. Of course, (1.11) implies (1.9) and (1.10). The limit (1.11) has been investigated numerically in [VC2], where the right-hand side of (1.11) was conjectured independently of [St2].

In [An], among other results, the lower bound of (1.10) has been extended to Markov functions of type x^α, *i.e.* to functions f of the form

$$f(z) = \int_0^\infty \frac{\varphi(x)\,dx}{z+x}, \tag{1.12}$$

where φ is a positive function satisfying

$$0 < c_1 \leq x^{-\alpha}\varphi(x) \leq c_2 < \infty \quad \text{for } x \in [0,\epsilon],\ \epsilon > 0.$$

We note that $-z^\alpha$ is of type (1.12).

Turning from diagonal sequences of best rational approximants to best polynomial approximants of $|x|^\alpha$ on $[-1,1]$, we mention the classical results of S. Bernstein [Be1], [Be2] that the limit

$$\lim_{m\to\infty} m^\alpha E_{m,0}(|x|^\alpha,[-1,1]) := \beta(\alpha) \tag{1.13}$$

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exists for $\alpha > 0$. The value of $\beta(1)$ has been calculated with a precision up to 100 digits in [VC1] and numerical investigations of $\beta(\alpha)$ for other values of α can be found in [VC3].

A comparison of (1.10) with (1.13) shows that the rational approximants converge substantially faster than the polynomial ones. Since ray sequences in the Walsh table constitute a bridge between both types of approximants, an essential question which we address in this paper is how the convergence behavior and especially the rate of convergence changes with the variation of the asymptotic numerator-denominator ratio c. The next theorem gives an answer and shows that for all ray sequences with $c \in (0, \infty)$ the rate of convergence is more similar to the diagonal case than to the polynomial one.

THEOREM 1.1. *Let $\alpha > 0$, and let $N_c \subseteq \mathbb{N}^2$ be a ray sequence with numerator-denominator ratio $c \in (0, \infty)$. For any pair of constants $(\underline{c}, \overline{c})$ with*

$$\underline{c} < \min(1, \sqrt{c}) \quad \textit{and} \quad \overline{c} > \max(1, \sqrt{c}), \tag{1.14}$$

we have

$$\left|\sin\left(\frac{\pi}{2}\alpha\right)\right| e^{-\pi\overline{c}\sqrt{\alpha n}} \leq E_{m,n}(|x|^\alpha, [-1, 1]) \leq e^{-\pi\underline{c}\sqrt{\alpha n}} \tag{1.15}$$

for $(m, n) \in N_c$ and $m + n$ sufficiently large.

REMARK. In the theorem we have not excluded the case $\alpha \in 2\mathbb{N}$, although it is a trivial one as mentioned earlier. The sine function on the left-hand side of (1.15) can be replaced by any other bounded function that is zero for $\alpha \in 2\mathbb{N}$ and positive elsewhere. Contrary to the other theorems that will be formulated and discussed in the sequel, we have in Theorem 1.1 not excluded ray sequences from the upper triangle of the Walsh table, *i.e.* $c < 1$.

Theorem 1.1 is an immediate consequence of Ganelius' result stated in (1.10) and the observation that $R_{ll} \subseteq R_{mn} \subseteq R_{kk}$ if $\ell = \min(m, n)$ and $k = \max(m, n)$.

If $c \neq 1$, then $\underline{c} < \overline{c}$ and therefore Theorem 1.1 does not give the precise coefficient in the exponent of the error estimate. The determination of the correct exponent remains open. However, it will turn out that the estimate in (1.15) is good enough for the investigations of the present paper.

It is conjectured that for every ray sequence N_c, $c \in (0, \infty)$, the limit

$$\lim_{(m,n)\in N_c} \frac{-1}{\pi\sqrt{\alpha n}} \log\Big(E_{mn}(|x|^\alpha, [-1, 1])\Big) \tag{1.16}$$

exists. From Theorem 1.1 we only know that if this is true, then the limit has to lie between $\min(1, \sqrt{c})$ and $\max(1, \sqrt{c})$.

For the special case of the function $f(x) = |x|$ it has been proved in [BIS] by Blatt, Iserles, and Saff that the diagonal sequence $\{r^*_{nn}\}$ converges not only on the interval $[-1, 1]$ but also in the two half-planes $H_+ := \{z : \operatorname{Re}(z) > 0\}$ and $H_- := \{z : \operatorname{Re}(z) < 0\}$. We have

$$\lim_{n\to\infty} r^*_{nn}(z) = \begin{cases} z & \text{for } z \in H_+ \\ -z & \text{for } z \in H_-, \end{cases} \tag{1.17}$$

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i.e., there is an overconvergence, which is maximal since the closure of $H_+ \cup H_-$ is $\mathbb{C}$. Further, it has been shown that all poles and zeros of the approximants r^*_{nn} lie on the imaginary axis, and poles and zeros interlace on each imaginary half-axis.

On the other hand, in the polynomial case, there is no overconvergence (see *e.g.* [Sa]). Outside of the interval $[-1,1]$ it can be shown that

$$\lim_{m\to\infty} r^*_{m0}(z) = \infty \tag{1.18}$$

uniformly on every compact subset of $\mathbb{C} \setminus [-1,1]$. Thus, the question arises to what happens in the intermediate case of a ray sequence with $c \in (1,\infty)$. Do the approximants in the intermediate case behave more like the diagonal or more like the best polynomial approximants, or is there some specific intermediate form of behavior? Corollary 1.3 provides an answer to this question.

THEOREM 1.2. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and $r^*_{mn} = r^*_{mn}(|x|^\alpha, [-1,1]; \cdot)$. Then for any ray sequence $N_c \subseteq \mathbb{N}^2$, with $c > 1$, and for the paradiagonal sequence N_c with $c = 1$ and $m = n + 2[\alpha/2]$ for all $(m,n) \in N_c$, we have*

$$\lim_{\substack{m+n\to\infty\\(m,n)\in N_c}} |r^*_{mn}(z)|^{1/(m+n)} = \exp\Big[\frac{c-1}{c+1} g_{\bar{\mathbb{C}}\setminus[-1,1]}(z,\infty)\Big], \tag{1.19}$$

where

$$g_{\bar{\mathbb{C}}\setminus[-1,1]}(z,\infty) := \log\left|z + \sqrt{z^2-1}\right| \tag{1.20}$$

is the Green function of the domain $\overline{\mathbb{C}} \setminus [-1,1]$ with logarithmic pole at ∞. The limit (1.19) holds uniformly on compact subsets of $\mathbb{C} \setminus [-1,1]$ if $c > 1$, and uniformly on compact subsets of $\mathbb{C} \setminus i\mathbb{R}$ if $c = 1$ and $m = n + 2[\alpha/2]$. (By $[\alpha/2]$ we denote the largest integer not greater than $\alpha/2$.)

REMARK. If $c = 1$ and $m = n + 2[\alpha/2]$ for all $(m,n) \in N_1$, then the right-hand side of (1.19) is identically 1 in $\mathbb{C} \setminus i\mathbb{R}$. Hence, in this case zeros or poles of r^*_{mn} can have no cluster points in $\mathbb{C} \setminus i\mathbb{R}$. The situation is different in case of a near-to-diagonal sequence, *i.e.* $c = 1$ with $m > n + 2[\alpha/2]$, for then (1.19) may hold only in some weaker form, because cluster points of zeros of r^*_{mn} off the imaginary axis can no longer be excluded.

COROLLARY 1.3. *If $1 < c \le \infty$, then $(c-1)/(c+1) > 0$, and therefore*

$$\lim_{\substack{m+n\to\infty\\(m,n)\in N_c}} |r^*_{mn}(z)| = \infty \quad \text{for } z \notin [-1,1]. \tag{1.21}$$

REMARK. Since the analytic continuation of $f(z) = |z|^\alpha$ is equal to z^α in H_+ and equal to $(-z)^\alpha$ in H_-, Corollary 1.3 shows that in case of ray sequences with $c > 1$ the phenomenon of overconvergence no longer exists.

Theorem 1.2 as well as the later theorems stated here will be proved in Section 4 after preparations in the next two sections.

It turns out that the overconvergence stated in (1.17) for the function $f(z) = |z|$ has an analogue for the wider class of functions $f(\alpha; z) = z^\alpha$, $\alpha > 0$. Since $f(\alpha;\cdot)$ has a zero of order α at $z = 0$, a natural extension of (1.17) is obtained by using the paradiagonal sequence $\{(n + 2[\alpha/2], n)\}_{n=1}^{\infty}$.

THEOREM 1.4. *Let* $\alpha > 0$ *and* r^*_{mn} *denote the best rational approximant* $r^*_{mn}(|x|^\alpha, [-1,1]; \cdot)$ *for* $m, n \in \mathbb{N}$.

(a) We have

$$\lim_{n\to\infty} r^*_{n+2[\alpha/2],n}(z) = \begin{cases} z^\alpha & \text{for } z \in H_+ \\ (-z)^\alpha & \text{for } z \in H_-, \end{cases} \tag{1.22}$$

uniformly on compact subsets of $H_- \cup H_+ = \mathbb{C} \setminus i\mathbb{R}$.

(b) Let n *be even and* $\alpha \notin 2\mathbb{N}$. *Then the* n *poles and* $n-2$ *of the* $n+2[\alpha/2]$ *zeros of* $r^*_{n+2[\alpha/2],n}$ *lie on the imaginary axis* $i\mathbb{R}$, *the poles are simple, and* $n/2$ *poles and* $n/2-1$ *zeros interlace on each half-axis.*

REMARKS. (1) In case $\alpha = 0, 2, 4, \ldots$, we have $r^*_{n+2[\alpha/2],n}(z) \equiv z^\alpha$ for all $n \in \mathbb{N}$, and part (b) of the theorem does not hold. However, there is a nice intuitive interpretation for these special cases. If α approaches one of the numbers in $2\mathbb{N}$, then the n poles and n of the $n+\alpha$ zeros become pairwise identical and cancel out. The remaining α zeros converge to $z = 0$. The limit (1.22) then holds trivially for all $z \in \mathbb{C}$.

(2) If n is odd, then it follows from (1.4) that $r^*_{n+2[\alpha/2],n} = r^*_{n-1+2[\alpha/2],n-1}$, and therefore part (b) of the theorem is applicable with n replaced by $n-1$. It can be shown that for $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ the poles and zeros of $r^*_{n+2[\alpha/2],n}$ are asymptotically dense in $i\mathbb{R}$ for $n \longrightarrow \infty$ (see [SaSt2]).

Next, we investigate the asymptotic distribution of the extreme points of the error function $f - r^*_{mn}$ on $[-1,1]$. It will turn out that the shape of this distribution depends on the numerator-denominator ratio c of the ray sequence $N_c \subseteq \mathbb{N}^2$. We have seen in (1.5) that all approximants $r^*_{mn}(z) = r^*_{mn}(|x|^\alpha, [-1,1]; z)$ are even functions; hence, we can assume without loss of generality that $m, n \in \mathbb{N}$ are even.

LEMMA 1.5. *For* $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ *and* $m, n \in \mathbb{N}$ *even, there exist* $m+n+3$ *points*

$$-1 = x_1 < x_2 < \cdots < x_{m+n+3} = 1 \tag{1.23}$$

such that

$$(-1)^{(m+n+2)/2+[\alpha/2]+k}\Big[|x_k|^\alpha - r^*_{mn}(x_k)\Big] = E_{m,n}(|x|^\alpha, [-1,1]) \tag{1.24}$$

for $k = 1, \ldots, m+n+3$, *and* $x = x_k$, $k = 1, \ldots, m+n+3$, *are the only points of* $[-1,1]$ *at which the error* $\big|\,|x|^\alpha - r^*_{mn}(x)\big|$ *attains its* sup *norm* $E_{m,n}$.

The lemma follows from Chebyshev's theorem on alternation points (see [Me], Theorem 23, or [Ri], Theorem 5.2) and the fact that

$$W_{mn} := \operatorname{span}\{1, x, \ldots, x^{m/2}, x^{\alpha/2}, x^{1+\alpha/2}, \ldots, x^{(n+\alpha)/2}\} \tag{1.25}$$

forms a Haar space of dimension $(m+n)/2+1$ on $[0,1]$ if $\alpha \notin 2\mathbb{N}$. More details can be found in Section 2, where we give a complete proof of Lemma 1.5.

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The points in (1.23) are called *extreme points*, the set of all such points $\{x_1, \ldots, x_{m+n+3}\}$ is denoted by A_{mn}, and the counting measure of the set A_{mn} is denoted by $\nu_{A_{mn}}$ and defined as

$$\nu_{A_{mn}} := \sum_{x \in A_{mn}} \delta_x, \tag{1.26}$$

where δ_z is Dirac's measure for $z \in \mathbb{C}$. By $\omega_{[-1,1]}$ we denote the equilibrium distribution of the set $[-1, 1]$, *i.e.* $d\omega_{[-1,1]}(x) = (1/\pi)\, dx/\sqrt{1-x^2}$, $x \in [-1, 1]$, and by $\stackrel{*}{\longrightarrow}$ we denote convergence in the weak-star topology on the Riemann sphere; *i.e.* we say that a sequence of measures $\{\nu_n\}$ converges weak-star to a measure ν, written $\nu_n \stackrel{*}{\longrightarrow} \nu$, if $\int f\, d\nu_n \longrightarrow \int f\, d\nu$ as $n \longrightarrow \infty$ for every function f continuous on $\overline{\mathbb{C}}$. The next theorem contains our main result about the asymptotic distribution of extreme points.

THEOREM 1.6. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$. For any ray sequence $N_c \subseteq \mathbb{N}^2$ with numerator-denominator ratio $c \in [1, \infty]$ and $m \geq n + 2[\alpha/2]$ for all $(m, n) \in N_c$ we have*

$$\frac{1}{m+n+3}\nu_{A_{mn}} \stackrel{*}{\longrightarrow} \frac{2}{c+1}\delta_0 + \frac{c-1}{c+1}\omega_{[-1,1]} \tag{1.27}$$

as $m + n \longrightarrow \infty$, $(m, n) \in N_c$.

REMARKS. (1) We see that the asymptotic distribution of the extreme points changes continuously with the numerator-denominator ratio c. Actually, it is a convex combination of the two measures δ_0 and $\omega_{[-1,1]}$, which are the asymptotic distributions in the two extreme cases of the near-to-diagonal sequence with $m = n + 2[\alpha/2]$ and the sequence of best polynomial approximants.

(2) In the special case $m = n + 2[\alpha/2]$ (paradiagonal case) we have $c = 1$, and therefore in this case "almost all" extreme points of $f - r^*_{nn}$ converge to $z = 0$.

(3) In the case of polynomial approximants ($c = \infty$), assertion (1.27) is a special case of a theorem of Kadec [Ka]. However, Theorem 1.6 is somewhat more general since it also covers sequences $\{r^*_{m,n_m}\}_{m=1}^{\infty}$ with $m/n_m \longrightarrow \infty$.

We come to the last group of results in this paper, the asymptotic distribution of zeros and poles of the approximants $r^*_{mn} = r^*_{mn}(|x|^\alpha, [-1, 1]; \cdot)$.

THEOREM 1.7. *Let $\alpha \in \mathbb{R}_+ \setminus 2\mathbb{N}$ and let N_c be a ray sequence with numerator-denominator ratio $c \in [1, \infty)$, $m \geq n + 2[\alpha/2]$, and m, n even for all $(m, n) \in N_c$.*

(a) All n poles of r^*_{mn} are simple and lie on the imaginary axis.

(b) Let $P_{mn} := \{\pi_1, \ldots, \pi_n\}$ and $Z_{mn} = \{\zeta_1, \ldots \zeta_m\}$ denote the sets of (finite) poles and zeros r^*_{mn}. The points can be ordered in such a way that

$$\frac{1}{i}\pi_1 > \frac{1}{i}\zeta_1 > \frac{1}{i}\pi_2 > \frac{1}{i}\zeta_2 > \cdots > \frac{1}{i}\pi_{n/2} > 0 > \frac{1}{i}\pi_{n/2+1} > \cdots > \frac{1}{i}\zeta_{n-2} > \frac{1}{i}\pi_n, \tag{1.28}$$

i.e. there are at least $n-2$ zeros on the imaginary axis, with at least one on each segment joining adjacent poles on each half-axis.

(c) We have

$$\frac{1}{n}\nu_{P_{mn}} \xrightarrow{*} \delta_0, \tag{1.29}$$
$$\frac{1}{m}\nu_{Z_{mn}} \xrightarrow{*} \frac{1}{c}\delta_0 + \left(1 - \frac{1}{c}\right)\omega_{[-1,1]}$$

as $m + n \longrightarrow \infty$, $(m, n) \in N_c$.

(d) If $c > 1$, *then all poles of* r^*_{mn} *converge to* $z = 0$, *and all zeros of* r^*_{mn} *cluster on* $[-1, 1]$, *i.e.*

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{(m,n)\in N_c\\ m+n\geq k}} P_{mn}} = \{0\}, \quad \bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{(m,n)\in N_c\\ m+n\geq k}} Z_{mn}} = [-1, 1]. \tag{1.30}$$

REMARK. Part (d) of Theorem 1.7 holds only for numerator-denominator ratios $c > 1$. It can be shown (see [SaSt2]) that in the special case of the paradiagonal sequence $\{(n+2[\alpha/2], n)\}_{n\in\mathbb{N}}$, poles and zeros are asymptotically dense on the imaginary axis. It follows from Theorem 1.4 that in this case the poles and zeros have no cluster points outside of the imaginary axis. This last result is perhaps not true for all other sequences with numerator-denominator ratio $c = 1$. For the case $m > n+2[\alpha/2]$ and $c = 1$ the results (1.28) and (1.29) are the only ones that we can prove so far. We remark that in the survey paper [SaSt1], formulas (20), (23) and (25) of that paper should be replaced by (1.24), (1.27) and (1.29), respectively.

2. **Connections with Stieltjes functions.** In the present paragraph we establish a connection between best rational approximants and multipoint Padé approximants. This connection will enable us to prove several properties of the best rational approximants r^*_{mn}.

From (1.5) we know that all r^*_{mn} are even functions. It is therefore possible to replace z^2 by z simultaneously in the approximant r^*_{mn} and in the function $|z|^\alpha$. The substitution corresponds to a mapping

$$\varphi: H_+ \longrightarrow D := \overline{\mathbb{C}} \setminus \mathbb{R}_- \quad (\mathbb{R}_- := \{x \in \mathbb{R} \mid x < 0\}) \tag{2.1}$$

with $\varphi(z) = z^2$. Throughout the following sections we denote by f the function

$$f(z) := f(\alpha; z) := z^\alpha \quad \text{for } z \in D \text{ and } \alpha \in \mathbb{R}. \tag{2.2}$$

and by r^*_{mn} we denote the best rational approximant

$$r^*_{mn} = r^*_{mn}(x^\alpha, [0, 1]; \cdot) \in R_{mn} \quad \text{for } m, n \in \mathbb{N},\ \alpha \geq 0. \tag{2.3}$$

In the new setting the best rational approximants r^*_{mn} are identical to $f(\alpha; \cdot)$ for all $\alpha \in \mathbb{N}$ and $m \geq \alpha$. Since these special cases are trivial, we will exclude them from further considerations and assume that

$$\alpha \in \mathbb{R}_+ \setminus \mathbb{N}. \tag{2.4}$$

As as immediate consequence of Cauchy's integral formula we have the representation

$$f(\alpha; z) = -z^{[\alpha]+1} \frac{\sin\big(\pi(\alpha - [\alpha] - 1)\big)}{\pi} \int_0^\infty \frac{x^{(\alpha-[\alpha]-1)}\, dx}{x + z} \quad \text{for all } z \in D. \tag{2.5}$$

Since $-1 < \alpha - [\alpha] - 1 < 0$, the integral in (2.5) defines a Stieltjes function. It is easy to check that the integral exists for all $z \in D$.

Before we turn to the study of the connection of r^*_{mn} with the theory of multipoint Padé approximants, we prove a lemma that covers most of the assertions of Lemma 1.5. The full proof of Lemma 1.5 will be given at the end of the present section.

LEMMA 2.1. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, and set $r^*_{mn} = p_{mn}/q_{mn}$ with p_{mn} and q_{mn} coprime and q_{mn} monic, $m \geq n+[\alpha]$, $m, n \in \mathbb{N}$. Then $\deg(p_{mn}) = m$ and $\deg(q_{mn}) = n$, and there exists a set of $m + n + 2$ alternation points in $[0, 1]$, i.e., we have $0 \leq x_1 < \cdots < x_{m+n+2} \leq 1$, and the error function*

$$e_{mn} := f(\alpha; \cdot) - r^*_{mn} \tag{2.6}$$

satisfies

$$e_{mn}(x_j) = \lambda(-1)^j E_{mn}(x^\alpha, [0, 1]), \quad j = 1, \ldots, m + n + 2, \tag{2.7}$$

with $\lambda = 1$ or $\lambda = -1$.

PROOF. It has already been mentioned in the introduction that the best rational approximant r^*_{mn} exists and is unique for each m, $n \in \mathbb{N}$. Let $x_1, \ldots, x_{m+n+2-d}$ be a sequence of alternation points of maximal length. From Chebyshev's Theorem on alternation points (see [Me], Theorem 23, or [Ri], Theorem 5.2) we know that

$$d \leq \min\big(m - \deg(p_{mn}), n - \deg(q_{mn})\big). \tag{2.8}$$

We shall show that $d = 0$.

From the intermediate value theorem and the definition of alternation points it follows that there exist at least $m + n + 1 - d$ zeros z_j of e_{mn} satisfying

$$x_j < z_j < x_{j+1} \tag{2.9}$$

and

$$e_{mn}(z_j) = 0 \quad \text{for } j = 1, \ldots, m + n + 1 - d. \tag{2.10}$$

From (2.10) we learn that r^*_{mn} interpolates the function $f(\alpha; \cdot)$ in the points $z_1, \ldots, z_{m+n+1-d}$, and the expression

$$q_{mn} e_{mn} = q_{mn} f(\alpha; \cdot) - p_{mn} \tag{2.11}$$

has zeros at the points $z_1, \ldots, z_{m+n+1-d}$. For $z \in [0, 1]$ the right-hand side of (2.11) is an element of the space

(2.12)

$$W_{m'n'} := \operatorname{span}\{1, z, \ldots, z^{m'}, z^\alpha, z^{\alpha+1}, \ldots, z^{\alpha+n'}\}, \quad m' := \deg(p_{mn}), \quad n' := \deg(q_{mn}).$$

Since $W_{m'n'}$ forms a Chebyshev system on $[0,1]$ of dimension $m' + n' + 2$ (see [KaSt], Chapter I, §3), we know that (2.11) and therefore also e_{mn} has at most $m' + n' + 1$ zeros in $[0,1]$. Hence, $m + n + 1 - d \leq 1 + \deg(p_{mn}) + \deg(q_{mn})$.

On the other hand, it follows from (2.8) that $2d+\deg(p_{mn})+\deg(q_{mn}) \leq m+n$. The last two inequalities together imply that $d = 0$. Thus, we have proved that there exist $m+n+2$ alternation points, and further we have shown that $\deg(p_{mn}) = m$ and $\deg(q_{mn}) = n$. ∎

From (2.7), (2.9) and (2.10) it follows that the rational approximant r^*_{mn} interpolates $f(\alpha;\cdot)$ in the $m + n + 1$ points of the set

$$B_{mn} := \{z_1, \ldots, z_{m+n+1}\}. \tag{2.13}$$

We shall see below that these are the only zeros of e_{mn} on $\mathbb{R}_+$. The polynomial

$$\omega_{mn}(z) := \prod_{x\in B_{mn}} (z - x) \in \Pi_{m+n+1}, \tag{2.14}$$

will be frequently used.

The next lemma contains most of the results about the structure of the denominator and numerator polynomials of r^*_{mn} that can be deduced from the interpolatory property of r^*_{mn}.

LEMMA 2.2. *Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$, and let the rational function $r_{mn} = p_{mn}/q_{mn} \in R_{mn}$ interpolate $f(\alpha;\cdot)$ in the points of $B_{mn} = B_{mn}(\alpha) \subseteq (0,1)$, p_{mn} and q_{mn} coprime, and assume that $m \geq n + [\alpha]$.*

(a) The denominator polynomial q_{mn} satisfies the orthogonality relation

$$\int_0^\infty x^\ell \frac{q_{mn}(-x)}{\omega_{mn}(-x)} x^\alpha \, dx = 0 \quad \text{for } \ell = 0, \ldots, n-1, \tag{2.15}$$

where ω_{mn} is the polynomial defined in (2.14). All zeros of q_{mn} are simple, contained in $(-\infty, 0)$, and their total number is exactly n.

(b) The numerator polynomial p_{mn} is of exact degree m, and in the segment between two adjacent zeros of q_{mn} there is at least one zero of p_{mn}.

(c) For the error function $e_{mn} := f(\alpha;\cdot) - r^*_{mn}$ we have the representation

$$e_{mn}(z) = -\frac{\sin \pi\alpha}{\pi} \frac{\omega_{mn}(z)}{q_{mn}(z)^2} \int_0^\infty \frac{q_{mn}(-x)^2 x^\alpha \, dx}{\omega_{mn}(-x)(x+z)} \quad \text{for } z \in D. \tag{2.16}$$

PROOF. (a) We will write p, q, ω, f instead of p_{mn}, q_{mn}, ω_{mn}, $f(\alpha;\cdot)$, respectively. Since r_{mn} interpolates f in the points of B_{mn}, we have

$$qf - p = \omega g \tag{2.17}$$

with g a function analytic in D. Let D_0 be a simply connected domain with rectifiable boundary ∂D_0, $\overline{D}_0 \subseteq D$, $B_{mn} \subseteq D_0$, and let C be the positively oriented boundary ∂D_0.

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Multiplying (2.17) by z^ℓ, $\ell = 0, \ldots, n-1$, and dividing by ω yields with Cauchy's theorem that

$$(2.18)\qquad \oint_C \zeta^\ell \frac{q(\zeta)}{\omega(\zeta)}\zeta^\alpha\, d\zeta - \oint_C \zeta^\ell \frac{p(\zeta)}{\omega(\zeta)}\, d\zeta = \oint_C \zeta^\ell g(\zeta)\, d\zeta = 0.$$

Since $\deg(\omega) = m+n+1 \geq m+\ell+2 \geq \deg(p)+\ell+2$ if $\ell \leq n-1$, we see that in the second integral on the left-hand side of (2.18) the integrand has a zero of order at least 2 at infinity, and the integrand is analytic outside C. Hence, this integral vanishes, and we have

$$(2.19)\qquad \oint_C \zeta^\ell \frac{q(\zeta)}{\omega(\zeta)}\zeta^\alpha\, d\zeta = 0 \quad \text{for } \ell = 0, \ldots, n-1.$$

If we let C deform to the boundary of an annulus slit along the negative real axis and let its inner radius tend to 0 and its outer radius tend to ∞, then the integral in (2.19) converges to (2.15), provided that the integral in (2.15) exists. Here only the behavior of the integrand near $\zeta = 0$ and $\zeta = \infty$ is critical. The modulus of the integrand at $\zeta = 0$ behaves like $|\zeta|^{\alpha+l}$ since $B_{mn} \subseteq (0,1)$. At $\zeta = \infty$ it behaves like $|\zeta|^\beta$

$$(2.20)\qquad \beta \leq (2n-1)-(m+n+1)+\alpha \leq -[\alpha]+\alpha-2 < -1.$$

From (2.20) we deduce that the integral in (2.15) exists. In (2.20) we have used the assumption $m \geq n+[\alpha]$.

From the orthogonality relation (2.14) it follows rather immediately that $\deg(q) = n$, and that all its zeros are simple and contained in $(-\infty, 0)$ (see [Sz], Chapter III).

(c) We continue with the proof of assertion (c) and defer the proof of assertion (b) until later. Let $h \in \Pi_n$ be an arbitrary polynomial, and let the domain $D_0 \subseteq D$, its contour C, the polynomials p, q, ω, and the function g be the same as those used in (2.17) and (2.18). By Cauchy's integral formula we deduce from (2.17) that

$$(2.21)\qquad \frac{1}{2\pi i}\oint_C \frac{hq}{\omega}(\zeta)\frac{\zeta^\alpha}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\oint_C \frac{hp}{\omega}(\zeta)\frac{d\zeta}{\zeta - z} = \frac{1}{2\pi i}\oint_C \frac{(hg)(\zeta)}{\zeta - z}\, d\zeta = (hg)(z)$$

for all $z \in D_0$. Since the degree of hp is smaller than that of ω, the rational function hp/ω is analytic outside of D_0 and has a zero at infinity. The second integral in (2.21) is therefore identically zero. With the identity $qhe_{mn} = qhf - hp$ we have thus proved that

$$(2.22)\qquad e_{mn}(z) = \frac{\omega(z)}{(qh)(z)2\pi i}\oint_C \frac{qh}{\omega}(\zeta)\frac{\zeta^\alpha\, d\zeta}{\zeta - z} \quad \text{for } z \in D_0.$$

Choosing $h = q$ and letting C deform to $[-\infty, 0]$ as before yields (2.16). We note that the integral in (2.16) exists for all $z \in D$, which can be proved in the same way as the existence of the integral (2.15) in part (a) has been verified. Further, we note that the factor $(\sin \pi\alpha)/\pi$ arises from the analytic continuation of ζ^α to $(-\infty, 0)$ from both sides. The technique is the same as that in the derivation of formula (2.5).

(b) We use the same notation as in the proof of part (c). From (2.22) and the identity $qe_{mn} = qf(\alpha;\cdot) - p$ it follows that

$$\begin{aligned} p(z) = q[f(\alpha;\cdot) - e_{mn}](z) &= \frac{q(z)}{2\pi i}\oint_C \frac{\zeta^\alpha\, d\zeta}{\zeta - z} - \frac{\omega(z)}{h(z)2\pi i}\oint_C \frac{qh}{\omega}(\zeta)\frac{\zeta^\alpha\, d\zeta}{\zeta - z} \\ &= \frac{1}{h(z)2\pi i}\oint_C \frac{(qh)(z)\omega(\zeta) - (qh)(\zeta)\omega(z)}{\zeta - z}\frac{\zeta^\alpha\, d\zeta}{\omega(\zeta)} \end{aligned} \tag{2.23}$$

for $z \in D_0$. If we choose $h = q$ and develop the right-hand side of (2.23) in powers of z, and if we assume that q is monic, *i.e.* $q(z) = z^n + \cdots$, then it follows from the identity

$$\frac{q(z)^2\omega(\zeta) - q(\zeta)^2\omega(z)}{\zeta - z} = \frac{\omega(\zeta) - \omega(z)}{\zeta - z}q(\zeta)^2 - \frac{q(\zeta)^2 - q(z)^2}{\zeta - z}\omega(\zeta) \tag{2.24}$$

that in the development $p(z) = a_m z^m + \cdots$ the leading coefficient a_m is given by

$$a_m = \frac{1}{2\pi i}\oint_C \frac{q(\zeta)^2}{\omega(\zeta)}\zeta^\alpha\, d\zeta. \tag{2.25}$$

Note that because of $m \geq n + [\alpha] \geq n$ we have $\deg(\omega) = m + n + 1 \geq 2n + 1 > 2n = \deg(q^2)$. As in (2.20) we can show that the curve C in (2.25) can be deformed to $[-\infty, 0]$, and we have

$$a_m = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \frac{q(-x)^2}{\omega(-x)}x^\alpha\, dx \neq 0, \tag{2.26}$$

which proves $\deg(p) = m$.

Next, we show that between two adjacent zeros of q there is at least one zero of p. Let $y_1, \ldots, y_n$ be the zeros of q numbered according to their value. From part (a) we know that all zeros of q lie in $(-\infty, 0)$. Since all zeros are simple, we know that the partial fraction representation of p/q has the form

$$\frac{p}{q}(z) = \sum_{i=1}^{n} \frac{\lambda_i}{z - y_i} + P(z) \tag{2.27}$$

with $P \in \Pi_{m-n}$. Since $f(\alpha;\cdot)$ has an analytic continuation across $(-\infty, 0)$ from both sides and since $f(\alpha;\cdot)$ is therefore bounded at each zero y_j of q, the rational function (2.27) is dominant in the error function $e_{mn} = f(\alpha;\cdot) - p/q$ near every pole y_j, $j = 1, \ldots, n$, of p/q. From the analytic continuation property of $f(\alpha;\cdot)$ we note that in (2.22) teh contour C can be deformed so as to contain y_j in its interior. We therefore deduce with $h_j(z) := q(z)/(z - y_j) \in \Pi_{n-1}$ that

$$\begin{aligned} -\lambda_j = \operatorname{Res}_{z=y_j} e_{mn}(z) &= \lim_{z\to y_j} e_{mn}(z)(z - y_j) \\ &= \frac{\omega(y_j)}{2\pi i q'(y_j)^2}\oint_C \left(\frac{q(\zeta)}{\zeta - y_j}\right)^2 \frac{\zeta^\alpha\, d\zeta}{\omega(\zeta)} \\ &= \frac{\omega(y_j)}{q'(y_j)^2}\frac{\sin \pi\alpha}{\pi}\int_0^\infty h_j(-x)^2 \frac{x^\alpha\, dx}{\omega(-x)} \neq 0. \end{aligned} \tag{2.28}$$

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The last equality is a result of deforming C to $[-\infty, 0]$, and the integral cannot vanish since the integrand does not change its sign on $[0, \infty]$.

Since α as well as the sign of $\sin \pi\alpha$ is fixed, it follows from (2.28) that all $\lambda_1, \ldots, \lambda_n$ are of same sign. It therefore follows from (2.27) that p has at least one zero in each interval (y_j, y_{j+1}), $j = 1, \ldots, n-1$. ∎

PROOF OF LEMMA 1.5. In view of Lemma 2.1 it remains only to be shown that the alternation points $x_1, \ldots, x_{m+n+2}$ in (2.7) are the only extreme points of e_{mn} on $[0, 1]$ and further that $x_1 = 0$, $x_{m+n+2} = 1$, and $\lambda = (-1)^{[\alpha]}$ in (2.7). All assertions of Lemma 1.5 then follow from Lemma 2.1 by using identity (1.5), which describes the connection between the problem of approximating x^α on $[0, 1]$ and $|x|^{2\alpha}$ on $[-1, 1]$.

Set $r^*_{mn} = p_{mn}/q_{mn}$. For $c \in \mathbb{R}$ we have

$$e_{mn}(z) = c \tag{2.29}$$

for some $z \in [0, 1]$ if and only if

$$z^\alpha q_{mn}(z) - c q_{mn}(z) - p_{mn}(z) = 0. \tag{2.30}$$

Since the left-hand side of (2.30) is an element of W_{mn} defined in (2.12), for each $c \in \mathbb{R}$ there exist at most $m+n+1$ zeros of (2.29) in $[0, 1]$. The extreme points of e_{mn} satisfy (2.29) with $c = E_{m,n}(x^\alpha, [0, 1])$ or $c = -E_{m,n}(x^\alpha, [0, 1])$. If an extreme point lies in $(0, 1)$, then it is at least a double zero of (2.29), while at the endpoints 0 and 1 it may correspond to a simple zero of (2.29). Since we know from Lemma 2.1 that there are at least $m+n+2$ alternation points, it follows that these are the only extreme points, and more than that, it follows that the two endpoints 0 and 1 have to be among these points.

In order to prove $\lambda = (-1)^{[\alpha]}$ in (2.7), we consider e_{mn} near infinity. For $m > n + [\alpha]$ the approximant $r^*_{mn}(z) = a_m z^{m-n} + \cdots$ is dominant in e_{mn} near infinity. From (2.26) we then know that

$$\operatorname{sign}(a_m) = (-1)^{[\alpha]} \operatorname{sign}\{\omega_{mn}(x) \mid x \in \mathbb{R}_-\} = (-1)^{[\alpha]+m+n+1}, \tag{2.31}$$

which is the same as $-\operatorname{sign} e_{mn}(z)$ for $z \in \mathbb{R}_+$ near infinity. From (2.16) we know that e_{mn} has exactly $m+n+1$ sign changes on $\mathbb{R}_+$ which shows that

$$\lambda = -\operatorname{sign} e_{mn}(0) = (-1)^{[\alpha]} \tag{2.32}$$

for $m > n + [\alpha]$. The case $m = n + [\alpha]$ follows in the same way if we observe that $\operatorname{sign}\big(e_{mn}(z)\big) = 1$ for $z > 1$. ∎

3. **Some results involving logarithmic potentials.** In order to simplify notation we assume that in every ray sequence $N_c = \{(m, n)\} \subseteq \mathbb{N}^2$ each index n appears only once and therefore we can consider m as a function of n, *i.e.* $m = m_n$, $n \in N \subseteq \mathbb{N}$, and $\{(m, n)\} = \{(m_n, n)\}_{n \in N}$. It is obvious that such an assumption can be made without loss

of generality. We also will change the parameterization of the sequences N_c. Instead of the numerator-denominator ratio c we now use the parameter Θ, which is defined by

$$\lim_N \frac{m_n}{n} =: 1 + 2\Theta. \tag{3.1}$$

From (1.3) we see that $\Theta = (c-1)/2$. We shall write $n \in N_\Theta$ if $(m_n, n) \in N_c$ and call N_Θ the index sequence of N_c. A ray sequence belongs asymptotically to the lower triangle of the Walsh table if $\Theta \geq 0$. Throughout the present paragraph, we make the assumption that

$$m_n \geq n + [\alpha] \quad \text{for all } n \in N_\Theta. \tag{3.2}$$

The assumption (3.2) implies $\Theta \geq 0$.

In order to have a further simplification of notation we write q_n, p_n, ω_n instead of $q_{m_n,n}$, $p_{m_n,n}$, $\omega_{m_n,n}$, respectively. Since zeros of q_n can tend to $-\infty$ as $n \longrightarrow \infty$, we have to normalize q_n in a way that avoids q_n from "blowing up" if zeros tend to $-\infty$. The monic polynomials are not appropriate for this purpose. In the present section we assume that the polynomials q_n are normalized so that

$$q_n(z) = \prod_{|y_j|\leq 1} (z - y_j) \prod_{|y_j|>1} \frac{z - y_j}{|y_j|} = \prod_{j=1}^{n} \frac{z - y_j}{\max(1, |y_j|)} \tag{3.3}$$

holds true, where $y_1, \ldots, y_n$ are the n zeros of q_n.

For any polynomial $p \in \Pi_n$ we denote by ν_p the counting measure of its zeros, *i.e.* ν_p associates a mass to each zero of p that is equal to the order of the zero. From the weak compactness of the unit ball of positive measures it follows that any ray sequence N_Θ contains an infinite subsequence, which we continue to denote by N_Θ, so that measures ν, ω and a constant c_0 exist with

$$\frac{1}{n}\nu_{q_n} \xrightarrow{\;*\;} \nu,\ \frac{1}{2n}\nu_{\omega_n} \xrightarrow{\;*\;} \omega,\ \frac{1}{2n}\log|I_n(1)| \longrightarrow c_0 \in \overline{\mathbb{R}} \quad \text{as } n \longrightarrow \infty,\ n \in N_\Theta, \tag{3.4}$$

where I_n denotes the integral

$$I_n(z) := \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^{0} \frac{q_n(x)^2 |x|^\alpha \, dx}{(x - z)\, \omega_n(x)}, \quad z \in \mathbf{C} \setminus [-\infty, 0]. \tag{3.5}$$

In Lemma 2.2(a) it has been shown that all zeros of q_n are contained in $(-\infty, 0)$, and that $\deg(q_n) = n$. Hence, ν is a probability measure with

$$\operatorname{supp}(\nu) \subseteq [-\infty, 0]. \tag{3.6}$$

Since the polynomial ω_n has $m_n + n + 1$ zeros and since all these zeros are contained in $(0, 1)$, it follows from (3.1) and (3.3) that ω is a positive measure with

$$\operatorname{supp}(\omega) \subseteq [0, 1] \quad \text{and} \quad \omega([0, 1]) = 1 + \Theta. \tag{3.7}$$

By

$$p(\mu;z) := \int \log \frac{\max(1,|x|)}{|z-x|}\, d\mu(x) \tag{3.8}$$

we denote the logarithmic potential of a given signed measure μ.

Definition (3.8) differs somewhat from the usual one (see for instance [La], Chapter I, or [StTo], Appendix), but it has the advantage that it is not affected by a strong growth of the measure μ near infinity. The definition can be reduced to a combination of logarithmic potentials that are defined in the usual way. Indeed, let $\mu = \mu_1 + \mu_2$ be a decomposition of μ such that $\operatorname{supp}(\mu_1) \subseteq \{z \mid |z| \le 1\}$ and $\operatorname{supp}(\mu_2) \subseteq \{z \mid |z| \ge 1\}$. Let μ_2^* denote the image of μ_2 under the mapping $x \longmapsto 1/x$. Then $\operatorname{supp}(\mu_2^*) \subseteq \{z \mid |z| \le 1\}$, and we have the identity

$$\begin{aligned} p(\mu;z) &= \int \log \frac{\max(1,|x|)}{|z-x|}\, d\mu(x) = \int \log \frac{1}{|z-x|}\, d\mu_1(x) + \int \log \frac{|x|}{|z-x|}\, d\mu_2(x) \\ &= \int \log \frac{1}{|z-x|}\, d\mu_1(x) + \int \log \frac{1}{|1/z-x|}\, d\mu_2^*(x) + \|\mu_2(\overline{\mathbb{C}})\| \log \frac{1}{|z|} \\ &= p(\mu_1;z) + p\Big(\mu_2^*; \frac{1}{z}\Big) + \|\mu_2(\overline{\mathbb{C}})\| \log \frac{1}{|z|}. \end{aligned} \tag{3.9}$$

Note that if $\operatorname{supp}(\mu) \subseteq \{z \mid |z| \le 1\}$, then we have

$$p(\mu;z) = \int \log \frac{1}{|z-x|}\, d\mu(x), \tag{3.10}$$

which is the usual definition of a logarithmic potential.

LEMMA 3.1. *We have*

$$\lim_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z)}{q_n(z)^2} \right| = p(\nu-\omega;z) \tag{3.11}$$

locally uniformly $z \in \mathbb{C} \setminus [-\infty, 1]$,

$$\limsup_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z)}{q_n(z)^2} \right| = p(\nu-\omega;z) \tag{3.12}$$

for quasi-every $z \in \mathbb{C} \setminus [-\infty, 0]$, *and for every sequence of points* $z_n \in \mathbb{C}$, $n \in N_\Theta$, *with* $z_n \longrightarrow z_0 \in \mathbb{C} \setminus [-\infty, 0]$ *as* $n \longrightarrow \infty$, $n \in N_\Theta$, *we have*

$$\limsup_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z_n)}{q_n(z_n)^2} \right| \le p(\nu-\omega;z_0). \tag{3.13}$$

Further we have

$$\liminf_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z)}{q_n(z)^2} \right| = p(\nu-\omega;z) \tag{3.14}$$

for quasi-every $z \in \mathbb{C} \setminus [0,1]$, *and for every sequence of points* $z_n \in \mathbb{C}$, $n \in N_\Theta$, *with* $z_n \longrightarrow z_0 \in \mathbb{C} \setminus [0,1]$ *as* $n \longrightarrow \infty$, $n \in N_\Theta$, *we have*

$$\liminf_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z_n)}{q_n(z_n)^2} \right| \geq p(\nu - \omega; z_0). \tag{3.15}$$

REMARK. We write $\lim_{N_\Theta}$ instead of $\lim_{n\to\infty, n\in N_\Theta}$. A property is said to hold *quasi-everywhere* on a set $V \subseteq \mathbb{C}$ if it holds for every $z \in V$ with possible exceptions on a set of capacity zero (see [La], Chapter II, No. 6).

PROOF. The limit (3.11) follows from the first two limits in (3.4) and the fact that all zeros of q_n and ω_n are contained in $(-\infty, 1)$. The asymptotic inequality (3.13) follows from the first two limits in (3.4) together with the principle of descent (see [La], Theorem 1.3), which has to be applied to the sequence of potentials $(1/2n)\log[1/\omega_n(z)]$. This sequence converges to $p(\omega; z)$. In the same way the asymptotic inequality (3.15) follows from the first two limits in (3.4) and the principle of descent, but now the principle of descent has to be applied to the functions $(1/n)\log|1/q_n(z)|$, which are not of the form covered by Theorem 1.3 in [La] because of the normalization (3.3). However, by using the decomposition described in (3.9), it is immediate that the principle of descent proved in Theorem 1.3 of [La] is also applicable to the sequence $\{(1/n)\log|1/q_n(z)|\}$. The limits (3.12) and (3.14) follow from the first two limits in (3.4) and the lower envelope theorem of potential theory (see [La], Theorem 3.8), where in case of the limit (3.14) the decomposition (3.9) again has be used in order to justify the applicability of Theorem 3.8 of [La]. ∎

In (3.4) we have only assumed the convergence of the sequence $(1/2n)\log|I_n(z)|$, $n \in N_\Theta$, at $z = 1$, where we allow the limits $\pm\infty$. The integral I_n has been defined in (3.5).

LEMMA 3.2. *(a) We have*

$$I_n(z) \neq 0 \quad \textit{for all } z \in D = \overline{\mathbb{C}} \setminus \mathbb{R}_-. \tag{3.16}$$

(b) For every compact set $V \subseteq D$ *we have*

$$\lim_{N_\Theta} \frac{1}{2n} \log|I_n(z)| = c_0 \tag{3.17}$$

uniformly for $z \in V$.

REMARK. The value of c_0 may depend on the subsequence N_Θ. Note that as yet we do not know whether c_0 is finite.

PROOF. (a) The first assertion follows from the observation that

$$\operatorname{Im}(z) \cdot \operatorname{Im}\left(\frac{1}{z+x}\right) < 0 \tag{3.18}$$

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for all $\mathrm{Im}(z) \neq 0$ and $x \in \mathbb{R}_+$ and

$$\mathrm{Re}\Big(\frac{1}{z+x}\Big) > 0 \tag{3.19}$$

for all $z, x \in \mathbb{R}_+$.

(b) The function $q_n(-x)^2 x^\alpha / \omega_n(-x)$ does not change sign on $(0,\infty)$, and it is positive or negative depending on whether $m_n + n + 1$ is even or odd. Since we have assumed in (3.2) that $m_n \geq n + [\alpha]$, for fixed $z \in D$ we have the estimate

$$\Big|\frac{q_n(-x)^2 x^\alpha}{\omega_n(-x)|x+z|}\Big| \leq c_3 x^{\alpha-[\alpha]-2} \quad \text{for all } x \in [0,\infty], \tag{3.20}$$

where c_3 is a constant that depends on z, q_n, ω_n, but is independent of x. Inequality (3.20) can easily be verified for large x by moving all zeros of the polynomials ω_n and q_n to the point $x = 0$ and using the inequality $2n + \alpha - (m_n + n + 1) - 1 \leq \alpha - [\alpha] - 2$. For finite x, inequality (3.20) follows from the fact that all zeros of ω_n are contained in $(0, 1)$. It follows from (3.20) that the integral in (3.5) exists for each $n \in N_\Theta$ and $z \in D$.

For a given z with $\mathrm{Im}(z) > 0$, the image of $(0,\infty)$ under the mapping $x \mapsto 1/(x+z)$ is contained in the lower half-plane $\{w \mid \mathrm{Im}(w) < 0\}$. Hence, for $\mathrm{Im}(z) > 0$ we have $\mathrm{Im}\big(I_n(z)/I_n(1)\big) < 0$. Note that $I_n(1)$ is real. For $\mathrm{Im}(z) < 0$, the opposite inequality $\mathrm{Im}\big(I_n(z)/I_n(1)\big) > 0$ holds. From these observations we deduce that

$$\big|\arg\big(I_n(z)\big) - \arg\big(I_n(1)\big)\big| < \pi \quad \text{for all } z \in D. \tag{3.21}$$

Hence,

$$\lim_{N_\Theta} \frac{1}{2n}\big|\arg\big(I_n(z)\big) - \arg\big(I_n(1)\big)\big| = 0 \tag{3.22}$$

locally uniformly in D. Since $\arg\big(I_n(z)\big)$ is the harmonic conjugate of $\log|I_n(z)|$, limit (3.17) follows from (3.22), the third limit in (3.4), and Schwarz's representation formula for the conjugate function. ∎

The inequality and the equality proved in the next lemma are of basic importance.

LEMMA 3.3. *We have*

$$p(\nu-\omega; z) + c_0 \begin{cases} = 0 & \text{for all } z \in [0,1] \\ \geq 0 & \text{for all } z \in [-\infty, 0]. \end{cases} \tag{3.23}$$

The constant $c_0 \in \mathbb{R}$ has been defined in (3.4), and it is finite.

PROOF. (a) We start by showing that the inequality

$$p(\nu-\omega; z) + c_0 \leq 0 \quad \text{for all } z \in [0,1] \tag{3.24}$$

is a consequence of the error estimate (1.15) in Theorem 1.1. From the upper estimate in (1.15) it follows that

$$\limsup_{N_\Theta} \frac{1}{2n} \log|e_n(z)| \leq \lim_{N_\Theta} \frac{1}{2n}(-c_1\sqrt{\alpha 4n}) = 0 \tag{3.25}$$

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uniformly for $z \in [0,1]$, where c_1 is a constant smaller than π. We note that because of the substitution $z^2 \longmapsto z$ we have to substitute the degree n by $2n$ in the error estimate (1.15).

From the remainder formula (2.16) of Lemma 2.2 together with (3.25) we know that

$$0 \geq \limsup_{N_\Theta} \frac{1}{2n} \log |e_n(z)| = \limsup_{N_\Theta} \frac{1}{2n} \left[\log \left| \frac{\omega_n(z)}{q_n(z)^2} \right| + \log |I_n(z)| \right] \tag{3.26}$$

uniformly for $z \in [0,1]$. The function I_n in (3.26) has been defined in (3.5). From (3.12) of Lemma 3.1 we know that

$$\limsup_{N_\Theta} \frac{1}{2n} \log \left| \frac{\omega_n(z)}{q_n(z)^2} \right| = p(\nu - \omega; z) \tag{3.27}$$

for quasi-every $z \in [0,1]$. Together with the limit (3.17) in Lemma 3.2 we deduce from (3.27) and (3.26) that the inequality (3.24) holds for quasi-every $z \in [0,1]$. The set $F := \{z \in [0,1] \mid p(\nu - \omega; z) + c_0 > 0\}$ is thin near every $x \in [0,1]$. Hence, in the fine to;ology, the set F belongs to the fine boundary of $[0,1] \setminus F$ (see [La, chapter V, Section 3]) and because of the continuity of $p(\nu - \omega; z)$ in the fine topology, the inequality (3.24) holds for all $z \in (0,1]$.

(b) Next we show the inequality in (3.23). Let x be an arbitrary point of $(-\infty, 0)$ and $\delta > 0$. Because of the upper semicontinuity of $p(-\nu; z)$ and the continuity of $p(\omega; z)$ on $(-\infty, 0)$ there exists an $\varepsilon > 0$ such that

$$\begin{aligned} p(-\nu; x) &\leq \max_{z \in I} p(-\nu; z), \\ p(\omega; z) - \delta &\leq \min_{z \in I} p(\omega; z) \end{aligned} \tag{3.28}$$

for all $z \in I := I_{\varepsilon,x} := [x - \varepsilon, x + \varepsilon] \subseteq (-\infty, 0)$. Denote by M_n the maximum

$$M_n := \max_{z \in I} |q_n(z)|. \tag{3.29}$$

Because of the first limit in (3.4) and the principle of descent (see [La], Theorem 1.3) we know that

$$\limsup_{N_\Theta} \frac{1}{n} \log |q_n(z_n)| \leq p(-\nu; z_0) \tag{3.30}$$

for any sequence $z_n \longrightarrow z_0 \in I$. (That the principle of descent and the lower envelope theorem holds for potentials of type (3.8) has been shown in the proof of Lemma 3.1.) From the lower envelope theorem (see [La], Theorem 3.8) we know that

$$\limsup_{N_\Theta} \frac{1}{n} \log |q_n(z)| = p(-\nu; z) \tag{3.31}$$

for quasi-every $z \in I$. It follows from (3.30) that

$$\limsup_{N_\Theta} \frac{1}{n} \log M_n \leq \max_{z \in I} p(-\nu; z). \tag{3.32}$$

Since (3.31) holds for any infinite subsequence of N_Θ, it follows from (3.30) and (3.31) that

$$M_n \ge e^{-n\delta} c_2^n \tag{3.33}$$

for all $n \in N_\Theta$ sufficiently large with

$$\log c_2 := \max_{z\in I} p(-\nu; z). \tag{3.34}$$

From the uniform convergence of $(1/2n)\log|\omega_n(z)| \longrightarrow p(-\omega; z)$ on I, it follows that there exists a constant c_3 such that

$$\left|\frac{\sin \pi\alpha}{\pi} \frac{|z|^\alpha}{\omega_n(z)} \operatorname{Im}\left(\frac{1}{-z+z_0}\right)\right| \ge c_3^{2n} \tag{3.35}$$

for all $z \in I$, n sufficiently large, $z_0 \in \mathbb{C} \setminus \mathbb{R}$ fixed, and the constant c_3 satisfies

$$\log c_3 \le \min_{z\in I} p(\omega; z) \le \log c_3 + 2\delta, \tag{3.36}$$

where we have used inequality (3.28).

Let $x_n \in I$ be such that $M_n = |q_n(x_n)|$. From Markov's inequality we know that

$$|q_n'(z)| \le \frac{n^2}{\varepsilon} M_n \quad \text{for all } z \in I. \tag{3.37}$$

Integrating q_n' shows that

$$|q_n(z)| \ge \frac{M_n}{2} \quad \text{for all } z \in I \text{ with } |z - x_n| \le \frac{\varepsilon}{2n^2} =: \varepsilon_n. \tag{3.38}$$

Since at least one half of the interval $[x_n - \varepsilon, x_n + \varepsilon_n]$ is contained in I we have the lower estimate

$$\begin{aligned} |I_n(z_0)| &\ge \left|\frac{\sin \pi\alpha}{\pi} \operatorname{Im} \int_0^\infty \frac{q_n(-x)^2 x^\alpha \, dx}{\omega_n(-x)(x+z_0)}\right| \\ &\ge c_3^{2n} \int_{x_n-\varepsilon_n}^{x_n+\varepsilon_n} |q_n(x)|^2 \, dx \\ &\ge c_3^{2n} \left(\frac{M_n}{2}\right)^2 \varepsilon_n \ge (c_3 c_2 e^{-\delta})^{2n} \frac{\varepsilon}{8n^2} \end{aligned} \tag{3.39}$$

for the function I_n defined in (3.5) if n is sufficiently large. We have used in (3.39) the inequalities (3.35), (3.38), and (3.33). With (3.34), (3.36), and (3.38) the estimate (3.39) implies that

(3.40)
$$\begin{aligned} \liminf_{N_\Theta} \frac{1}{2n} \log |I_n(z_0)| &\ge \liminf_{N_\Theta} \left[\frac{1}{2n} \log \frac{\varepsilon}{8n^2} + \log c_2 + \log c_3 - \delta\right] = \log c_2 + \log c_3 - \delta \\ &\ge \max_{z\in I} p(-\nu; z) + \min_{z\in I} p(\omega; z) - 3\delta \ge p(-\nu; x) + p(\omega; x) - 4\delta. \end{aligned}$$

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From the limit (3.17) and the arbitrariness of $\delta > 0$ we deduce from (3.40) that

$$p(\omega - \nu; x) \leq c_0. \tag{3.41}$$

Since $x \in (-\infty, 0)$ was arbitrary, (3.41) completes the proof of the inequality in (3.23) for all $z \in (-\infty, 0)$. That the inequality holds also for $x = 0$ and $x = -\infty$, follows from the continuity of $p(\nu - \omega; \cdot)$ in the fine topology (see [La], Chapter III, §1) and the fact that $-\infty$ and 0 are regular points of $[-\infty, 0]$.

(c) In order to prove the equality in (3.23) for all $z \in [0, 1]$ we first show that

$$p(\nu - \omega; z) + c_0 \geq 0 \quad \text{for all } z \in \operatorname{supp}(\omega). \tag{3.42}$$

From (3.7) we know that $\operatorname{supp}(\omega) \subseteq [0, 1]$. In part (b) it has been shown that (3.42) holds for $z = 0$. Now let x be an arbitrary element of $\operatorname{supp}(\omega) \setminus \{0\}$. Then from the second limit in (3.4) it follows that for every $n \in N_\Theta$ there exist at least two points $a_n, b_n \in B_n := B_{m_n,n}$ with $a_n < b_n$ and both limits $a_n \longrightarrow x$ and $b_n \longrightarrow x$ exist as $n \longrightarrow \infty$. Note that B_n is the set of zeros of the error function e_n (see (2.10)). Thus, there exists at least one extreme point $z_n \in A_n := A_{m_n,n}$ with $a_n < z_n < b_n$ for every $n \in N_\Theta$ and of course we also have

$$z_n \longrightarrow x \quad \text{as } n \longrightarrow \infty,\ n \in N_\Theta, \tag{3.43}$$

and

$$|e_n(z_n)| = \|e_n\|_{[0,1]} \quad \text{for all } n \in N_\Theta. \tag{3.44}$$

From the lower estimate in (1.15) of Theorem 1.1 together with (3.44) it follows that

$$\limsup_{N_\Theta} \frac{1}{2n} \log |e_n(z_n)| \geq \lim_{N_\Theta} \frac{1}{2n}(-c_4\sqrt{\alpha 4n}) = 0, \tag{3.45}$$

where c_4 is a constant larger than $\pi\sqrt{1 + 2\Theta}$. With remainder formula (2.16) of Lemma 2.2 it then follows from (3.45) in a similar way as in (3.26) that

$$0 \leq \limsup_{N_\Theta} \frac{1}{2n} \log |e_n(z_n)| = \limsup_{N_\Theta} \frac{1}{2n}\left[\log\left|\frac{\omega_n(z_n)}{q_n(z_n)^2}\right| + \log |I_n(z_n)|\right]. \tag{3.46}$$

With the limit (3.17) of Lemma 3.2 and (3.13) of Lemma 3.1 it then follows from (3.46) that $p(\nu - \omega; x) + c_0 \geq 0$. Since x was an arbitrary point of $\operatorname{supp}(\omega) \setminus \{0\}$, we have proved (3.42).

Since $p(\nu - \omega; \cdot)$ is superharmonic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\omega)$, it follows from the minimum principle and (3.42) that

$$p(\nu - \omega; \cdot) + c_0 \geq 0 \quad \text{for all } z \in \mathbb{C}. \tag{3.47}$$

Together with the inequality (3.24) this proves the equality in (3.23) for all $z \in (0, 1]$.

(d) It remains to be shown that $c_0 \in \mathbb{R}$ is finite, which will turn out to be a rather immediate consequence of the equality in (3.23). It follows from the definition of $p(\nu; z)$ in (3.8), $\operatorname{supp}(\nu) \subseteq [-\infty, 0]$, and $\|\nu\| = 1$ that for all $z \in (0, 1]$ we have

$$\log \frac{1}{2} \leq p(\nu; z) \leq \log \frac{1}{|z|}. \tag{3.48}$$

Because $\operatorname{supp} \omega \subseteq [0, 1]$, the potential $p(\omega; z)$ is bounded from below on $[0, 1]$ and $p(\omega; z) = \infty$ can only hold for a set of capacity zero on $[0, 1]$ (see [La], Chapter III, Section 1). Hence, from (3.48) and the equality in (3.23) it follows that c_0 is finite. ∎

The next lemma is to a large extent a corollary of Lemma 3.3.

LEMMA 3.4. *We have*

$$p(\nu - \omega; z) + c_0 \equiv \Theta g_{\overline{\mathbb{C}} \setminus [0,1]}(z, \infty), \tag{3.49}$$

$$\nu = \delta_0, \tag{3.50}$$

$$\omega = \delta_0 + \Theta \omega_{[0,1]}, \tag{3.51}$$

where δ_0 *is the Dirac measure at* $z = 0$ *and* $\omega_{[0,1]}$ *is the equilibrium distribution on* $[0, 1]$.

PROOF. In (3.6) and (3.7) it has been shown that ν and ω are positive measures with $\|\nu\| = 1$ and $\|\omega\| = 1 + \Theta$, therefore $(\omega - \nu)(\overline{\mathbb{C}}) = \Theta$. From the equality in (3.23) of Lemma 3.3 we then immediately deduce that

$$p(\nu - \omega; z) + c_0 = \Theta g_{\overline{\mathbb{C}} \setminus [0,1]}(z, \infty) + \int g_{\overline{\mathbb{C}} \setminus [0,1]}\neg\neg(z, x)\, d\nu(x). \tag{3.52}$$

We now proceed with a proof by contradiction. Assume that

$$\nu \neq \delta_0. \tag{3.53}$$

Since we know from (3.6) that $\operatorname{supp}(\nu) \subseteq [-\infty, 0]$, it follows from (3.53) that $\nu|_{[-\infty,0)} \neq 0$ and therefore it follows from (3.52) that

$$p(\nu - \omega; z) + c_0 > 0 \quad \text{for all } z \in \overline{\mathbb{C}} \setminus [0, 1]. \tag{3.54}$$

We shall show that inequality (3.54) implies that all zeros of the polynomials q_n converge to zero as $n \longrightarrow \infty$. This then contradicts (3.53), and thus (3.49) and (3.50) follow. Further, (3.51) follows from (3.49) since the Green function $g_{\overline{\mathbb{C}} \setminus [0,1]}(z, \infty)$ has the representation

$$g_{\overline{\mathbb{C}} \setminus [0,1]}(z, \infty) \equiv \log 4 - \int \log \frac{1}{|z - x|}\, d\omega_{[0,1]}(x) \tag{3.55}$$

(see [Ts], Theorem III.12, or [StTo], Appendix V).

Let us assume that y_n, $n \in N_\Theta$, is a sequence of zeros of q_n satisfying

$$y_n < -\varepsilon \tag{3.56}$$

for some $0 < \varepsilon < 1$. Let $\delta > 0$ be chosen in such a way that from (3.54) we can deduce

$$p(\nu - \omega; z) \geq -c_0 + 4\delta \quad \text{for } z \in [-\infty, -\varepsilon]. \tag{3.57}$$

It then follows from the asymptotic inequality (3.15) in Lemma 3.1 that

$$\frac{1}{2n} \log \frac{q_n(x)^2 |x|^{[\alpha]+2}}{|\omega_n(x)|\,|x-1|} \leq c_0 - 3\delta \quad \text{for all } x \in [-\infty, -\varepsilon],\ n \geq n_0, n \in N_\Theta, \tag{3.58}$$

for n_0 sufficiently large. Note that the function on the left of (3.58) is subharmonic in a neighborhood of infinity since from (3.2) we know that $m_n \geq n + [\alpha]$, which implies $2n + [\alpha] + 2 - m_n - n - 1 - 1 \leq \theta$.

From (3.58) we deduce that

$$\int_{-\infty}^{-\varepsilon} \frac{q_n(x)^2 |x|^\alpha \,dx}{|x-1|\,|\omega_n(x)|} \leq e^{2n(c_0 - 3\delta)} \int_{-\infty}^{-\varepsilon} |x|^{\alpha - [\alpha] - 2}\,dx \leq e^{2n(c_0 - 2\delta)} \quad \text{for } n \geq n_0,\ n \in N_\Theta, \tag{3.59}$$

if n_0 is sufficiently large. From the third limit in (3.4), the definition of I_n in (3.5), and the estimate (3.59) it then follows that

$$\begin{aligned} \int_{-\varepsilon}^{0} \frac{q_n(x)^2 |x|^\alpha \,dx}{|x-1|\,|\omega_n(x)|} &= \Big(\int_{-\infty}^{0} - \int_{-\infty}^{-\varepsilon} \Big) \frac{q_n(x)^2 |x|^\alpha \,dx}{|x-1|\,|\omega_n(x)|} \geq e^{2n(c_0-\delta)} - e^{2n(c_0 - 2\delta)} \\ &= e^{2n(c_0-\delta)}(1 - e^{-2n\delta}) \quad \text{for } n \geq n_0,\ n \in N_\Theta, \end{aligned} \tag{3.60}$$

if n_0 is sufficiently large.

We factor out the zero y_n from q_n by defining

$$\tilde{q}_n(z) := q_n(z) \frac{\max(1, |y_n|)}{z - y_n} \in \Pi_{n-1}, \tag{3.61}$$

which is a polynomial normalized in accordance with (3.3). The factoring out of one zero does not change the first limit in (3.4). Consequently, we have

$$\frac{1}{2n} \nu_{q_n \tilde{q}_n} \xrightarrow{\ *\ } \quad \text{as } n \to \infty,\ n \in N_\Theta. \tag{3.62}$$

It is easy to verify that the asymptotic inequality (3.15) in Lemma 3.1 remains true if q_n^2 is replaced by $q_n \tilde{q}_n$. Thus, it follows from (3.62) and (3.57) in exactly the same way as in (3.58) that

$$\frac{1}{2n} \log \left| \frac{(q_n \tilde{q}_n)(x) |x|^{[\alpha]+2}}{\omega_n(x)} \right| \leq c_0 - 3\delta \quad \text{for all } x \in [-\infty, -\varepsilon],\ n \geq n_0,\ n \in N_\Theta, \tag{3.63}$$

if n_0 is sufficiently large. As a consequence we have, as in (3.59),

$$\int_{-\infty}^{-\varepsilon} \frac{|(q_n \tilde{q}_n)(x)|\,|x|^\alpha \,dx}{|\omega_n(x)|} \leq e^{2n(c_0 - 2\delta)} \quad \text{for } n > n_0,\ n \in N_\Theta. \tag{3.64}$$

The equality

$$\frac{|x - y_n|}{\max(1, |y_n|)|x-1|} \leq 1 \tag{3.65}$$

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holds for all $x \in \mathbb{R}_-$. Because of (3.56) the polynomial $q_n\tilde{q}_n$ does not change its sign on $[-\varepsilon, 0]$. Hence, we deduce from (3.65) that

$$\left|\int_{-\varepsilon}^{0} \frac{(q_n\tilde{q}_n)(x)|x|^\alpha}{\omega_n(x)}\,dx\right| = \int_{-\varepsilon}^{0} \left|\frac{(q_n\tilde{q}_n)(x)|x|^\alpha}{\omega_n(x)}\right| dx \geq \left|\int_{-\varepsilon}^{0} \frac{q_n(x)^2|x|^\alpha\,dx}{(x-1)\,\omega_n(x)}\right| \tag{3.66}$$

for all $n \in N_\Theta$. Using (3.60), (3.64), and (3.66) we obtain the estimate

$$\left|\int_{-\infty}^{0} \frac{(q_n\tilde{q}_n)(x)|x|^\alpha\,dx}{\omega_n(x)}\right| \left(\int_{-\varepsilon}^{0} - \int_{-\infty}^{-\varepsilon}\right) \frac{|(q_n\tilde{q}_n)(x)|\,|x|^\alpha}{|\omega_n(x)|}\,dx \geq e^{2n(c_0-\delta)} - e^{2n(c_0-2\delta)} = e^{2n(c_0-\delta)}(1-e^{-2n\delta}) > 0 \tag{3.67}$$

for $n \in N_\Theta$ sufficiently large.

Since $\tilde{q}_n \in \Pi_{n-1}$, the strict inequality in (3.67) contradicts the orthogonality (2.15) in Lemma 2.2. Thus, we have shown that inequality (3.54) implies that all zeros of the polynomials q_n have to converge to zero as $n \in N_\Theta$ tends to infinity. This completes the proof. ∎

In the proof of Lemma 3.4 we have actually shown more than has been stated in Lemma 3.4. The identity (3.50) implies only that almost all zeros of the denominator polynomials q_n converge to $z = 0$ as $n \longrightarrow \infty$, $n \in N_\Theta$. The proof, however, shows that under certain conditions all zeros converge to $z = 0$. This stronger assertion is part of the next lemma.

For a polynomial $p \in \Pi_n$ we denote by $Z(p)$ the set of all zeros taking account of multiplicities.

LEMMA 3.5. *If $\Theta > 0$, then*

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{n\geq k\\ n\in N_\Theta}} Z(q_n)} = \{0\}, \tag{3.68}$$

$$\bigcap_{k=1}^{\infty} \overline{\bigcup_{\substack{n\geq k\\ n\in N_\Theta}} Z(p_n)} = [0,1], \tag{3.69}$$

where q_n and p_n are the denominator and numerator polynomials of the approximant r_n^*.

REMARK. The case $\Theta = 0$ has been excluded from Lemma 3.5. Theorem 1.4, which will be proved below, implies that for the case of the sequence $\{r^*_{n+[\alpha],n}\}_{n\in N}$ all zeros and poles of the approximants $r^*_{n+[\alpha],n}$ cluster on $[-\infty, 0]$. So at least in this case there is an asymptotic behavior different from that described in (3.68).

PROOF. If $\Theta > 0$, then it follows from (3.49) that inequality (3.54) holds. In the proof of Lemma 3.4 it has been shown that from this inequality it follows that all zeros of the denominator polynomials q_n converge to $z = 0$. This proves (3.68).

In order to prove (3.69) we need some preparation. Since $\Theta > 0$ implies inequality (3.54), we know that all results are true that have been proved in (3.58)–(3.67), and $\varepsilon > 0$ can be chosen arbitrarily small.

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Now, let $r > 0$, $\delta > 0$, and $\Gamma_1 = \{z \mid |z| = r, |\arg z| \le \pi - \delta\}$. There exists $\varepsilon > 0$ such that

$$(3.70)\qquad \Big|\frac{1}{x/z-1} - \frac{1}{x-1}\Big| \le \frac{\delta}{|x-1|} \quad \text{for all } x \in [-\varepsilon, 0],\ z \in \Gamma_1,$$

and there exists a constant $c_1 < \infty$ with

$$(3.71)\qquad \frac{1}{|x/z-1|} \le \frac{c_1}{|x-1|} \quad \text{for all } x \in \mathbb{R}_-,\ z \in \Gamma_1.$$

Define

$$(3.72)\qquad a_n := \int_{-\varepsilon}^{0} \frac{q_n(x)^2 |x|^\alpha\, dx}{(x-1)\,\omega_n(x)}, \quad n \in N_\Theta.$$

From (3.60) we know that

$$(3.73)\qquad |a_n| \ge e^{2n(c_0-\delta)} \quad \text{for } n \ge n_0,\ n \in N_\Theta,$$

if n_0 is sufficiently large. Because of the third limit in (3.4) we know that

$$(3.74)\qquad \lim_{N_\Theta} \frac{1}{2n} \log |a_n| = c_0.$$

From (3.71) together with (3.59) we deduce that

$$(3.75)\qquad \Big|\int_{-\infty}^{-\varepsilon} \frac{q_n(x)^2|x|^\alpha\, dx}{(x/z-1)\,\omega_n(x)}\Big| \le c_1 \int_{-\infty}^{-\varepsilon} \frac{q_n(x)^2|x|^\alpha\, dx}{|x-1|\,|\omega_n(x)|} \le e^{2n(c_0-2\delta)}$$

for $n \ge n_0$, $n \in N_\Theta$, $z \in \Gamma_1$, and n_0 sufficiently large. From (3.70) together with (3.72) we further deduce that

$$(3.76)\qquad -\delta |a_n| \le \Big|\int_{-\varepsilon}^{0} \frac{q_n(x)^2|x|^\alpha\, dx}{(x/z-1)\,\omega_n(x)} - a_n\Big| \le \delta |a_n| \quad \text{for } n \in N_\Theta.$$

Since $e_n(z) = z^\alpha - (p_n/q_n)(z)$, it follows from formula (2.16) in Lemma 2.2 that

$$(3.77)\qquad \frac{zp_n(z)\,q_n(z)}{a_n\omega_n(z)} = \frac{z^{\alpha+1}q_n(z)^2}{a_n\omega_n(z)} - \frac{\sin \pi\alpha}{\pi} \int_{-\infty}^{0} \frac{q_n(x)^2|x|^\alpha\, dx}{(x/z-1)\,\omega_n(x)}, \quad z \in \mathbb{C} \setminus [-\infty, 0].$$

As a consequence of the inequality (3.54), the three limits in (3.4), and limit (3.74) we have as in Lemma 3.1 that

$$(3.78)\qquad \lim_{N_\Theta} \frac{z^{\alpha+1}q_n(z)^2}{a_n\omega_n(z)} = 0$$

locally uniformly for $z \in \mathbb{C} \setminus [0,1]$.

From (3.76) together with (3.75) and (3.73) it follows that

(3.79)

$$-\delta - e^{-2n\delta} \le \Big|\frac{1}{a_n}\int_{-\infty}^{0} \frac{q_n(x)^2|x|^\alpha\, dx}{(x/z-1)\,\omega_n(x)} - 1\Big| \le \delta + e^{-2n\delta} \quad \text{for } n \ge n_0,\ n \in N_\Theta,\ z \in \Gamma_1,$$

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which proves that

$$\lim_{N_\Theta} \frac{zq_n(z)\,p_n(z)}{a_n\omega_n(z)} = \frac{\sin \pi\alpha}{\pi} \tag{3.80}$$

uniformly for $z \in \Gamma_1$. Since $r > 0$ and $\delta > \theta$ are arbitrary, it follows that (3.80) holds for all $z \in \mathbb{C} \setminus \mathbb{R}_-$.

Since we already know from the proof of Lemma 3.4 that the polynomials q_n and ω_n have all their zeros in the interval $[-\varepsilon, 1]$ for $n \in N_\Theta$ sufficiently large, and asymptotically all these zeros cluster on $[0, 1]$, it follows from (3.80) and the argument principle that all zeros of the polynomials $p_n, n \in N_\Theta$, have to cluster on $[0, 1]$. ■

4. **Proofs.** In the present section we shall prove all results of Section 1 except Theorem 1.1, which has already been verified as an immediate consequence of Ganelius' result (1.10), and Lemma 1.5, which has been proved in Section 2.

We note that in Section 1 notation has been used that differs from that in Sections 2 and 3. Thus, in Section 1 we have considered only even functions, which then have been transformed by the mapping (2.1), to the problem of approximating x^α on $[0, 1]$. The mapping (2.1) is basically a substitution of z^2 by z. As a consequence, approximation on $[-1, 1]$ is transformed to approximation on $[0, 1]$, the degree $2n$ is reduced to n, and the exponent α is reduced to $\alpha/2$. While in Section 1 nontrivial approximation problems arise if $\alpha \notin 2\mathbb{N}$, this condition is $\alpha \notin \mathbb{N}$ in the later sections.

In Section 3 there was a further change: we have switched from the numerator-denominator ratio c to the parameter Θ. The transition has been defined in (3.1). From (1.3) and (3.1) we deduced that

$$c = 1 + 2\Theta \quad \text{and} \quad \Theta = \frac{1}{2}(c - 1). \tag{4.1}$$

It follows that

$$\frac{m}{n} \longrightarrow c \text{ and } \frac{m-n}{m+n} \longrightarrow \frac{\Theta}{1+\Theta} = \frac{c-1}{c+1} \quad \text{as } m+n \longrightarrow \infty,\ (m,n) \in N_c. \tag{4.2}$$

We continue to write $n \in N_\Theta$ for $(m, n) \in N_c$ and $r_n^*, \omega_n, \ldots$ for $r_{m_n n}^*, \omega_{m_n n}, \ldots$ in Sections 2 and 3. In all proofs we shall consider the approximation problem on $[0, 1]$, and at the end of each proof we shall describe the transition to the versions given in Section 1.

PROOF OF THEOREM 1.2. Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and let N_Θ be a ray sequence with $\Theta > 0$. From (2.16) of Lemma 2.2(c) we know that

$$\begin{aligned} r_n^*(z) &= z^\alpha - e_n(z) \\ &= z^\alpha - \frac{\sin \pi\alpha}{\pi} \frac{\omega_n(z)}{q_n(z)^2} \int_{-\infty}^0 \frac{q_n(x)^2 |x|^\alpha\, dx}{\omega_n(x)(x-z)} \end{aligned} \tag{4.3}$$

for $z \in D = \mathbb{C} \setminus \mathbb{R}_-$. Since $f(\alpha; z) = z^\alpha$ is bounded on compact subsets of D, and since the same is true for the analytic continuation of z^α across $(-\infty, 0)$, it follows that the second term on the right-hand side of (4.3) is the dominant one; it is decisive for the asymptotic behavior of $|r_n^*|$ as $n \longrightarrow \infty$.

Without loss of generality we can assume that the limits in (3.4) exist; for otherwise we can choose an infinite subsequence of N_Θ for which these assumptions hold.

From the limits in (3.4), limit (3.11) in Lemma 3.1, limit (3.17) in Lemma 3.2, and identity (3.49) in Lemma 3.4 we deduce that

(4.4)
$$\lim_{N_\Theta} \frac{1}{2n} \log\left| \frac{\sin \pi\alpha}{\pi} \frac{\omega_n(z)}{q_n(z)^2} \int_{-\infty}^{0} \frac{q_n(x)^2 |x|^\alpha \, dx}{\omega_n(x)(x-z)} \right| = \lim_{N_\Theta} \frac{1}{2n} \left[\log\left| \frac{\omega_n(z)}{q_n(z)^2} \right| + \log |I_n(z)| \right]$$
$$= p(\nu - \omega; z) + c_0 = \Theta g_{\bar{C}\setminus[0,1]}(z,\infty),$$

locally uniformly for $z \in \mathbb{C} \setminus [-\infty, 1]$.

From (4.3), (4.4), and (4.2) together with the fact that $2n/(m_n + n) \longrightarrow 1/(1+\Theta)$ as $n \longrightarrow \infty$, $n \in N_\Theta$, it follows that

$$\lim_{N_\Theta} \frac{1}{m_n + n} \log |r_n^*(z)| = \frac{\Theta}{1+\Theta} g_{\bar{C}\setminus[01]}(z,\infty) = \frac{c-1}{c+1} g_{\bar{C}\setminus[0,1]}(z,\infty) \tag{4.5}$$

locally uniformly for $z \in \mathbb{C} \setminus [-\infty, 1]$. As an immediate consequence of (3.68) and (3.69) in Lemma 3.5 we see that (4.5) also holds locally uniformly for $z \in \mathbb{C} \setminus [0, 1]$.

By the transformation (2.1), limit (4.5) transforms to (1.19). Thus (1.19) is proved for $c > 1$.

In case that $c = 1$, and $m = n + 2[\alpha/2]$ for $(m, n) \in N_c$, the right-hand side of (1.19) is identically 1. The limit (1.19) follows then as a consequence of Theorem 1.4 part (a), which will be proved next. ■

PROOF OF THEOREM 1.4. The main work is to show that the sequence of error functions $e_n(z) = z^\alpha - r_n^*(z)$, $n \in \mathbb{N}$, is bounded on any compact subset $V \subseteq \mathbb{C} \setminus \mathbb{R}_-$. In the proof we shall use divided differences and iterated differences; the relevant properties of these notions will be assembled first.

Let f be a real function defined on $\mathbb{C}$, and let $x_0, \ldots, x_j \in \mathbb{C}$ be a finite sequence of distinct points. The *divided difference* $f(x_0, \ldots, x_j)$ of order j is recursively defined by

$$f(x_0, \ldots, x_j) := \frac{f(x_0, \ldots, x_{j-1}) - f(x_1, \ldots, x_j)}{x_0 - x_j}. \tag{4.6}$$

with $f(x_j)$ the divided difference of order 0. For points x_j that form an arithmetic progression

$$x_j = x_0 + jh, \quad x_0, h \in \mathbb{C},\ h \neq 0,\ j \in \mathbb{N}, \tag{4.7}$$

the *iterated differences* are defined recursively by

(4.8)
$$\Delta f(x) = f(x+h) - f(x)$$
$$\Delta^j f(x) = \Delta^{j-1} f(x+h) - \Delta^{j-1} f(x), \quad j = 1, 2, \ldots.$$

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We have the formulae

$$f(x_0,\dots,x_j)=\frac{1}{j!\,h^j}\Delta^j f(x_0) \tag{4.9}$$

and

$$\Delta^j f(x_0)=\sum_{l=0}^{j}(-1)^{j-l}\binom{j}{l}f(x_l) \tag{4.10}$$

for $j=1,2,\dots$ (see [Ge], Chapter 1).

Since under the assumption (4.7) all points $x_0,\dots,x_j$ lie on a straight line, for each real function f that has a j-th order continuous derivative $f^{(j)}$ there exists a $\xi\in(x_0,x_j)$ such that

$$f(x_0,\dots,x_j)=\frac{1}{j!}f^{(j)}(\xi). \tag{4.11}$$

From Theorem 1.1 we know that the error function e_n converges to 0 uniformly on $[0,1]$ as $n\longrightarrow\infty$. Therefore the e_n are bounded on $[0,1]$ for all $n\in\mathbb{N}$. Let $a\in(0,1)$ and h a real number with $0<h<(1-a)/([\alpha]+1)$. Further let the sequence $\{x_j\}$ be defined by $x_j=a+jh$ for $j=0,\dots,[\alpha]+1$. Then it follows from (4.9) and (4.10) that there exists a constant c_1 such that

(4.12)

$$|e_n(x_0,\dots,x_j)|\le\frac{1}{j!\,h^j}\sum_{l=0}^{j}\binom{j}{l}|e_n(x_l)|\le c_1\quad\text{for } j=0,\dots,[\alpha]+1 \text{ and all } n\in\mathbb{N}.$$

Since z^α is analytic in D, the same is true for the divided differences of z^α with respect to the sequences $\{x_0,\dots,x_j\}$, $j=0,\dots,[\alpha]+1$. Hence, these differences are bounded on compact subsets of D independent of $n\in\mathbb{N}$. The divided difference is a linear operator, and therefore it follows from (4.12) and from the identity $e_n=z^\alpha-r_n^*$ that there exists a constant c_2 so that

$$|r_n^*(x_0,\dots,x_j)|\le c_2\quad\text{for all } j=0,\dots,[\alpha]+1 \text{ and all } n\in\mathbb{N}. \tag{4.13}$$

With (4.11) it then follows that there exist points $\xi_n^{(j)}\in(x_0,x_j)\subseteq[a,1]$ with

$$|r_n^{*(j)}(\xi_n^{(j)})\le c_2\quad\text{for all } j=0,\dots,[\alpha+1] \text{ and all } n\in\mathbb{N}. \tag{4.14}$$

From Lemma 2.2(a) we know that the denominator polynomial q_n of r_n^* has exactly n zeros y_{jn}, $j=1,\dots,n$, which are all simple and contained in $(-\infty,0)$. From Lemma 2.2(b) we know that the numerator polynomial of r_n^* is exactly of degree $m_n=n+[\alpha]$. Hence, the partial fraction decomposition of r_n^* has the form

$$r_n^*(z)=\sum_{j=1}^{n}\frac{\lambda_{jn}}{z-y_{jn}}+P_n(z)=S_n(z)+P_n(z), \tag{4.15}$$

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where $P_n \in \Pi_{[\alpha]}$ and S_n is the fractional part of r_n^* (see also (2.27)). It has been shown in (2.28) that

$$\lambda_{jn} = \frac{\sin \pi\alpha}{\pi} \frac{\omega_n(y_{jn})}{q_n'(y_{jn})^2} \int_0^\infty \left(\frac{q_n(-x)}{x+y_{jn}}\right)^2 \frac{x^\alpha\, dx}{\omega_n(-x)}, \quad j = 1, \ldots, n, \tag{4.16}$$

which implies that all coefficients $\lambda_{1n}, \ldots, \lambda_{nn}$ are of identical sign for a given n. Since P_n is a polynomial of degree $[\alpha]$, we have $P_n^{([\alpha]+1)} \equiv 0$ and

$$r_n^{*([\alpha]+1)}(z) = S_n^{([\alpha]+1)}(z) = (-1)^{[\alpha]+1}([\alpha]+1)! \sum_{j=1}^{n} \frac{\lambda_{jn}}{(z-y_{jn})^{[\alpha]+2}}. \tag{4.17}$$

From elementary considerations it follows that for any compact set $V \subseteq D$ there exists a constant $c_3 = c_3(V)$ such that

$$\frac{1}{|z-x|^{[\alpha]+2}} \le \frac{c_3}{|\zeta - x|^{[\alpha]+2}} \quad \text{for all } z \in V,\ \zeta \in [a,1],\ x \in \mathbb{R}_-. \tag{4.18}$$

With (4.18) we derive from (4.17) that there exists a constant c_4 such that

$$\begin{aligned} |r_n^{*([\alpha]+1)}(z)| &\le ([\alpha]+1)! \sum_{j=1}^{n} \frac{|\lambda_{jn}|}{|z-y_{jn}|^{[\alpha]+2}} \\ &\le c_3([\alpha+1)! \sum_{j=1}^{n} \frac{|\lambda_{jn}|}{|\xi_n^{([\alpha]+1)} y_{jn}|^{[\alpha]+2}} \\ &\le c_3 |r_n^{*([\alpha]+1)}(\xi_n^{([\alpha]+1)})| \le c_4 \end{aligned} \tag{4.19}$$

for all $z \in V$ and $n \in \mathbb{N}$, where $\xi_n^{([\alpha]+1)} \in (x_0, x_{[\alpha]+1})$ are the points introduced in (4.14). Integrating (4.19) $[\alpha]+1$ times and using the initial values $\xi_n^{(j)}$, $j = 0, \ldots, [\alpha]+1$, which are assumed to satisfy (4.14), shows that there exists a constant c_5 so that

$$|r_n^*(z)| \le c_5 \quad \text{for all } z \in V \text{ and } n \in \mathbb{N}. \tag{4.20}$$

Since r_n^* converges to z^α uniformly on $[0,1]$ as $n \longrightarrow \infty$, it follows from (4.20) and Montel's theorem that

$$\lim_{n\to\infty} r_n^*(z) = z^\alpha \tag{4.21}$$

uniformly on each compact $V \subseteq \mathbb{C} \setminus [-\infty, 0]$.

With the mapping (2.1), it then is immediate that (4.21) implies (1.22). Thus, the proof of part (a) of Theorem 1.4 is complete.

Part (b) is basically a consequence of Lemma 2.2(b). There is has been shown that all poles lie on $\mathbb{R}_-$. By the mapping (2.1) these locations are moved to $i\mathbb{R}$ and their number is duplicated. Since in Lemma 2.2(b) it has been shown that between two poles there is always a zero, this guarantees that at least $n-2$ zeros of $r^*_{n+[\alpha],n}$ lie on $i\mathbb{R}$. ∎

PROOF OF THEOREM 1.6. By transformation (2.1) the assumption in Theorem 1.6 transforms to $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $m_n \ge n + [\alpha]$ for all $n \in \mathbb{N}_\Theta$. Hence, the assumptions (2.4) and (3.2) of Sections 2 and 3 are satisfied.

Let $A_n = A_{m_n n}$ be the set of $m_n + n + 2$ extreme points of the error function $e_n(z) = z^\alpha - r_n^*(z)$ on $[0, 1]$ for the transformed problem. It has been shown in (2.9) that between two adjacent points of A_n there is always a zero of the error function e_n. These are altogether $m_n + n + 1$ zeros, they form the set $B_n = B_{m_n n}$ defined in (2.13), and they are also the zeros of the polynomial ω_n defined in (2.14).

In the second limit of (3.3) we have assumed that the sequence N_Θ has been selected in such a way that the limit

$$\frac{1}{2n}\nu_{B_n} \xrightarrow{*} \omega \quad \text{as } n \longrightarrow \infty,\ n \in N_\Theta, \tag{4.22}$$

exists. From (4.2) we know that

$$\frac{m_n + n + 1}{2n} \longrightarrow 1 + \Theta \quad \text{as } n \longrightarrow \infty,\ n \in N_\Theta. \tag{4.23}$$

Since the points of A_n and B_n interlace, limit (4.22) still holds if the set B_n is replaced by the set A_n.

In (3.60) of Lemma 3.4 it has been shown that $\omega = \delta_0 + \Theta\omega_{[0,1]}$, where ω is the limit measure in (4.22). Putting (4.22) and (4.23) together with the last identity we have

(4.24)
$$\frac{1}{m_n + n + 2}\nu_{A_n} \xrightarrow{*} \frac{1}{1+\Theta}(\delta_0 + \Theta\omega_{[0,1]}) = \frac{2}{c+1}\delta_0 + \frac{c-1}{c+1}\omega_{[0,1]}, \quad \text{as } n \longrightarrow \infty,\ n \in N_\Theta,$$

where in the last equality we have used the identity $\Theta = (c-1)/2$ from (4.1).

With (4.24) the limit (1.27) is practically proved; it only remains to show that the inverse of transformation (2.1) transforms the measures δ_0 and $\omega_{[0,1]}$ in the measures δ_0 and $\omega_{[-1,1]}$, respectively. Indeed, the two branches of the mapping $z \longmapsto x = \varphi^{-1}(z) = \pm\sqrt{z}$ map the domain $D = \mathbb{C} \setminus [-\infty, 0]$ onto $H_+ = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ and $H_- = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$. The equilibrium distribution

$$d\omega_{[0,1]}(z) = \frac{dz}{\pi\sqrt{z(1-z)}}, \quad z \in [0, 1], \tag{4.25}$$

is mapped by both branches of φ^{-1} onto the measure

$$2d\omega_{[-1,1]}(x) = \frac{2dx}{\pi\sqrt{1-x^2}}, \quad x \in [-1, 1]. \tag{4.26}$$

This is twice the equilibrium distribution on $[-1, 1]$. Since φ^{-1} is 2-valued, we have to divide the measure (4.26) by 2. The same considerations hold for the transformation of the Dirac measure δ_0.

Thus, the limit (1.27) follows from (4.24) for the subsequence N_Θ, for which the limits in (3.3) hold. Since the right-hand sides of (4.24) and (1.27) are independent of the selected subsequence, the limit (1.27) holds also for the original sequence N_c. ■

PROOF OF THEOREM 1.7. The assumptions are the same as in Theorem 1.6, and again we transform the problem by (2.1) in a form that allows the application of results

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from Sections 2 and 3. The original degrees m and n have been assumed to be even. After the transformation of the problem these degrees are halved.

(a) From part (a) of Lemma 2.1 we know that the denominator polynomial q_n in the transformed problem has exactly n simple zeros on the negative real axis $(-\infty, 0)$. From part (b) of Lemma 2.1 it follows that each of these zeros is a pole of r_n^*. The inverse mapping φ^{-1} of transformation (2.1) then transforms these poles to a doubled number of poles on the imaginary axis $i\mathbb{R}$.

(b) From part (b) of Lemma 2.1 it follows that between two adjacent poles of r_n^* on the two imaginary halfaxes there is at least one zero of r_n^*. This proves (1.28).

(d) We now prove assertion (d) and continue with the proof of assertion (c) afterwards. The assumption $c > 1$ implies that in the transformed problem $\Theta > 0$. Hence, Lemma 3.5 is applicable. In (3.68) and (3.69) of this lemma it has been shown that all poles of the transformed approximant r_n^* converge to $z = 0$, and all zeros converge to $[0, 1]$ as $n \longrightarrow \infty$ and $n \in N_\Theta$. Transforming back via the inverse mapping φ^{-1} to the original problem, the set $[0, 1]$ is mapped on $[-1, 1]$ and the point $z = 0$ is mapped on $x = 0$. Thus, (3.68) and (3.69) of Lemma 3.5 imply (1.30).

(c) From (3.55) in Lemma 3.4 together with Lemma 2.2(b) and the first limit in (3.4), it follows that

$$\frac{1}{n}\nu_{P_{mn}} \xrightarrow{*} \delta_0 \quad \text{as } m+n \longrightarrow \infty,\ (m,n) \in N_c, \tag{4.27}$$

which proves the first limit (1.29).

In the proof of the second limit in (1.29) we distinguish the two cases $c = 1$ and $c > 1$. If $c = 1$, then it follows from the interlacing property (1.28) in part (b), that the limit (4.27) remains true if we substitute P_{mn} by Z_{mn}. This proves the second limit in (1.29) if $c = 1$.

If $c > 1$, then in the transformed problem we have $\Theta > 0$, and, as in the proof of Theorem 1.2, we can deduce that (4.5) holds. Transforming this limit back by the inverse mapping φ^{-1} of (2.1) and taking care of degrees and the effect of the transformation on the Green function, yields that

$$\lim_{\substack{m+n\longrightarrow\infty \\ (m,n)\in N_c}} \frac{1}{m+n}\log|r_{mn}^*(z)| = \frac{\Theta}{1+\Theta} g_{\bar{C}\setminus[-1,1]}(z,\infty) = \frac{c-1}{c+1} g_{\bar{C}\setminus[-1,1]}(z,\infty) \tag{4.28}$$

uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$. The last equality in (4.28) follows from (4.1). Since the Green function $g_{\bar{C}\setminus[-1,1]}(z, \infty)$ has the representation

$$g_{\bar{C}\setminus[-1,1]}(z,\infty) \equiv \log 2 - p(\omega_{[-1,1]}; z) \tag{4.29}$$

(*cf.* [Ts], Theorem III.12, or [StTo], Appendix V), and since

$$\frac{n}{m+n} \longrightarrow \frac{1}{1+c} \quad \text{as } m+n \longrightarrow \infty,\ (m,n) \in N_c, \tag{4.30}$$

it follows from (4.29) and the interlacing property (1.28) of poles and zeros proved in part (b) that

$$\left(\frac{1}{n}\nu_{Z_{mn}} - \frac{1}{n}\nu_{P_{mn}}\right) \xrightarrow{*} (c-1)\omega_{[-1,1]} \quad \text{as } m+n \longrightarrow \infty,\ (m,n) \in N_c. \tag{4.31}$$

With the limit (4.27) and the fact that $m/n \longrightarrow c$, we deduce from (4.31) that

$$\frac{1}{m}\nu_{Z_{mn}} \xrightarrow{*} \frac{1}{c}\delta_0 + \left(1 - \frac{1}{c}\right)\omega_{[-1,1]} \quad \text{as } m+n \longrightarrow \infty,\ (m,n) \in N_c. \tag{4.32}$$

This completes the proof of Theorem 1.7. ∎

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