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On Minimal and Maximal p-operator Space Structures

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Abstract. We show that $L^\infty(\mu)$, in its capacity as multiplication operators on $L^p(\mu)$, is minimal as a p-operator space for a decomposable measure μ. We conclude that $L^1(\mu)$ has a certain maximal type p-operator space structure that facilitates computations with $L^1(\mu)$ and the projective tensor product.

1 Introduction

In the theory of operator spaces, there are extremal operator space structures that can be assigned to any Banach space. These arose in the papers [3, 7] and are exposed in the monograph [8]. They have particular value when understanding mappings and tensor products.

In this article we examine minimal and maximal p-operator space structures. These structures' existences were noted in [10], where they were used to characterize certain algebras as algebras of operators on $\mathcal{SQ}_p$-spaces. Our primary motivation is to gain the isometric tensor product formula $L^1(\mu)\widehat{\otimes}^p\mathcal{V} \cong L^1(\mu,\mathcal{V})$ for the p-operator projective tensor product of [5]. Here $L^1(\mu)$ has a certain maximal operator space structure, which appears naturally via the embedding of $L^1(\mu) \hookrightarrow L^\infty(\mu)^*$, where $L^\infty(\mu)$ acts on $L^p(\mu)$ as multiplication operators. This is a less obvious task than we had initially hoped and seems worth an exposition in its own right. The techniques used in this article are all classical and elementary.

1.1 Background

Let $1 < p < \infty$, and let p' denote the conjugate index, so $1/p+1/p' = 1$. The theory of p-operator spaces is designed to give an analogue to the theory of operator spaces on a Hilbert space, which we might call 2-operator spaces. The theory of p-operator spaces has its origins in [12, 13] and was studied extensively in [10]. Daws [5] presents these spaces in the format we are using, a format also used extensively by An, Lee, and Ruan [1]. We closely follow the presentation of [5] and use some concepts from [1].

We let $\ell^p(n)$ denote $\mathbb{C}^n$ with the ℓ^p-norm. Given a Banach space $\mathcal{V}$, a p-operator space structure on $\mathcal{V}$ is a sequence of norms $\|\cdot\|_n$, each norm on $n\times n$-matricies with entries in $\mathcal{V}$, which satisfy the axioms below:

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(D_∞) For u in $\mathbb{M}_n(\mathcal{V})$ and v in $\mathbb{M}_m(\mathcal{V})$, $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$.
(M_p) For u in $\mathbb{M}_n(\mathcal{V})$ and α, β in

$$\mathbb{M}_n \cong \mathcal{B}\big(\ell^p(n)\big), \quad \|\alpha u \beta\|_n \leq \|\alpha\|_{\mathcal{B}(\ell^p(n))}\|u\|_n\|\beta\|_{\mathcal{B}(\ell^p(n))}.$$

A Banach space $\mathcal{V}$, equipped with a sequence of norms as above, will be called a *p-operator space*. In the sequel we will drop the subscript n from the norm on $\mathbb{M}_n(\mathcal{V})$. A linear map $T\colon \mathcal{V} \to \mathcal{W}$ gives rise to amplifications $T^{(n)}\colon \mathbb{M}_n(\mathcal{V}) \to \mathbb{M}_n(\mathcal{W})$, $T^{(n)}[v_{ij}] = [Tv_{ij}]$. Such a map is called *completely bounded* if $\|T\|_{p\mathrm{cb}} = \sup_n \|T^{(n)}\| < \infty$. Moreover it is called *completely contractive* if $\|T\|_{p\mathrm{cb}} \leq 1$ and a *complete isometry* if each $T^{(n)}$ is an isometry. The space of such maps will be denoted $\mathcal{CB}_p(\mathcal{V}, \mathcal{W})$.

We say a Banach space E is in the class $\mathcal{SQ}_p$ if it is a quotient of a subspace of $L^p(\phi)$ for some measure ϕ. The space $\mathcal{B}(E)$ is a p-operator space given identifications $\mathbb{M}_n(\mathcal{B}(E)) \cong \mathcal{B}(\ell^p(n) \otimes^p E) \cong \mathcal{B}(\ell^p(n, E))$. Here, $L^p(\phi) \otimes^p E$ is the completion with respect to the norm given by embedding $L^p(\phi) \otimes E \hookrightarrow L^p(\phi, E)$. Moreover, any p-operator space admits a complete isometry into $\mathcal{B}(E)$ for some E in $\mathcal{SQ}_p$ [12, 13]. Spaces that admit complete isometries into $\mathcal{B}(L^p(\phi))$ will admit better properties than general p-operator spaces. We will follow [1] and say that such spaces act on (some) L^p.

We follow [5] when assigning p-operator space structures to mapping spaces. We identify $\mathbb{M}_n(\mathcal{CB}_p(\mathcal{V}, \mathcal{W})) \cong \mathcal{CB}_p(\mathcal{V}, \mathbb{M}_n(\mathcal{W}))$, where $\mathbb{M}_n(\mathcal{W})$ is a p-operator space via the identifications $\mathbb{M}_m(\mathbb{M}_n(\mathcal{W})) \cong \mathbb{M}_{mn}(\mathcal{W})$. In particular, for the dual space, $\mathbb{M}_n(\mathcal{V}^*) \cong \mathcal{CB}_p(\mathcal{V}, \mathcal{B}(\ell^p(n)))$ completely isometrically. We have a p-version of the projective tensor product $\otimes^\gamma$ and the injective tensor product $\otimes^\lambda$, namely the *p-projective tensor product* $\widehat{\otimes}^p$ of [5] and the *p-injective tensor product* $\check{\otimes}^p$ of [1]. The p-projective tensor product enjoys all of the usual functorial properties that are analogues of $\otimes^\gamma$, while the theory of $\check{\otimes}^p$ is not as well understood. However, we do have that $\mathbb{M}_n(\mathcal{V}) \cong \mathcal{V}\check{\otimes}^p\mathcal{B}(\ell^p(n))$ completely isometrically.

As observed in [10, p. 89], for a p-operator space $\mathcal{V}$, the algebraic identification $\mathcal{V} \otimes \mathcal{B}(\ell^p(n)) \cong \mathbb{M}_n(\mathcal{V})$ allows us to view $\|\cdot\|_n$ as a reasonable cross-norm on $\mathcal{V} \otimes \mathcal{B}(\ell^p(n))$; see the terminology in [14], for example. Indeed, an application of (M_p) then of (D_∞) shows that $\|[\alpha_{ij} v]\|_n \leq \|\alpha\|_{\mathcal{B}(\ell^p(n))}\|v\|$ for α in $\mathbb{M}_n$ and v in $\mathcal{V}$; while [5, Lem. 4.2] (stating that contractive linear functions are automatically completely contractive) shows that $|\varphi \otimes \psi(v)| \leq \|\varphi\|_{\mathcal{V}^*}\|\psi\|_{\mathcal{B}(\ell^p(n))^*}\|v\|_n$ for any φ and ψ, where $v \in \mathcal{V} \otimes \mathcal{B}(\ell^p(n))$. Moreover, if $\mathcal{X}$ is any Banach space, then the algebraic identifications

$$\mathbb{M}_n(\mathcal{X}) \cong \mathcal{X} \otimes^\lambda \mathcal{B}\big(\ell^p(n)\big) \tag{1.1}$$

(injective tensor product on the right) are easily verified to produce a p-operator space structure on $\mathcal{X}$ that is minimal in the sense that $\|\cdot\|_n \leq \|\cdot\|_n'$ (for each n) with any other operator space structure on $\mathcal{X}$. We call this operator space structure the *minimal p-operator space* structure on $\mathcal{X}$. If $\mathcal{V}$ is a p-operator space and $T\colon \mathcal{V} \to \mathcal{X}$ is bounded, then T is completely bounded, with $\|T\|_{p\mathrm{cb}} = \|T\|$. Indeed, by reasonableness of the injective tensor product, we see that

$$T^{(n)} \cong (T \otimes \mathrm{id}) \circ \iota_n\colon \mathcal{V}\check{\otimes}^p\mathcal{B}\big(\ell^p(n)\big) \to \mathcal{V} \otimes^\lambda \mathcal{B}\big(\ell^p(n)\big) \to \mathcal{X} \otimes^\lambda \mathcal{B}\big(\ell^p(n)\big)$$

is bounded with norm at most $\|T\|$, where ι_n is the identity on $\mathcal{V} \otimes \mathcal{B}(\ell^p(n))$, which is a contraction, as $\check{\otimes}^p$ gives a reasonable cross-norm. Any p-operator space $\mathcal{V}$ whose p-operator structure is the minimal one, *i.e.,* such that $\mathcal{V} = \min \mathcal{V}$ completely isometrically, is called a *minimal p-operator space.*

Proposition 1.1 *The following are equivalent for a p-operator space $\mathcal{V}$:*

(i) *$\mathcal{V}$ is minimal;*
(ii) *for any p-operator space $\mathcal{W}$, $\mathcal{CB}_p(\mathcal{W}, \mathcal{V}) = \mathcal{B}(\mathcal{W}, \mathcal{V})$ isometrically;*
(iii) *for any p-operator space $\mathcal{W}$, $\mathcal{W}\check{\otimes}^p\mathcal{V} = \mathcal{W} \otimes^\lambda \mathcal{V}$.*

Proof Since $\mathcal{CB}_p(\mathcal{W}, \mathcal{V}) \subset \mathcal{B}(\mathcal{W}, \mathcal{V})$ contractively, the observation above gives that (i) implies (ii). Condition (ii) implies that $\mathrm{id}\colon \min \mathcal{V} \to \mathcal{V}$ is completely contractive. Since the converse is automatic, (i) holds.

If (ii) holds, then $\mathcal{CB}_p(\mathcal{W}^*, \mathcal{V}) = \mathcal{B}(\mathcal{W}^*, \mathcal{V})$ isometrically. Thus, by virtue of the definition of the p-operator injective tensor product ([1, §3]) and the well-known injection $\mathcal{W} \otimes \mathcal{V} \hookrightarrow \mathcal{B}(\mathcal{W}^*, \mathcal{V})$, the p-operator injective and injective tensor norms agree on $\mathcal{W} \otimes \mathcal{V}$. ■

The definition of maximal p-operator space will be given in Section 3.

The following rudimentary fact will be referred to a couple of times in the sequel and is an obvious consequence of the density of simple functions in $L^{p'}(\phi)$ and duality.

Lemma 1.2 *For any finite subset $F \subset L^p(\phi)$ and $\varepsilon > 0$, there is an m in $\mathbb{N}$ and a contraction $V\colon L^p(\phi) \to \ell^p(m)$ for which*

$$(1-\varepsilon)\|f\|_{L^p} \leq \|Vf\|_{\ell^p} \leq (1+\varepsilon)\|f\|_{L^p}$$

for f in F.

2 On Minimal p-operator Spaces

In the theory of 2-operator spaces, a special role is played by commutative C*-algebras and completely isometric copies of their subspaces. These are the *minimal operator spaces*. Classical theory tells us that any representation of a commutative C*-algebra $\mathcal{A} \cong \mathcal{C}_0(\Omega)$ on a Hilbert space can be realized as a direct sum of representations on cyclic subspaces, where each in turn produces a Radon measure ν on Ω by which the representation is unitarily equivalent to a representation by multiplication operators on $L^2(\nu)$. We are not aware of any analogue of this result for representation on $\mathcal{SQ}_p$-spaces, or even on L^p-spaces. This reduces our study to representations that are already multiplication representations on L^p-spaces. This gives rise to a more robust theory than might be anticipated.

2.1 On the Space of Continuous Functions as a Minimal p-operator Space

We begin with the continuous bounded functions $\mathcal{C}_b(\Omega)$ on a locally compact space Ω. In this case a familiar formula for the injective tensor product gives, for each n, an

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isometric identification

$$\mathbb{M}_n\big(\min \mathcal{C}_b(\Omega)\big) \cong \mathcal{C}_b\big(\Omega, \mathcal{B}(\ell^p(n))\big), \quad [f_{ij}] \mapsto \big(\omega \mapsto [f_{ij}(\omega)]\big). \tag{2.1}$$

Indeed, the Stone–Čech compactification satisfies $\mathcal{C}(\beta\Omega, M) \cong \mathcal{C}_b(\Omega, M)$ for any finite dimensional Banach space M. We let ν be a Radon measure on Ω and

$$M_\nu \colon \mathcal{C}_b(\Omega) \to \mathcal{B}\big(L^p(\nu)\big)$$

the contractive injection given by $M_\nu(f)\xi(\omega) = f(\omega)\xi(\omega)$ for ν-a.e. ω. We say that ν is faithful if $\nu(U) > 0$ for any open set U. If ν is faithful, then M_ν is an isometry.

The following simple result is required for the next section. The result seems as if it ought to hold for more general L^∞-spaces, except for a certain localization of norm argument at the end of the proof.

Proposition 2.1 *Given a faithful Radon measure ν on Ω,*

$$M_\nu \colon \min \mathcal{C}_b(\Omega) \to \mathcal{B}\big(L^p(\nu)\big)$$

is a complete isometry.

Proof It suffices to verify that each amplification $M_\nu^{(n)}$ is an isometry. We identify $\mathbb{M}_n(\mathcal{B}(L^p(\nu))) \cong \mathcal{B}(L^p(\nu, \ell^p(n)))$, and observe that under this identification, $M_\nu^{(n)}(F)\xi(\omega) = F(\omega)\xi(\omega)$, for F in $\mathcal{C}_b(\Omega, \mathcal{B}(\ell^p(n)))$, ξ in $L^p(\nu, \ell^p(n))$ and ν-a.e. ω. We compute

$$\begin{aligned}
\|M_\nu^{(n)}(F)\xi\|_{L^p(\nu,\ell^p)} &= \left(\int_\Omega \|F(\omega)\xi(\omega)\|_{\ell^p}^p \, d\nu(\omega)\right)^{1/p} \\
&\le \left(\int_\Omega \|F(\omega)\|_{\mathcal{B}(\ell^p)}^p \|\xi(\omega)\|_{\ell^p}^p \, d\nu(\omega)\right)^{1/p} \\
&\le \|F\|_{\mathcal{C}_b(\Omega,\mathcal{B}(\ell^p))} \|\xi\|_{L^p(\nu,\ell^p)}.
\end{aligned}$$

Thus $M_\nu^{(n)}$ is a contraction.

Conversely, given $\varepsilon > 0$, find ω_0 for which $\|F(\omega_0)\|_{\mathcal{B}(\ell^p)} > \|F\|_{\mathcal{C}_b(\Omega,\mathcal{B}(\ell^p))} - \varepsilon$, and then ξ_0 in $\ell^p(n)$ with $\|\xi_0\|_{\ell^p} = 1$ and for which $\|F(\omega_0)\xi_0\|_{\ell^p} = \|F(\omega_0)\|_{\mathcal{B}(\ell^p)}$. Find a compact neighbourhood K of ω_0 such that $\|F(\omega) - F(\omega_0)\|_{\mathcal{B}(\ell^p)} < \varepsilon$ for ω in K. (This is the "localization of norm argument" to which we alluded above.) Then $\xi = \nu(K)^{-1/p} 1_K(\,\cdot\,)\xi_0$ in $L^p(\nu, \ell^p(n))$ is of norm 1 and satisfies

$$\|M_\nu^{(n)}(F)\xi - F(\omega_0)\xi\|_{L^p(\nu,\ell^p)} < \varepsilon.$$

It is immediate that $M_\nu^{(n)}$ is an isometry. ∎

Of course, the above result applies to $\ell^\infty(\Omega)$ for any set Ω. Let $\mathcal{X}$ be a Banach space. We let Ω denote any subset of the unit ball of $\mathcal{X}^*$ that is norming for $\mathcal{X}$, and consider the isometric embedding

$$\mathcal{X} \hookrightarrow \ell^\infty(\Omega), \quad x \mapsto \big(\omega \mapsto \omega(x)\big). \tag{2.2}$$

As already observed in [10], this is a complete isometry of minimal spaces, hence $\min \mathcal{X}$ acts on L^p.

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2.2 L^∞ as a Minimal p-operator Space

We show that, for a suitable measure μ, $L^\infty(\mu)$ attains its minimal p-operator space structure as multiplication operators on $L^p(\mu)$.

We say a measure μ is decomposable if we can write $\mu = \sum_{\iota\in I}\mu_\iota$, where each μ_ι is finite and μ_ι and $\mu_{\iota'}$ are mutually singular for distinct indices. For such measures, we have the duality $L^1(\mu)^* \cong L^\infty(\mu)$, provided we define $L^\infty(\mu)$ to be certain equivalence classes of locally essentially bounded functions; see [9, p. 192]. We will hereafter assume μ is a decomposable measure.

We require that a certain p-analogue of a familiar result in representation theory of commutative C*-algebras holds; see [4, II.1.1], for example, whose standard proof we modify. We let $M_\mu\colon L^\infty(\mu)\to \mathcal{B}(L^p(\mu))$ be the representation given by multiplication operators.

Lemma 2.2 *There is a locally compact space Ω such that $L^\infty(\mu) \cong \mathcal{C}_b(\Omega)$ via a $*$-algebra isomorphism $f \mapsto \widehat{f}$, a faithful Radon measure ν on Ω, and a surjective isometry $U\colon L^p(\nu)\to L^p(\mu)$ such that $UM_\nu(\widehat{f}) = M_\mu(f)U$.*

Proof We first assume that μ is finite. (The proof will work for the σ-finite case as well.) In this case there is a norm 1 cyclic and separating vector ξ for M_μ; indeed, let ξ be any fully supported norm 1 element. We let Ω denote the Gelfand spectrum of $L^\infty(\mu)$ and $f\mapsto \widehat{f}$ the Gelfand transform. We observe that $\widehat{|f|^p} = |\widehat{f}|^p$.

We define ν on Ω by

$$\int_\Omega \widehat{f}\,d\nu = \int f|\xi|^p\,d\mu.$$

Since ξ is fully supported, ν is faithful. We then define $U\colon \mathcal{C}(\Omega)\to L^p(\mu)$ by $U\widehat{f} = f\xi$. We observe that

$$\|U\widehat{f}\|^p_{L^p(\mu)} = \int |f|^p|\xi|^p\,d\mu = \int_\Omega |\widehat{f}|^p\,d\nu = \|\widehat{f}\|^p_{L^p(\nu)}.$$

Since $\mathcal{C}(\Omega)$ is dense in $L^p(\nu)$, and ξ is a cyclic vector, U extends to a surjective isometry on $L^p(\nu)$. Finally, if $f,g\in L^\infty(\mu)$, then

$$UM_\nu(\widehat{f})\widehat{g} = U\widehat{fg} = fg\xi = M_\mu(f)U\widehat{g}$$

which, again by density of $\mathcal{C}(\Omega)$ in $L^p(\nu)$, shows that $UM_\nu(\widehat{f}) = M_\mu(f)U$.

Now consider general decomposable $\mu = \sum_{\iota\in I}\mu_\iota$. For each ι, let Ω_ι denote the Gelfand spectrum of $L^\infty(\mu_\iota)$ and we have C*-isomorphisms

$$L^\infty(\mu)\cong \ell^\infty - \bigoplus_{\iota\in I} L^\infty(\mu_\iota) \cong \ell^\infty - \bigoplus_{\iota\in I}\mathcal{C}(\Omega_\iota)\cong \mathcal{C}_b(\Omega),$$

where $\Omega = \bigsqcup_{\iota\in I}\Omega_\iota$ is the topological coproduct. Let $f\mapsto \widehat{f}$ denote the composite isomorphism. We observe, moreover, that $L^p(\mu)\cong \ell^p\text{-}\bigoplus_{\iota\in I} L^p(\mu_\iota)$, where each $L^p(\mu_\iota)$ is an M_μ-invariant subspace. We let ν_ι be a measure supported on Ω_ι given as above, and we let $U_\iota\colon L^p(\nu_\iota)\to L^p(\mu_\iota)$ be the associated surjective isometry intertwining $M_{\mu_\iota} = M_\mu|_{L^p(\mu_\iota)}$ and M_{ν_ι}. Then $U = \bigoplus_{\iota\in I} U_\iota$ is the desired isometry intertwining M_μ and M_ν. ■

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Theorem 2.3 *The map* $M_\mu\colon \min L^\infty(\mu) \to \mathcal{B}(L^p(\mu))$ *is a complete isometry.*

Proof The above lemma provides a map $f \mapsto \widehat{f}\colon L^\infty(\mu) \to \mathcal{C}_b(\Omega)$, which is a complete isometry for the minimal p-operator space structure on both spaces, a faithful Radon measure ν on Ω, and a surjective isometry $U\colon L^p(\nu) \to L^p(\mu)$ such that $M_\mu(f) = U^{-1}M_\nu(\widehat{f})U$. Since M_ν is completely isometric by Proposition 2.1, we find that M_μ is a complete isometry. ∎

On the topic of $L^\infty(\mu)$, we record the following useful result, aspects of which are folklore. This will be used in Section 3.3.

Lemma 2.4

(i) $M_\mu(L^\infty(\mu))$ *is its own commutant in* $\mathcal{B}(L^p(\mu))$ *and hence a weak*-closed subalgebra.*

(ii) *There is a a completely contractive expectation* $E\colon \mathcal{B}(L^p(\mu)) \to M_\mu(L^\infty(\mu))$, i.e., $E(M_\mu(f)TM_\mu(g)) = M_\mu(f)E(T)M_\mu(g)$ *for* f, g *in* $L^\infty(\mu)$ *and* T *in* $\mathcal{B}(L^p(\mu))$.

Proof (i) Let $\mathcal{F}$ be the family of μ-finite sets. If $F \in \mathcal{F}$, then $1_F \in L^\infty \cap L^p(\mu)$. Fix T in the commutant of $M_\mu(L^\infty(\mu))$ in $\mathcal{B}(L^p(\mu))$ and let $h_F = T1_F$ for F in $\mathcal{F}$. We observe that for ξ in $L^\infty \cap L^p(\mu)$, the space of which is dense in $L^p(\mu)$, that $T(1_F\xi) = T(1_F)\xi = h_F\xi$, from which it easily follows that $h_F \in L^\infty(\mu)$ with $\|h_F\|_\infty \le \|T\|$. It is clear that $1_F h_{F'} = 0$ and $h_F + h_{F'} = h_{F\cup F'}$ if $F \cap F'$ is μ-null. We let $\{F_\iota\}_{\iota\in I}$ be a family of sets witnessing the decomposability of μ. We observe that the net $(\sum_{\iota\in J} h_{F_\iota})_J$, indexed over the increasing family of finite subsets of I, converges weak* to an element h of $L^\infty(\mu)$. Indeed if $\psi \in L^1(\mu)$, then there is a σ-finite set S, so $1_S\psi = \psi$ and

$$\lim_J \int \sum_{\iota\in J} h_{F_\iota}\psi\, d\mu = \int \sum_{\iota\in I_S} h_{F_\iota}\psi\, d\mu,$$

where $I_S = \{\iota : \mu(F_\iota \cap S) > 0\}$ is countable. In particular, $h\psi = \sum_{\iota\in I_S} h_{F_\iota}\psi$. Now if $\xi \in L^p \cap L^\infty(\mu)$ and $\eta \in L^{p'}(\mu)$, we let S be σ-finite so $1_S\xi = \xi$ and we have

$$\int (T\xi)\eta\, d\mu = \int T\Big(\sum_{\iota\in I_S} 1_{F_\iota}\xi\Big)\eta\, d\mu = \int \sum_{\iota\in I_S} T(1_{F_\iota}\xi)\eta\, d\mu$$
$$= \int \sum_{\iota\in I_S} h_{F_\iota}\xi\eta\, d\mu = \int h\xi\eta\, d\mu.$$

Thus $T = M_\mu(h)$. The commutant of any set in $\mathcal{B}(L^p(\mu))$ is weak*-closed.

(ii) We let $U^\infty(\mu) = \{u \in L^\infty(\mu) : u^*u = 1\}$. Let m be an invariant mean on $\ell^\infty(U^\infty(\mu))$, which we may consider, notationally, as a finitely additive measure. We define E by

$$E(T) = \int_{U^\infty(\mu)} M_\mu(u)TM_\mu(u^*)\, dm(u),$$

where the "integral" is understood in the weak* sense. Since $\operatorname{span} U^\infty(\mu) = L^\infty(\mu)$, it is immediate that E is a contractive expectation. If

$$T \in \mathbb{M}_n(\mathcal{B}(L^p(\mu))) \cong \mathcal{B}(\ell^p(n) \otimes^p L^p(\mu)),$$

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then we observe that

$$E^{(n)}(T) = \int_{U^\infty(\mu)} \big(I \otimes M_\mu(u)\big)\, T\big(I \otimes M_\mu(u^*)\big)\ dm(u).$$

Hence E is completely contractive. ∎

3 Maximal p-operator Spaces

3.1 Definitions and Basic Properties

For a Banach space $\mathcal{X}$, we consider the p-operator space structures on $\mathcal{X}$ whose norms on x in $\mathbb{M}_n(\mathcal{X})$ are given by

$$\begin{aligned}\|x\|_{\max_{L^p}} &= \sup\big\{\, \|\pi^{(n)}(x)\| : \pi\colon \mathcal{X} \to \mathcal{B}(L^p(\phi)) \text{ is a contraction, } \phi \text{ is a measure}\big\}\\ &= \sup\big\{\, \|\pi^{(n)}(x)\| : \pi\colon \mathcal{X} \to \mathcal{B}(\ell^p(m)) \text{ is a contraction, } m \in \mathbb{N}\big\}\\ \|x\|_{\max} &= \sup\big\{\, \|\pi^{(n)}(x)\| : \pi\colon \mathcal{X} \to \mathcal{B}(E) \text{ is a contraction, } E \in \mathcal{SQ}_p\big\}.\end{aligned}$$

The equality of the two descriptions of $\|\cdot\|_{\max_{L^p}}$ is an immediate consequence of Lemma 1.2. It is clear that these norms give p-operator space structures on $\mathcal{X}$, which we call the *maximal structure on L^p* and the *maximal structure*, respectively. We denote the associated operator spaces by $\max_{L^p}\mathcal{X}$ and $\max\mathcal{X}$. There is an equivalent formulation of $\max\mathcal{X}$ given in [10, p. 95], presented in a local context. It is clear that $\mathrm{id}\colon \max\mathcal{X} \to \max_{L^p}\mathcal{X}$ is a complete contraction. There is no loss of generality if we replace contractions π, above, by isometries; simply consider the isometry $\mathrm{id}\colon \mathcal{X} \to \min\mathcal{X}$ that acts on L^p by (2.2).

It is clear that $\|v\| \le \|v\|_{\max}$ for every operator space $\mathcal{V}$ and v in $\mathbb{M}_n(\mathcal{V})$. It is unknown to the authors whether the operator space structures max and $\max_{L^p}$ coincide on any non-trivial Banach space. We thus use the following definition. We say that a p-operator space $\mathcal{V}$ is of *maximal type* if for v in $\mathbb{M}_n(\mathcal{V})$ we have

$$\|v\|_{\max_{L^p}} \le \|v\|.$$

Lemma 3.1 *Let $\mathcal{V}$ be a p-operator space. Then the following are equivalent:*

(i) *$\mathcal{V}$ is of maximal type;*
(ii) *$\mathcal{CB}_p(\mathcal{V},\mathcal{Z}) = \mathcal{B}(\mathcal{V},\mathcal{Z})$ isometrically for any p-operator space $\mathcal{Z}$ acting on L^p;*
(iii) *$\mathcal{CB}_p(\mathcal{V},\mathcal{B}(\ell^p(n))) = \mathcal{B}(\mathcal{V},\mathcal{B}(\ell^p(n)))$ isometrically for each n.*

Proof It is the case for any operator space $\mathcal{V}$ that

$$\mathcal{CB}_p\big(\mathcal{V},\mathcal{B}(\ell^p(n))\big) \subset \mathcal{B}\big(\mathcal{V},\mathcal{B}(\ell^p(n))\big)$$

contractively. We obtain the converse inclusion, contractively, only for maximal type p-operator spaces, by definition. Thus (i) is equivalent to (ii). That (ii) implies (iii) is obvious. That (iii) implies (ii) is a consequence of Lemma 1.2. ∎

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Corollary 3.2 *Let $\mathcal{V}$ be a p-operator space. The following are equivalent:*

(i′) *$\mathcal{V}$ is of maximal type;*

(ii′) *$\mathcal{V}\widehat{\otimes}^p\mathcal{W} = \mathcal{V}\otimes^\gamma\mathcal{W}$, isometrically, for any p-operator space $\mathcal{W}$;*

(iii′) *$\mathcal{V}\widehat{\otimes}^p\mathcal{N}(\ell^p(m)) = \mathcal{V}\otimes^\gamma\mathcal{N}(\ell^p(m))$, isometrically, for any m.*

Proof We will show that each statement (n′) of the present result is equivalent to statement (n) of Lemma 3.1

We have that $\mathcal{W}^*$ represents completely isometrically on some L^p by [5, Thm. 4.3]. Hence, thanks to the well-known dual paring $\langle v\otimes w, T\rangle = Tv(w)$ of $\mathcal{V}\otimes\mathcal{W}$ with $\mathcal{B}(\mathcal{V},\mathcal{W}^*)$ and its p-operator space analogue ([5, Prop. 4.9]), if (ii) of the lemma holds, then the p-operator projective and projective tensor norms agree on $\mathcal{V}\otimes\mathcal{W}$. If (ii′) holds, then statement (ii) of the lemma holds whenever $\mathcal{Z}=\mathcal{W}^*$, *i.e.,* for any p-operator dual space. Hence statement (ii) holds with $\mathcal{Z}^{**}$ in place of $\mathcal{Z}$. We let $\kappa_{\mathcal{Z}}\colon\mathcal{Z}\to\mathcal{Z}^{**}$ denote the canonical embedding and have that $\mathcal{B}(\mathcal{V},\mathcal{Z})\cong\kappa_{\mathcal{Z}}\circ\mathcal{B}(\mathcal{V},\mathcal{Z})\subset\mathcal{B}(\mathcal{V},\mathcal{Z}^{**})=\mathcal{CB}_p(\mathcal{V},\mathcal{Z}^{**})$ isometrically. If $\mathcal{Z}$ acts on L^p, then, by [5, Prop. 4.4], $\kappa_{\mathcal{Z}}$ is a complete isometry so $\mathcal{CB}_p(\mathcal{V},\mathcal{Z})\cong\kappa_{\mathcal{Z}}\circ\mathcal{CB}_p(\mathcal{V},\mathcal{Z})\subset\mathcal{CB}_p(\mathcal{V},\mathcal{Z}^{**})$ isometrically, hence $\mathcal{B}(\mathcal{V},\mathcal{Z})\cong\kappa_{\mathcal{Z}}\circ\mathcal{B}(\mathcal{V},\mathcal{Z})=\kappa_{\mathcal{Z}}\circ\mathcal{CB}_p(\mathcal{V},\mathcal{Z})\cong\mathcal{CB}_p(\mathcal{V},\mathcal{Z})$ isometrically, hence statement (ii) holds generally.

Just as above, (iii′) holds if and only if (iii) of the lemma holds. ■

We observe that if $\mathcal{V}$ and $\mathcal{W}$ are each maximal type p-operator spaces, then $\mathcal{V}\widehat{\otimes}^p\mathcal{W}$ is also of maximal type. Indeed, if $\mathcal{Z}$ acts on L^p, then [5, Prop. 4.9] provides isometric identifications

$$\mathcal{CB}_p(\mathcal{V}\widehat{\otimes}^p\mathcal{W},\mathcal{Z})\cong\mathcal{CB}_p(\mathcal{V},\mathcal{CB}_p(\mathcal{W},\mathcal{Z}))=\mathcal{B}(\mathcal{V},\mathcal{B}(\mathcal{W},\mathcal{Z}))=\mathcal{B}(\mathcal{V}\otimes^\gamma\mathcal{W},\mathcal{Z})$$

and we appeal to statements (ii) and (ii′) above. We do not know whether $\max_{L^p}\mathcal{V}\widehat{\otimes}^p\max_{L^p}\mathcal{W}$ is completely isometric to $\max_{L^p}(\mathcal{V}\otimes^\gamma\mathcal{W})$, but this does hold for L^1-spaces, as we will see in Section 3.3.

3.2 Duality and Quotients

Proposition 3.3 (i) *If $\mathcal{V}$ is a maximal type p-operator space, then the dual structure is minimal,* i.e., $\mathcal{V}^*=\min\mathcal{V}^*$. *In particular,* $(\max\mathcal{V})^*=\min\mathcal{V}^*=(\max_{L^p}\mathcal{V})^*$.

(ii) *If $\mathcal{V}$ is a complete quotient of a maximal type p-operator space, then $\mathcal{V}$ is of maximal type.*

Proof (i) We follow the proof from classical operator spaces (see [2, Cor. 2.8] or [8, (3.3.13)]) and use Lemma 3.1. Letting Ω be a dense subset of the unit ball of $\mathcal{V}$, we have complete isometries

$$\mathbb{M}_n(\mathcal{V}^*)\cong\mathcal{CB}\big(\mathcal{V},\mathcal{B}(\ell^p(n))\big)=\mathcal{B}\big(\mathcal{V},\mathcal{B}(\ell^p(n))\big)\widetilde{\subset}\ell^\infty\big(\Omega,\mathcal{B}(\ell^p(n))\big)$$

whose composition is given by $[\psi_{ij}]\mapsto(\omega\mapsto[\psi_{ij}(\omega)])$. By (2.2) this is the minimal p-operator structure on $\mathcal{V}^*$.

(ii) If $q\colon\mathcal{V}\to\mathcal{Z}$ is a complete quotient map, and $T\colon\mathcal{Z}\to\mathcal{B}(\ell^p(n))$ is a linear contraction, then $T\circ q\colon\mathcal{V}\to\mathcal{B}(\ell^p(n))$ is a contraction, hence a complete contraction

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by (i). Thus if z is in the open unit ball of $\mathbb{M}_n(\mathcal{Z})$, there exists v in the open unit ball of $\mathbb{M}_n(\mathcal{V})$ so $z = q^{(n)}(v)$. Then for any linear contraction $T\colon \mathcal{Z} \to \mathcal{B}(\ell^p(n))$ we have $\|T^{(n)}(z)\|_{\mathcal{B}(\ell^p)} = \|(T \circ q)^{(n)}(v)\|_{\mathcal{B}(\ell^p)} < 1$, so T is a complete contraction. ■

We aim to obtain the dual statement to (i), above. We note that unlike in the 2-operator space setting, it is not a priori obvious that $(\min \mathcal{C}(\Omega))^{**} = \min \mathcal{C}(\Omega)^{**}$ completely isometrically, though we will establish this fact below.

We require a preparatory idea from the theory of vector measures. For a compact Hausdorff space Ω we let $M(\Omega)$ denote the space of complex Borel measures on Ω. Furthermore, if E is a Banach space we let $M(\Omega, E)$ denote the E-valued Borel measures on Ω of bounded variation. If E satisfies the Radon-Nikodym property of [6, p. 61], we have

$$M(\Omega, E) = \bigcup_{\nu \in M^+(\Omega)} L^1(\nu, E) \cong \bigcup_{\nu \in M^+(\Omega)} L^1(\nu) \otimes^\gamma E \cong M(\Omega) \otimes^\gamma E \tag{3.1}$$

where the implied isomorphism is isometric. Indeed, if $G \in M(\Omega, E)$, there is ν in $M^+(\Omega)$ and g in $L^1(\nu, E)$ for which $G(B) = \int_B g\, d\nu$, with $\|G\|_{M(\Omega,E)} = |G|(B) = \|g\|_{L^1(\nu,E)}$. It is well-known that $L^1(\nu, E) \cong L^1(\nu) \otimes^\gamma E$ isometrically. Since, by Lebesgue decomposition, $L^1(\nu)$ is contractively complemented in $M(\Omega)$, we have that $L^1(\nu) \otimes^\gamma E$ embeds isometrically into $M(\Omega) \otimes^\gamma E$. Moreover, each element in $M(\Omega) \otimes^\gamma E$ is an element of some $L^1(\nu) \otimes^\gamma E$. Indeed, write an element of the former as $\sum_{k=1}^\infty \nu_k \otimes x_k$, where each $\|x_k\|_E = 1$ and $\sum_{k=1}^\infty \|\nu_k\|_M < \infty$. Then let $\nu = \sum_{k=1}^\infty |\nu_k|$ and observe that each $\nu_k \ll \nu$, so the element is in $L^1(\nu) \otimes^\gamma E$.

Theorem 3.4 *If $\mathcal{W}$ is a minimal operator space, then its dual operator space is maximal on L^p*, i.e., $(\min \mathcal{W})^* = \max_{L^p} \mathcal{W}^*$.

Proof We begin with $\min \mathcal{C}(\Omega)$ for a compact space. From the formula

$$\mathcal{V} \check{\otimes}^p \mathcal{B}(\ell^p(n)) \cong \mathbb{M}_n(\mathcal{V})$$

on one hand, and from (2.1) on the other, we obtain for each n, isometric identifications

$$\min \mathcal{C}(\Omega) \check{\otimes}^p \mathcal{B}\big(\ell^p(n)\big) \cong \mathbb{M}_n\big(\min \mathcal{C}(\Omega)\big) \cong \mathcal{C}\big(\Omega, \mathcal{B}(\ell^p(n))\big).$$

Taking duals, we have from [1, Theo. 3.6] on one hand, and [15, 16] (or see [6, p. 182]) on the other, that

$$\big(\min \mathcal{C}(\Omega)\big)^* \widehat{\otimes}^p \mathcal{N}\big(\ell^p(n)\big) \cong M\big(\Omega, \mathcal{N}(\ell^p(n))\big).$$

Thanks to the fact that finite dimensional spaces enjoy the Radon–Nikodym property, we can use (3.1) on the right-hand side of the above identification to see that

$$\big(\min \mathcal{C}(\Omega)\big)^* \widehat{\otimes}^p \mathcal{N}\big(\ell^p(n)\big) = M(\Omega) \otimes^\gamma \mathcal{N}\big(\ell^p(n)\big)$$

isometrically for each n. By Corollary 3.2 we see that $M(\Omega)$, in is capacity as the dual of $\min \mathcal{C}(\Omega)$, admits a maximal type p-operator space structure. Since this is a dual space, it follows from [5, Thm. 4.3] that this is the maximal structure on L^p.

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Now we consider $\min \mathcal{W} \tilde{\subset} \min \mathcal{C}(\Omega)$, where Ω is the unit ball of $\mathcal{W}^*$ with weak* topology. Hence $\mathcal{W}^* \cong \max_{L^p} M(\Omega)/\{\nu : \langle \nu, w\rangle = 0 \text{ for } w \in \mathcal{W}\}$ completely isometrically. By Proposition 3.3(iii), $\mathcal{W}^*$ is of p-maximal type. But by [5, Thm. 4.3], $\mathcal{W}^*$ acts on some L^p, hence the operator space structure is $\max_{L^p}$. ■

We observe that it is an immediate consequence of Theorem 3.4 and Proposition 3.3(i), that $(\min \mathcal{V})^{**} = \min \mathcal{V}^{**}$ completely isometrically.

As another consequence we see that for any Banach space $\mathcal{X}$ and $\mathcal{Y}$

$$\min \mathcal{X} \check{\otimes}^p \min \mathcal{Y} = \min(\mathcal{X} \otimes^\lambda \mathcal{Y}) \tag{3.2}$$

completely isometrically. Indeed, we have from Lemma 3.1(ii) and (1.1), that

$$\begin{aligned} \mathbb{M}_n\big(\mathcal{CB}_p(\max \mathcal{X}^*, \min \mathcal{Y})\big) &\cong \mathcal{CB}_p\big(\max \mathcal{X}^*, \mathbb{M}_n(\min \mathcal{Y})\big) \\ &= \mathcal{B}\big(\mathcal{X}^*, \mathbb{M}_n(\min \mathcal{Y})\big) \cong \mathcal{B}\big(\mathcal{X}^*, \mathcal{Y} \otimes^\lambda \mathcal{B}(\ell^p(n))\big) \end{aligned}$$

isometrically. Thus the embedding of $\mathbb{M}_n(\mathcal{X} \otimes \mathcal{Y}) \cong \mathcal{X} \otimes \mathcal{Y} \otimes \mathcal{B}(\ell^p(n))$ into the space above establishes that

$$\mathbb{M}_n(\min \mathcal{X} \check{\otimes}^p \min \mathcal{Y}) = \mathcal{X} \otimes^\lambda \mathcal{Y} \otimes^\lambda \mathcal{B}(\ell^p(n))$$

isometrically, for each n. Then (3.2) follows from (1.1).

3.3 L^1 Spaces

Spaces $L^1(\mu)$, for a decomposable measure μ, are the most natural class of maximal p-operator spaces.

Theorem 3.5 *The operator space structure on $L^1(\mu)$, as a subspace of $(\min L^\infty(\mu))^*$, is the maximal structure on L^p, i.e., $\max_{L^p} L^1(\mu)$.*

Proof We will establish that with the operator space structure given by $L^1(\mu) \hookrightarrow (\min L^\infty(\mu))^*$, we have $\mathcal{CB}_p(L^1(\mu), \mathcal{V}) = \mathcal{B}(L^1(\mu), \mathcal{V})$ isometrically, for any p operator space $\mathcal{V}$ acting on some L^p. By Lemma 3.1, this implies that $L^1(\mu)$ is of maximal type. However, since $(\min L^\infty(\mu))^*$ acts on L^p ([5, Thm. 4.3]), this is the $\max_{L^p}$ structure.

The assumption that $\mathcal{V}$ acts on L^p implies that the embedding $\kappa_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}^{**}$ is a complete isometry ([5, Prop. 4.4]). We also note that $L^1(\mu)^* \cong \min L^\infty(\mu)$ completely isometrically. Indeed, as noted in [11, Prop. 1.6.13], it is sufficient, by virtue of [5, Prop. 5.5] to observe that $\min L^\infty(\mu) \cong M_\mu(L^\infty(\mu))$ is weak* closed. This was shown in Lemma 2.4.

We consider, first, the adjoint $S^* : \mathcal{V}^* \to L^1(\Omega)^* \cong L^\infty(\Omega)$, which is completely bounded with $\|S^*\|_{p\mathrm{cb}} = \|S^*\| = \|S\|$ by Proposition 1.1(ii). We then have that $S = S^{**} \circ \kappa_{L^1(\mu)} : L^1(\mu) \to \kappa_{\mathcal{V}}(\mathcal{V}) \cong \mathcal{V}$ satisfies $\|S\|_{p\mathrm{cb}} \le \|S^{**}\|_{p\mathrm{cb}}$, which, by [5, Lem. 4.5], is no greater than $\|S^*\|_{p\mathrm{cb}} = \|S\|$. ■

The following is an immediate consequence of Lemma 2.4 and [5, Prop. 5.6].

Corollary 3.6 *The map $\eta \otimes \xi \mapsto \eta\xi$ extends to a complete quotient map from* $\mathcal{N}(L^p(\mu)) = L^{p'}(\mu) \otimes^\gamma L^p(\mu)$ *onto* $\max_{L^p} L^1(\mu)$.

We obtain the following useful tensor product formulas. If $\mathcal{V}$ is a p-operator space, Corollary 3.2 provides the isometric identifications

$$\max_{L^p} L^1(\mu) \widehat{\otimes}^p \mathcal{V} = L^1(\mu) \otimes^\gamma \mathcal{V} \cong L^1(\mu, \mathcal{V}).$$

We also obtain a completely isometric identification

$$\max_{L^p} L^1(\mu) \widehat{\otimes}^p \max_{L^p} L^1(\nu) = \max_{L^p}(L^1(\mu) \otimes^\gamma L^1(\nu)) \cong \max_{L^p} L^1(\mu \times \nu). \tag{3.3}$$

Indeed, we have an isometric identification $\max_{L^p} L^1(\mu) \widehat{\otimes}^p \max_{L^p} L^1(\nu) = L^1(\mu) \otimes^\gamma L^1(\nu) \cong L^1(\mu \times \nu)$. The first space has dual $\min L^\infty(\mu) \overline{\otimes}_F \min L^\infty(\nu)$ (Fubini product) in $\mathcal{B}(L^p(\mu) \otimes^p L^p(\nu))$ by [5, Thm. 6.3], while the third has dual $L^\infty(\mu \times \nu)$. The latter space acts as multiplication operators on $L^p(\mu \times \nu) \cong L^p(\nu) \otimes^p L^p(\nu)$. This dual identification shows that $\min L^\infty(\mu) \overline{\otimes}_F \min L^\infty(\nu) \cong \min L^\infty(\mu \times \nu)$. Hence (3.3) follows.

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