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ALGEBRAIC CONVERGENCE THEOREMS OF COMPLEX KLEINIAN GROUPS

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Abstract. Let $\{G_{r,i}\}$ be a sequence of r-generator subgroups of $U(1, n; \mathbb{C})$ and G_r be its algebraic limit group. In this paper, two algebraic convergence theorems concerning $\{G_{r,i}\}$ and G_r are obtained. Our results are generalisations of their counterparts in the n-dimensional sense-preserving Möbius group.

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1. Introduction. Let $\mathbb{G}$ be the n-dimensional sense-preserving Möbius group $M(\bar{\mathbb{R}}^n)$ or the unitary group $U(1, n; \mathbb{C})$.

DEFINITION 1.1. Let $\{G_{r,i}\}$ be a sequence of subgroups in group $\mathbb{G}$ and each be generated by $g_{1,i},\ g_{2,i}, \dots,\ g_{r,i}$, where $r = 1, 2, \dots$. If for each t $(1 \le t \le r)$,

$$g_{t,i} \to g_t \in \mathbb{G} \text{ as } i \to \infty,$$

then we say that $\{G_{r,i}\}$ algebraically converges to $G_r = \langle g_1, g_2, \dots, g_r \rangle$.

If for each i, $G_{r,i}$ is a Kleinian group, the problem that when G_r is still a Kleinian group has been investigated by a number of authors.

When $n = 2$, Jørgensen and Klein [**7**] proved the following.

THEOREM JK. *If each $G_{r,i}$ is a r-generator Kleinian group, then the limit group G_r is also a Kleinian group.*

Examples in [**12**] show that the Theorem JK could not be extended to n-dimensional cases ($n \ge 3$) without any modifications. The reason for this phenomenon is that there is a great difference in the fixed point set of elliptic elements between $M(\bar{\mathbb{R}}^2)$ and $M(\bar{\mathbb{R}}^n)$ when $n \ge 3$. Several authors have obtained their analogues in $M(\bar{\mathbb{R}}^n)$ when $n \ge 3$ by adding some condition(s) to control the fixed point set of elliptic elements.

Apanasov [**1**] proved the following.

THEOREM A. *If for each $G_{r,i}$, its generators are of infinite order and $G_{r,i}$ is discrete, then for each t $(1 \le t \le r)$, $g_t = \lim_{i\to\infty} g_{t,i}$ is different from the identity. Furthermore, if each $G_{r,i}$ is a torsion-free Kleinian group, then G_r is also a torsion-free Kleinian group.*

Martin [**9**] proved this.

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THEOREM M. *Let G_r be an algebraic limit group of a sequence of r-generator Kleinian groups of $M(\bar{\mathbb{R}}^n)$ of uniformly bounded torsion. Then G_r is a Kleinian group.*

Wang [**11**] proved this.

THEOREM W. *Let $r < \infty$ and G_r be the algebraic limit group of a sequence of r-generator Kleinian groups $\{G_{r,i}\}$ of $M(\bar{\mathbb{R}}^n)$. If $\{G_{r,i}\}$ satisfies EP-condition, then G_r is a Kleinian group.*

See details in [**11**] for the definitions of uniformly bounded torsion, EP-condition and $WY(G_r)$.

A complex hyperbolic space is more complicated than a real hyperbolic space. For example, it has variable negative curvature, it is a K*ä*hler manifold with biholomorphic automorphisms and its boundary has a natural contact structure, which is locally modelled on the Heisenberg geometry. Because of a closed connection between real and complex hyperbolic geometry, the road map of analogy frequently points the way towards potentially interesting questions. It is interesting to investigate analogous results of a real hyperbolic space in the setting of a complex hyperbolic space.

The purpose of this paper is to find analogous results mentioned above in the setting of a complex hyperbolic space. In order to state our main results, we first recall some notations and facts about a complex hyperbolic space.

The complex hyperbolic n-space $H^n_{\mathbb{C}}$ may be identified with a unit ball in $\mathbb{C}^n$ with the Bergman metric [**6**, **8**]. The group of its holomorphic isometries is the group $U(1, n; \mathbb{C})$ acting on $H^n_{\mathbb{C}}$ and on its boundary $\partial H^n_{\mathbb{C}}$. For a non-trivial element g of $U(1, n; \mathbb{C})$, we say that g is *parabolic* if it has exactly one fixed point and this lies on $\partial H^n_{\mathbb{C}}$; g is *loxodromic* if it has exactly two fixed points and they lie on $\partial H^n_{\mathbb{C}}$ and g is *elliptic* if it has a fixed point in $H^n_{\mathbb{C}}$.

For elliptic element g, let Λ_0 and Λ_i, $i = 1, 2, \ldots, n$ be its negative and positive classes of eigenvalues, respectively. Then the fixed point set of g in $H^n_{\mathbb{C}}$ contains only one fixed point if $\Lambda_0 \neq \Lambda_i$, $i = 1, 2, \ldots, n$ and is a totally geodesic sub-manifold, which is equivalent to $H^m_{\mathbb{C}}$(for some $m \leq n$) if Λ_0 coincides with exactly m of class Λ_i, $i = 1, 2, \ldots, n$. We call an elliptic element g an irrational rotation if $e^{\mathbf{i}\theta} \in \Lambda_t$ with irrational θ for some t. We remark that $U(1, n; \mathbb{C})$ has an elliptic element, which has no fixed point in the boundary $\partial H^n_{\mathbb{C}}$. Such elements are the counterparts of *fixed-point-free* elements in $M(\bar{\mathbb{R}}^n)$.

For subgroup $G \subset U(1, n; \mathbb{C})$, the *limit set* $L(G)$ of G is defined as

$$L(G) = \overline{G(p)} \cap \partial H^n_{\mathbb{C}}, \quad p \in H^n_{\mathbb{C}}.$$

As in [**3**], for subgroup G of $U(1, n; \mathbb{C})$ containing a loxodromic element, let

$$W(G) = \cap_{f \in h(G)} G_{fix(f)},$$

where $h(G)$ is a set of all loxodromic elements in G and $G_{fix(f)} = \{g \in G : fix(f) \subset fix(g)\}$.

A subgroup G of $U(1, n; \mathbb{C})$ is called *non-elementary* if it contains two non-elliptic elements of infinite order with distinct fixed points that are not irrational rotations; otherwise G is called *elementary*. We call a non-elementary and discrete subgroup G of $U(1, n; \mathbb{C})$ a complex Kleinian group.

As in [**11**], a subset H of $U(1, n; \mathbb{C})$ is said to have *uniformly bounded torsion* if there exists an integer M such that

$$ord(g) \leq M \text{ or } ord(g) = \infty \text{ if } g \in H.$$

We refer to [**4–6**, **8**] for more details of these concepts and some properties of a complex hyperbolic space.

When $\mathbb{G} = U(1, n; \mathbb{C})$ in Definition 1.1, we assume that $r < \infty$, and for $\{G_{r,i}\}$ we introduce the following conditions:

We say that $\{G_{r,i}\}$ *satisfies I-condition if any sequence* $\{f_{i_k}\}$ $(f_{i_k} \in G_{r,i_k})$ *satisfying that for each* k, $card[fix(f_{i_k})] = \infty$ *and* $f_{i_k} \to$ *the identity as* $k \to \infty$, *has uniformly bounded torsion. Here* $card(M)$ *denotes the cardinality of set* M.

We say that $\{G_{r,i}\}$ *satisfies IP-condition if* $\{G_{r,i}\}$ *satisfies the following conditions: for any sequence* $\{f_{i_k}\}$ $(f_{i_k} \in G_{r,i_k})$, *if* $card(fix(f_{i_k})) = \infty$ *for each* k, *and* $f_{i_k} \to f$ *as* $k \to \infty$ *with* f *being the identity or parabolic, then* $\{f_{i_k}\}$ *has uniformly bounded torsion.*

As should be apparent, our exposition and results here owe a great deal to Martin's [**9**] and Wang's papers [**11**]. Our main results are the following theorems.

THEOREM 1.1. *Let* $\{G_{r,i}\}$ *be a sequence of groups of* $U(1, n; \mathbb{C})$. *If each* $G_{r,i}$ *is discrete, then the algebraic limit group* G_r *of* $\{G_{r,i}\}$ *is either a complex Kleinian group, or it is elementary, or* $W(G_r)$ *is not finite.*

THEOREM 1.2. *Let* G_r *be the algebraic limit group of complex Kleinian groups* $\{G_{r,i}\}$ *of* $U(1, n; \mathbb{C})$. *If* $\{G_{r,i}\}$ *satisfies IP-condition, then* G_r *is a complex Kleinian group.*

2. Several lemmas.

The following lemma is crucial for us.

LEMMA 2.1. (cf. [**5**]). *Suppose that* f *and* $g \in U(1, n; \mathbb{C})$ *generate a discrete and non-elementary group. Then*

(1) *if* f *is parabolic or loxodromic, we have*

$$max\{N(f), N([f, g])\} \geq 2 - \sqrt{3},$$

where $[f, g] = fgf^{-1}g^{-1}$ *is a commutator of* f *and* g, $N(f) = \|f - I_{n+1}\|$ *and* $\|.\|$ *is the Hilbert–Schmidt norm;*

(2) *if* f *is elliptic, we have*

$$max\{N(f), N([f, g^i]) \mid i = 1, 2, \ldots, n+1\} \geq 2 - \sqrt{3}.$$

The following lemma is a classification of elementary subgroups of $U(1, n; \mathbb{C})$.

LEMMA 2.2. (cf. [**2**]).

(1) *If* G *contains a parabolic element but no loxodromic element, then* G *is elementary if and only if it fixes a point in* $\partial H^n_{\mathbb{C}}$*;*

(2) *If* G *contains a loxodromic element, then* G *is elementary if and only if it fixes a point in* $\partial H^n_{\mathbb{C}}$ *or a point-pair* $\{x, y\} \subset \partial H^n_{\mathbb{C}}$*;*

(3) G *is purely elliptic, i.e. each non-trivial element of* G *is elliptic, then* G *is elementary and fixes a point in* $\overline{H^n_{\mathbb{C}}}$.

LEMMA 2.3. (cf. [**10**]). *Let* G *be a discrete subgroup of* $U(1, n; \mathbb{C})$ *such that every element has finite order, then* G *is finite.*

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LEMMA 2.4. *Let $f \in U(1, n; \mathbb{C})$ be an elliptic element of order m. If $2 \leq m < M$, then there is a constant $\delta(M)$ such that*

$$N(f) > \delta(M).$$

Proof. Let the eigenvalues of f be $\lambda_j = e^{i\theta_j}$ $(j = 1, \dots, n+1)$. By Schur's unitary triangularization theorem, there is a matrix $U \in U(n+1; \mathbb{C})$ such that

$$Uf\bar{U}^T = \begin{pmatrix} \lambda_1 & * & * & * \\ 0 & \lambda_2 & * & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & \lambda_{n+1} \end{pmatrix}.$$

Hence, $\|f - I_{n+1}\|^2 \geq \sum_{j=1}^{n+1} |\lambda_j - 1|^2 = 2(n+1) - 2\sum_{j=1}^{n+1} \cos\theta_j$. It follows from $f^m = I_{n+1}$ that there is a j such that $|\cos\theta_j| \neq 1$ and $\theta_j = \frac{2p\pi}{m}$ (here p and m are prime). Hence,

$$1 - \cos\theta_j \geq 1 - |\cos\theta_j| > 1 - \left|\cos\frac{\pi}{m}\right|.$$

Set $\delta(M) = \sqrt{1 - |\cos\frac{\pi}{m}|}$. Then $\delta(M)$ is the desired number. □

From Lemma 2.4, we have the following.

COROLLARY 2.1. *If $f_j \to I_{n+1}$ as $j \to \infty$ and f_j are elliptic elements with ord$(f_j) < m$, then for all large enough j, $f_j = I_{n+1}$.*

COROLLARY 2.2. *If f_j are elliptic elements with ord$(f_j) \leq M$ and $f_j \to f$ as $j \to \infty$, then f is an elliptic element with order m $(2 \leq m \leq M)$, and for all large enough j, order$(f_j) = m$.*

Proof. By the Pigeonhole Principle, we can choose a subsequence f_{jk} such that each element with order m $(2 \leq m \leq M)$. Then $f_{jk}^m \to f^m$, i.e. f is an elliptic element with order m. □

LEMMA 2.5. (cf. [**9**, Lemma 2.8]). *Let x and y be two distinct points in $\overline{H_{\mathbb{C}}^n}$. If $f \in U(1, n; \mathbb{C})$ interchanges x and y, then*

$$N(f) \geq \sqrt{2}.$$

Proof. Since f interchanges $x, y \in \overline{H_{\mathbb{C}}^n}$, we can find $\lambda_i \neq 0$ $(i = 1, 2)$ such that

$$f\begin{pmatrix} 1 \\ x \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ y \end{pmatrix}, \quad f\begin{pmatrix} 1 \\ y \end{pmatrix} = \lambda_2 \begin{pmatrix} 1 \\ x \end{pmatrix}. \tag{2.1}$$

By the linear algebra theory, we can find $U \in U(n+1; \mathbb{C})$ such that

$$U\begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} t_1 \\ 0 \end{pmatrix}, \quad U\begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}, \tag{2.2}$$

where $t_1, c_1, c_2 \in \mathbb{C}$ and $|t_1|^2 = 1 + \|x\|^2$, $|c_1|^2 + |c_2|^2 = 1 + \|y\|^2$. Since $x \neq y$, we have that $c_2 \neq 0$.

Let $g = (b_{ij})_{i,j=1,\ldots,n+1} = Uf\bar{U}^T$. It follows from (2.1) and (2.2) that

$$b_{11}t_1 = c_1\lambda_1,\ b_{21}t_1 = c_2\lambda_1 \quad \text{and} \quad b_{21}c_1 + b_{22}c_2 = 0,$$

which implies that $b_{11} = -b_{22}$.

Thus,

$$\begin{aligned}\|f - I_{n+1}\|^2 = \|g - I_{n+1}\|^2 &\geq |b_{11} - 1|^2 + |b_{22} - 1|^2 \\ &\geq \frac{1}{2}\,|b_{11} + b_{22} - 2|^2 = 2,\end{aligned}$$

i.e. $N(f) \geq \sqrt{2}$. □

Lemma 2.6. *Let $\{f_i\}$ and $\{g_i\}$ be two sequences of $U(1, n; \mathbb{C})$, which converge to f and g, respectively. Suppose that each group $\langle f_i, g_i\rangle$ is a complex Kleinian group and each f_i is of infinite order. Then f is of infinite order and $\langle f, g\rangle$ is a complex Kleinian group if $\{\langle f_i, g_i\rangle\}$ satisfies I-condition.*

Proof. We first prove that $\langle f, g\rangle$ is discrete.

Suppose that $\langle f, g\rangle$ is not discrete. Then there is a sequence $\{h_j\}$ of $\langle f, g\rangle$ such that $h_j \to I_{n+1}$ as $j \to \infty$. Let $h_{j,i}$ be the corresponding elements in $\langle f_i, g_i\rangle$ such that

$$h_{j,i} \to h_j \text{ as } i \to \infty.$$

These elements form a sequence $h_{j_k,i_k} \in \langle f_{i_k}, g_{i_k}\rangle$ satisfying

$$h_{j_k,i_k} \to I_{n+1} \text{ as } k \to \infty.$$

Since $\langle f_{i_k}, g_{i_k}\rangle$ is a complex Kleinian and $\{\langle f_i, g_i\rangle\}$ satisfies I-condition, by Lemma 2.4, we conclude that h_{j_k,i_k} is parabolic or loxodromic.

Let q_{1,i_k} and q_{2,i_k} be two loxodromic elements of $\langle f_{i_k}, g_{i_k}\rangle$ having no common fixed point. Since $h_{j_k,i_k} \to I_{n+1}$, there is a positive M such that for all $k > M$,

$$max\{N(h_{j_k,i_k}), N([h_{j_k,i_k}, q_{t,i_k}])\} < 2 - \sqrt{3}, \quad (t = 1, 2).$$

By Lemma 2.1 and the discreteness of $\langle q_{t,i_k}, h_{j_k,i_k}\rangle$, we know that $\langle q_{t,i_k}, h_{j_k,i_k}\rangle$ is elementary, which is a contradiction to Lemma 2.2. The above shows that $\langle f, g\rangle$ is discrete.

We now come to prove that $\langle f, g\rangle$ is non-elementary.

We first show that f is parabolic or loxodromic. Since $\langle f, g\rangle$ is discrete, f cannot be an irrational rotation. Suppose that there is a positive M such that $f^M = I_{n+1}$. Then $f_i^M \neq I_{n+1}$ and

$$f_i^M \to I_{n+1} \quad \text{as } i \to \infty.$$

Hence, for sufficiently large i,

$$max\big\{N\big(f_i^M\big), N\big(\big[f_i^M, g_i^t\big]\big) \mid t = 1, 2, \ldots, n+1\big\} < 2 - \sqrt{3}.$$

By Lemma 2.1, $\langle f_i^M, g_i\rangle$, which are subgroups of discrete group $\langle f_i, g_i\rangle$, are elementary for sufficiently large i. This implies that $\langle f_i, g_i\rangle$ is elementary. This is a contradiction.

We then show that $\langle f, g\rangle$ is non-elementary.

Suppose that $\langle f, g\rangle$ is elementary. As in [**9**, Proposition 2.7], we can show that $\langle f, g\rangle$ is virtually abelian. Thus, there exist two integers t and s such that

$$[f^t, gf^s g^{-1}] = I_{n+1}.$$

Let $h_i = [f_i^t, g_i f_i^s g_i^{-1}]$. Then,

$$h_i \in \langle f_i, g_i\rangle, \quad h_i \neq I_{n+1} \ \text{ and } \ h_i \to I_{n+1} \ \text{ as } \ i \to \infty.$$

As in the proof of discreteness of $\langle f, g\rangle$, we can get a contradiction. Thus, $\langle f, g\rangle$ is non-elementary. □

3. Proofs of convergence theorems.

Proof of Theorem 1.1. Assume that G_r is non-elementary and $W(G_r)$ is finite. We need to prove that G_r is discrete.

Suppose that G_r is not discrete. Then there is a sequence $\{g_j\}$ of G_r such that

$$g_j \to I_{n+1} \ \text{ as } \ j \to \infty.$$

We will get a contradiction by showing that each g_j belongs to $W(G_r)$ for large enough j.

Since G_r is non-elementary, G_r contains two loxodromic elements f_1 and f_2 sharing no common fixed point. Then for large enough j,

$$N(g_j) + \sum_{k=1}^{n+1} N\big([g_j, f_m^k]\big) < 2 - \sqrt{3} \ \ (m = 1, 2).$$

Let $g_{j,t}$ and $f_{m,t}$ be the corresponding entries in $G_{r,t}$. That is $g_{j,t} \to g_j$ and $f_{m,t} \to f_m$ as $t \to \infty$. Then for large enough t and j,

$$N(g_{j,t}) + \sum_{k=1}^{n+1} N\big([g_{j,t}, f_{m,t}^k]\big) < 2 - \sqrt{3} \ \ (m = 1, 2).$$

Lemma 2.1 implies that $< f_{m,t}, g_{j,t} >$ $(m = 1, 2)$ are elementary for large enough t and j. Since $f_{m,t}$ is a loxodromic element and $g_{j,t}$ cannot interchange the two fixed points of $f_{m,t}$ for large enough t and j, it follows from Lemma 2.2 that $fix(f_{m,t}) \subset fix(g_{j,t})$ holds for each $m = 1, 2$ and sufficiently large t and j. Hence, there is an integer k_1 such that for all $j \geq k_1, fix(f_m) \subset fix(g_j)$ holds for each $m = 1, 2$.

Let $T(k_1) = \bigcap_{j \geq k_1} fix(g_j)$. Then $T(k_1)$ contains the linear span of fixed points of f_m and so has dimension of at least 1 for large positive integer k_1. Thus, by passing to a subsequence of $\{g_j\}$ (denoted by the same manner), we have

$$T(k_1) \neq \emptyset \ \text{ and } \ 1 \leq dim[T(k_1)] \leq n - 1.$$

Suppose that there exists a loxodromic element $h \in G_r$ such that

$$fix(h) \cap T(k_1) = \emptyset.$$

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As an above reasoning (if needed, passing to a subsequence), there exists k_2 $(> k_1)$ such that

$$fix(h) \subset T(k_2) \text{ and } dim[T(k_1)] + 1 \leq dim[T(k_2)] \leq n - 1.$$

By repeating this step finite times, we can find k such that

$$fix(g) \subset T(k)$$

holds for any loxodromic element $g \in G_r$. Then $g_j \in W(G_r)$ for all $j > k$. This is a contradiction to the fact that $W(G_r)$ is finite.

The proof is complete. □

Proof of Theorem 1.2. We divide our proof into three parts.

(1) First we prove that G_r is discrete.

Suppose that G_r is not discrete. Then there is a sequence $\{g_j\}$ of G_r such that

$$g_j \to I_{n+1} \text{ as } j \to \infty,$$

and we can find a corresponding sequence $\{g_{j_k,i_k}\}$ such that

$$g_{j_k,i_k} \in G_{r,i_k} \text{ and } g_{j_k,i_k} \to I_{n+1} \text{ as } k \to \infty. \tag{3.1}$$

Since $\{G_{r,i}\}$ satisfies *IP-condition* and $G_{r,i}$ is discrete for each i, we may assume that for each k, g_{j_k,i_k} is parabolic or loxodromic. For each k, there is at least one generator of G_{r,i_k}, say f_{1,i_k}, such that $\langle f_{1,i_k}, g_{j_k,i_k}\rangle$ is non-elementary, which is a contradiction to Lemma 2.1. The above proves the discreteness of G_r.

(2) Then we prove that G_r is infinite.

Suppose that G_r is finite. As in the proof of part (1) in [**9**, Proposition 5.8], we can find a sequence $\{h_i\}$ such that $\{h_i\} \in G_{r,i}$ and $h_i \to I_{n+1}$ as $i \to \infty$. Similar discussions as in the proof of part (1) show that this is impossible. Hence, G_r is infinite.

(3) We prove that G_r is non-elementary.

Suppose that G_r is elementary. It follows from the infiniteness of G_r and Lemma 2.3 that G_r contains some element h of infinite order, i.e. h is parabolic or loxodromic. Let $\{h_i\}$ be the corresponding elements in $\{G_{r,i}\}$. Then

$$h_i \to h \text{ as } i \to \infty.$$

Suppose that h is loxodromic. Then h_i is loxodromic for all sufficiently large i. For each generator f_s $(s = 1, 2, \dots, r)$ of G_r, as $\langle f_s, h\rangle$ is discrete and elementary, there exist k_s and p_s such that $[h^{k_s}, f_s h^{p_s} f_s^{-1}] = I_{n+1}$. Since $\{G_{r,i}\}$ satisfies *IP-condition* and $G_{r,i}$ is discrete, we have

$$[h_i^{k_s}, f_{s,i} h_i^{p_s} f_{s,i}^{-1}] \neq I_{n+1} \text{ and } [h_i^{k_s}, f_{s,i} h_i^{p_s} f_{s,i}^{-1}] \to I_{n+1} \text{ as } i \to \infty.$$

Let $g_i = [h_i^{k_s}, f_{s,i} h_i^{p_s} f_{s,i}^{-1}] \to I_{n+1}$ as $i \to \infty$. Then, as in the proof of part (1), we get a contradiction.

Thus, we may assume that h is parabolic. Since $\{G_{r,i}\}$ satisfies *IP-condition*, by Corollary 2.2, we know that h_i is parabolic or loxodromic.

Suppose that there is a subsequence of $\{h_i\}$ such that each h_i is parabolic or loxodromic. Then for i, there is a generator, say $f_{1,i}$, such that the group $\langle f_{1,i}, h_i\rangle$ is

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non-elementary. By Lemma 2.6, the limit group of the sequence $\{\langle f_{1,i}, h_i\rangle\}$ is non-elementary. This implies that G_r is non-elementary. This is a contradiction.

The proof is complete. □

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