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A boson-fermion correspondence in cohomological Donaldson–Thomas theory

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Abstract

We introduce and study a fermionisation procedure for the cohomological Hall algebra $\mathcal{H}_{\Pi_Q}$ of representations of a preprojective algebra, that selectively switches the cohomological parity of the BPS Lie algebra from even to odd. We do so by determining the cohomological Donaldson–Thomas invariants of central extensions of preprojective algebras studied in the work of Etingof and Rains, via deformed dimensional reduction. Via the same techniques, we determine the Borel–Moore homology of the stack of representations of the μ-deformed preprojective algebra introduced by Crawley–Boevey and Holland, for all dimension vectors. This provides a common generalisation of the results of Crawley-Boevey and Van den Bergh on the cohomology of smooth moduli schemes of representations of deformed preprojective algebras and my earlier results on the Borel–Moore homology of the stack of representations of the undeformed preprojective algebra.

1. Introduction

Given a quiver Q, in [26] Kontsevich and Soibelman define the cohomological Hall algebra

$$\mathcal{A}_Q = \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}(\mathfrak{M}_{\mathbf{d}}(Q), \mathbb{Q})[-\chi_Q(\mathbf{d}, \mathbf{d})], \tag{1.1}$$

which has as underlying vector space the singular cohomology of the stack of finite-dimensional complex representations of Q, shifted in cohomological degree by the Euler form (see (2.1) for the definition). As indicated by (1.1), this algebra is graded by the dimension vectors of representations of Q. The multiplication is defined by taking push-forward and pull-back of cohomology in the usual correspondence diagram

$$\mathfrak{M}(Q) \times \mathfrak{M}(Q) \xleftarrow{\pi_1 \times \pi_3} \mathfrak{Exact}(Q) \xrightarrow{\pi_2} \mathfrak{M}(Q). \tag{1.2}$$

In the diagram (1.2), $\mathfrak{M}(Q) = \coprod_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathfrak{M}_{\mathbf{d}}(Q)$ is the stack of finite-dimensional Q-representations, $\mathfrak{Exact}(Q)$ is the stack of short exact sequences of Q-representations, and π_i is the morphism taking a short exact sequence to its ith entry.

If Q is moreover symmetric (i.e. for every pair of vertices i,j, there are as many arrows from i to j as from j to i), then the multiplication respects the cohomological degree, and a theorem of Efimov [15] states that (a slight modification of) $\mathcal{A}_Q$ is a free *super*commutative algebra; the 'super' here means that elements in odd cohomological degrees anti-commute with each other.

There are a couple of instances in which this result is possible to check by hand; indeed, it was observed in [26, Section 2.5] that if $Q^{(l)}$ denotes the quiver with one vertex and l loops, then

$$\mathcal{A}_{Q^{(0)}} \cong \mathbf{Sym}((\mathbb{Q}[u])\,[-1]) \tag{1.3}$$

$$\mathcal{A}_{Q^{(1)}} \cong \mathbf{Sym}(\mathbb{Q}[u])\,, \tag{1.4}$$

where $[-1]$ denotes a cohomological shift, and u^i is placed in cohomological degree $2i$. Both algebras are free supercommutative algebras generated by a countable set of symbols α_i for $i \geq 0$, with α_i placed in cohomological degree $1 + 2i$ in the zero-loop case, and in cohomological degree $2i$ in the one-loop case. In other words, (1.3) states that $\mathcal{A}_{Q^{(0)}}$ is a free exterior algebra with countably many generators, while (1.4) states that $\mathcal{A}_{Q^{(1)}}$ is a free commutative algebra with the same generators. As observed in [26], the fact that the underlying vector spaces of the two algebras are the same (since $\mathfrak{M}_n(Q^{(l)})$ is homotopic to BGl_n, regardless of l) can be seen as a consequence of the boson-fermion correspondence in representation theory.

Now let Q be an arbitrary finite quiver. We form the tripled quiver $\tilde{Q}$, by adjoining to the quiver Q an arrow a^* with the opposite orientation to a, for each a an arrow of Q, and also adjoining a loop ω_i at each vertex i of Q. So for instance, $\widetilde{Q^{(1)}} \cong Q^{(3)}$. The quiver $\tilde{Q}$ carries a canonical cubic potential

$$\tilde{W} = \sum_{a \in Q_1} [a, a^*] \sum_{i \in Q_0} \omega_i, \tag{1.5}$$

and one may define (again as in [26]) the critical cohomological Hall algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}$. Again, the multiplication respects cohomological degree. The underlying vector space of this algebra is the vanishing cycle cohomology of the function $\mathrm{Tr}(\tilde{W})$ on the stack $\mathfrak{M}\!\left(\tilde{Q}\right)$. The multiplication is defined via pull-back and push-forward of vanishing cycle cohomology along the same correspondence diagram (1.2). Via dimensional reduction [8] the algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ is isomorphic [38, 45] to the cohomological Hall algebra structure on the Borel–Moore homology of the stack of representations of the preprojective algebra Π_Q constructed by Schiffmann and Vasserot in [40] and studied in [46].

In general, the algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ is not supercommutative; for example, $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ contains the universal enveloping algebra of the Kac–Moody Lie algebra associated with the quiver Q' obtained by removing all vertices from Q that support 1-cycles [10]. On the other hand, via the cohomological integrality theorem [11], the entire algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ satisfies a Poincaré–Birkhoff–Witt theorem, meaning that we have an $\mathbb{N}^{Q_0}$-graded isomorphism of cohomologically graded vector spaces (but *not* of algebras):

$$\mathcal{A}_{\tilde{Q}, \tilde{W}} \cong \mathbf{Sym}\left(\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \left(\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \otimes \mathbb{Q}[u] \right)[-1] \right). \tag{1.6}$$

In the isomorphism (1.6), $\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}$ is a cohomologically graded vector space, the whole of $\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \otimes \mathbb{Q}[u]$ is placed in $\mathbb{N}^{Q_0}$-degree $\mathbf{d}$, u^i has cohomological degree $2i$, and the symbol $[-1]$ denotes the cohomological shift as before. Moreover by [9], the Poincaré polynomial of the cohomologically graded vector space $\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}$ satisfies the relation

$$\mathrm{p}\!\left(\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}, q^{1/2} \right) := \sum_{i \in \mathbb{Z}} \dim\!\left(\mathrm{BPS}^i_{\tilde{Q}, \tilde{W}, \mathbf{d}} \right) q^{i/2} = q^{-1/2} \mathtt{a}_{Q, \mathbf{d}}\left(q^{-1} \right), \tag{1.7}$$

where $\mathtt{a}_{Q, \mathbf{d}}(q)$ is the *Kac polynomial* [20], counting absolutely indecomposable $\mathbf{d}$-dimensional Q-representations over a field of order q. For example, we observe that $Q^{(1)} \cong \widetilde{Q^{(0)}}$ and calculate

$$\mathtt{a}_{Q^{(0)}, \mathbf{d}}(q) = \begin{cases} 1 & \text{if } \mathbf{d} = 1 \\ 0 & \text{if } \mathbf{d} \geq 2 \end{cases}$$

recovering (1.4) from (1.6) and (1.7).

In particular, we see that for arbitrary finite Q, the algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}}$ is *bosonic*, in the sense that it is situated entirely in even cohomological degree. Having observed that $\mathcal{A}_{Q^{(1)}} = \mathcal{A}_{\widetilde{Q^{(0)}}, \tilde{W}}$ has a fermionic counterpart $\mathcal{A}_{Q^{(0)}}$, we may ask the following

Question 1.1. For a finite quiver Q, is there a 'fermionic' version of $\mathcal{A}_{\tilde{Q},\tilde{W}}$?

It is a consequence of the *purity* of the cohomological BPS invariants for the quiver with potential $\left(\tilde{Q}, \tilde{W}\right)$ (again, see [9]) that there is an equality

$$\mathrm{p}\left(\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}, q^{1/2}\right) = \chi_{q^{1/2}}\left(\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}\right) \tag{1.8}$$

between the Poincaré polynomial and the virtual Poincaré polynomial[1] of $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}$. The polynomials $\chi_{q^{1/2}}(\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}})$ are essentially by definition the *refined* BPS invariants for the quiver $\tilde{Q}$ with potential $\tilde{W}$. Expressed in the language of refined Donaldson–Thomas theory, the analogue of Question 1.1 is

Question 1.2. Is there quiver Q' with potential W' such that the refined BPS invariants satisfy

$$\chi_{q^{1/2}}\left(\mathrm{BPS}_{Q',W',\mathbf{d}}\right) = q^{1/2}\chi_{q^{1/2}}\left(\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}\right)?.$$

In this paper, we will answer these two questions in the affirmative.

1.1 Counting rational curves

At least in the case in which Q is an affine Dynkin quiver, there are strong hints from the McKay correspondence that the answer to Question 1.2 should be at least a partial 'yes'. We recall some geometric background; see [3, 24] for more details.

Fix a finite subgroup $G \subset SL_2(\mathbb{C})$ and denote by $X_0 = \mathbb{C}^2/G$ the associated Kleinian singularity. We denote by Γ the McKay graph of G, and by Γ' the full ADE type sub-graph of Γ obtained by removing the vertex corresponding to the trivial representation. We denote by $p: Y_0 \to X_0$ the minimal resolution of X_0. The surface X_0 contains an isolated singularity x, and the exceptional fibre $p^{-1}(x)$ consists of a chain of rational curves, with incidence graph Γ'. The space Y_0 has a universal deformation Y parametrised by $\mathfrak{h}$, the Cartan subalgebra of the simple Lie algebra corresponding to Γ', so that we have a Cartesian diagram

$$
\begin{array}{ccc}
Y_0 & \hookrightarrow & Y \\
\downarrow & & \downarrow \pi \\
0 & \hookrightarrow & \mathfrak{h}.
\end{array}
$$

The generic fibre of π contains no rational curves, while if $h \in \mathfrak{h}$ lies in a root hyperplane corresponding to a vertex i of Γ', the rational curve C_i corresponding to i deforms along the line $t \cdot h$ for $t \in \mathbb{C}$ [3, Prop.2.2]. We pick $h \in \mathfrak{h}$ and form the Cartesian diagram

$$
\begin{array}{ccc}
Y^h & \longrightarrow & Y \\
\downarrow \pi' & & \downarrow \pi \\
\mathbb{A}^1 & \xrightarrow{t \mapsto t \cdot h} & \mathfrak{h}.
\end{array} \tag{1.9}
$$

For $\beta \in \mathbf{H}_2(Y^h, \mathbb{Z})$ and $n \in \mathbb{N}$, let $\mathcal{M}_{\beta,n}\left(Y^h\right)$ denote the moduli space of semistable coherent sheaves $\mathcal{F}$ on Y', with fundamental class of the support of $\mathcal{F}$ equal to β, and with $\chi(\mathcal{F}) = n$.

Consider the case in which $\beta = [C_i]$ for some $i \in Q_0$, so that stability is equivalent to semistability. Since a stable coherent sheaf $\mathcal{F}$ cannot split as a direct sum, it is supported on a single fibre of the morphism π'. If C_i deforms along $\mathbb{A}^1$, with curve over $t \in \mathbb{A}_1$ labelled $C_{i,t}$, then in each fibre $\pi'(t)$ there is a unique semistable coherent sheaf $\mathcal{O}_{C_{i,t}}(n-1)$ with Euler characteristic n and $\left[\mathcal{O}_{C_{i,t}}(n-1)\right] = \beta$. If C_i does not deform, then there is a unique semistable coherent sheaf on the whole of Y^h of class (β, n), supported above $0 \in \mathbb{A}^1$. So

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[1] See [32] for the virtual Poincaré polynomial analogue of (1.7); any two of purity, (1.7), and the virtual analogue of (1.7) imply the third.

$$\mathcal{M}_{[C_i],n}\left(Y^h\right) = \begin{cases} \text{pt} & \text{if } h \notin i^\perp \\ \mathbb{A}^1 & \text{if } h \in i^\perp. \end{cases}$$

The *cohomological BPS invariants* $\text{BPS}_{\beta,n}$ of the 3-fold Y^h are in general hard to define rigorously, involving vanishing cycle cohomology, d-critical structures [19], orientation data, perverse filtrations, etc. However, for simple classes like the ones we are considering here, the definition/calculation boils down to something more straightforward:

$$\text{BPS}_{[C_i],n}\left(Y^h\right) = \mathbf{H}\big(\mathcal{M}_{[C_i],n}\big(Y^h\big), \mathbb{Q}\big)_{\text{vir}}$$

$$\cong \begin{cases} \mathbb{Q} & \text{if } h \notin i^\perp \\ \mathbb{Q}[1] & \text{if } h \in i^\perp. \end{cases}$$

The subscript vir denotes the cohomological shift $[\dim(\mathcal{M}_{[C_i],n})]$ by the dimension of the space we are taking the singular cohomology of and accounts for the shift in the $h \in i^\perp$ case. So in both cases, the cohomological BPS invariants for simple curve classes are one-dimensional vector spaces and are concentrated in even or odd cohomological degree, depending on whether the choice of h means that C_i is rigid or not.

Similarly, the definition and calculation of the simplest degree zero cohomological BPS invariant $\text{BPS}_{0,1}\big(Y^h\big)$ is much easier than the general case, and we have

$$\text{BPS}_{0,1}\big(Y^h\big) \cong \mathbf{H}(Y^h, \mathbb{Q})_{\text{vir}}.$$

It is easy to verify that the singular cohomology of Y^h does not depend on the choice of h at all. In general, one expects $\text{BPS}_{0,n} \cong \text{BPS}_{0,1}$ (compare with [2]), so it turns out that the degree zero[2] cohomological DT theory of Y^h does not depend on h.

To put this geometric discussion in very leading language: for the most degenerate case $h = 0$, all of the cohomological DT theory is bosonic, since the vanishing cycle cohomology ends up living in even cohomological degree (taking into account the shift $[-1]$ in (1.6)). On the other hand, modifying the deforming family defined by $h \in \mathfrak{h}$ to be more generic, a portion of the cohomological DT theory is fermionised, depending on which root hyperplanes h avoids.

1.2 The noncommutative conifold and central extensions of the preprojective algebra

Let Q' be the A_1 quiver, and let Q be its affine extension, which we label as follows:

$$Q = \qquad 0 \overset{a}{\underset{b}{\rightleftarrows}} 1. \tag{1.10}$$

Then as a special case of [21], there is a derived equivalence between the category of finitely generated modules for the preprojective algebra Π_Q of Q, and the category of coherent sheaves on Y_0, the minimal resolution of the type A_1 singularity defined by the equation $x^2 + y^2 = z^2$:

$$\mathcal{D}^{\text{b}}_{\text{fg}}\big(\Pi_Q - \text{Mod}\big) \overset{\cong}{\to} \mathcal{D}^{\text{b}}(Coh(Y_0)).$$

We have that

$$\tilde{Q} = \qquad \omega_0 \,\circlearrowleft\, 0 \, \substack{\overset{b^*}{\frown} \\ \overset{a}{\underset{b}{\rightrightarrows}} \\ \underset{a^*}{\smile}} \, 1 \,\circlearrowright\, \omega_1$$

---

[2]By which we mean the Donaldson–Thomas theory of coherent sheaves with zero-dimensional support.

The Kac polynomials of Q are possible to calculate by hand, so that we can calculate the cohomological BPS invariants for $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right)$ via (1.7)

$$\mathrm{BPS}_{\tilde{Q},\tilde{W},(m,n)} \cong \begin{cases} \mathbb{Q}[3] \oplus \mathbb{Q}[1] & \text{if } m = n \\ \mathbb{Q}[1] & \text{if } m = n \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.11}$$

There is an isomorphism $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right) \cong \Pi_Q[\omega] = \Pi_Q \otimes \mathbb{C}[\omega]$, so that we have in addition a derived equivalence (see e.g. [42])

$$\mathcal{D}^{\mathrm{b}}_{\mathrm{fg}}\left(\mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right) - \mathrm{Mod}\right) \overset{\cong}{\to} \mathcal{D}^{\mathrm{b}}\left(\mathrm{Coh}(Y^0)\right),$$

where Y^0 is defined via the construction in the previous subsection by setting $h = 0 \in \mathfrak{h}$, i.e. $Y^0 = Y_0 \times \mathbb{A}^1$.

For the A_1-singularity, the Cartan subalgebra $\mathfrak{h}$ is one-dimensional, so aside from 0 there is an essentially unique choice of $h \in \mathfrak{h}$. Defining $Y_{\mathrm{con}} = Y^h$ for a *nonzero* choice of $h \in \mathfrak{h}$ in diagram (1.9), we obtain the resolved conifold. As noted in the previous section, instead of having an $\mathbb{A}^1$-family of rational curves giving rise to cohomological BPS invariants in odd degrees, the resolved conifold contains a unique rigid curve. It follows that the cohomological BPS invariants corresponding to sheaves supported on the curve flip parity and are supported in even cohomological degrees, so that they contribute to the fermionic part of the cohomological DT theory of the resolved conifold (as ever, taking into account the shift defined as in (1.6))

The resolved conifold also has a noncommutative model, studied in this context by Szendrői, which we recall (see [41] for details on the noncommutative Donaldson–Thomas theory of the conifold, and also [17, 27, 37] for more recent work on CoHAs related to toric 3-folds). We consider the double

$$\overline{Q} = \qquad 0 \underset{a^*}{\overset{b^*}{\underset{b}{\overset{a}{\rightrightarrows}}}} 1.$$
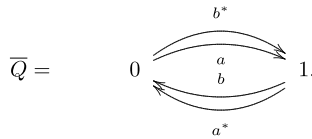
Set $W_{\mathrm{KW}} = aa^*bb^* - a^*ab^*b$ to be the Klebanov–Witten potential [25]. Then (e.g. as a special case of a result due to Van den Bergh [43]), there is a derived equivalence

$$\mathcal{D}^{\mathrm{b}}_{\mathrm{fg}}\left(\mathrm{Jac}\left(\overline{Q}, W_{\mathrm{KW}}\right) - \mathrm{Mod}\right) \overset{\cong}{\to} \mathcal{D}^{\mathrm{b}}(\mathrm{Coh}(Y_{\mathrm{con}})).$$

To interpolate between the two cases ($h = 0, h \neq 0$), it turns out to be more instructive to consider the quiver $\tilde{Q}$ with the potential

$$\tilde{W}^{(1,-1)} = \tilde{W} + \omega_0^2 - \omega_1^2.$$

As we recall in Section 2.1, the resulting Jacobi algebra has already been studied: it is a special case of the central extensions of Π_Q introduced by Etingof and Rains in [16]. There is an isomorphism (see Example 2.2)

$$\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{(1,-1)}\right) \cong \mathrm{Jac}\left(\overline{Q}, W_{\mathrm{KW}}\right) \tag{1.12}$$

and the cohomological DT theory of the Jacobi algebras in (1.12) turns out to be the same. The cohomological BPS invariants for the noncommutative conifold can be deduced from [30] and purity (proved as in [12, Thm.4.7]):

$$\mathrm{BPS}_{\tilde{Q},\tilde{W}^{(1,-1)},m,n} \cong \begin{cases} \mathbf{H}(Y_{\mathrm{con}}, \mathbb{Q})[3] \cong \mathbb{Q}[3] \oplus \mathbb{Q}[1] & \text{if } m = n \\ \mathbb{Q} & \text{if } m = n \pm 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1.13}$$

Comparing (1.11) with (1.13), we see exactly the same pattern as in the commutative case of Section 1.1, with the passage from trivial $\mathbb{A}^1$-deformations of Y_0 to nontrivial ones replaced in the context of noncommutative algebraic geometry by the passage from the trivial central extension of the algebra Π_Q to the nontrivial ones constructed by Etingof and Rains; this change provokes a flip in the parity of some, but not all, of the cohomological BPS invariants.

1.3 Main results

In the rest of the paper, we prove that the above discussion regarding the (noncommutative) conifold forms part of a general procedure for (selectively) 'fermionising' the cohomological Hall algebras of preprojective algebras. This culminates in the following theorem:

Theorem A. *Let* $\mu \in \mathbb{C}^{Q_0}$. *Set* $\tilde{W}^\mu = \sum_{a \in Q_1} [a, a^*] \sum_{i \in Q_0} \omega_i + \frac{1}{2} \sum_{i \in Q_0} \mu_i \omega_i^2$. *Then the cohomological BPS invariants for the quiver* $\tilde{Q}$ *with potential* $\tilde{W}^\mu$ *satisfy*

$$
\mathrm{p}\big(\mathrm{BPS}_{\tilde{Q},\tilde{W}^\mu,\mathbf{d}}, q^{1/2}\big) = \begin{cases} \mathsf{a}_{Q,\mathbf{d}}\big(q^{-1}\big) & \text{if } \mathbf{d} \cdot \mu \neq 0 \\ q^{-1/2} \mathsf{a}_{Q,\mathbf{d}}\big(q^{-1}\big) & \text{if } \mathbf{d} \cdot \mu = 0, \end{cases} \tag{1.14}
$$

where $\mathsf{a}_{Q,\mathbf{d}}(q)$ *are the Kac polynomials for* Q. *In addition, the natural mixed Hodge structure on the cohomological BPS invariants* $\mathrm{BPS}_{\tilde{Q},\tilde{W}^\mu,\mathbf{d}}$ *is pure, of Tate type, so that we have isomorphisms of Hodge theoretic BPS invariants*

$$
\mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q},\tilde{W}^\mu,\mathbf{d}} \cong \begin{cases} \bigoplus_{i \in \mathbb{Z}} (\mathscr{L}^i)^{\oplus \mathsf{a}_{Q,\mathbf{d},-i}} & \text{if } \mathbf{d} \cdot \mu \neq 0 \\ \bigoplus_{i \in \mathbb{Z}} \big(\mathscr{L}^{i-1/2}\big)^{\oplus \mathsf{a}_{Q,\mathbf{d},-i}} & \text{if } \mathbf{d} \cdot \mu = 0, \end{cases} \tag{1.15}
$$

where $\mathscr{L} := \mathbf{H}_c(\mathbb{A}^1, \mathbb{Q})$. *By purity, the virtual Poincaré polynomials of the cohomological BPS invariants agree with the above Poincaré polynomials, so that we have equalities for the refined BPS invariants*

$$
\chi_{q^{1/2}}\big(\mathrm{BPS}_{\tilde{Q},\tilde{W}^\mu,\mathbf{d}}\big) = \begin{cases} \mathsf{a}_{Q,\mathbf{d}}\big(q^{-1}\big) & \text{if } \mathbf{d} \cdot \mu \neq 0 \\ q^{-1/2} \mathsf{a}_{Q,\mathbf{d}}\big(q^{-1}\big) & \text{if } \mathbf{d} \cdot \mu = 0. \end{cases} \tag{1.16}
$$

Comparing with (1.7), we see that for dimension vectors $\mathbf{d}$ satisfying $\mathbf{d} \cdot \mu \neq 0$ the cohomological BPS invariants have switched parity. In particular, for generic choices of μ, the algebra $\mathcal{A}_{\tilde{Q},\tilde{W}^\mu}$ is a fermionised version of $\mathcal{A}_{\tilde{Q},\tilde{W}}$, so (1.15) and (1.16) answer Questions 1.1 and 1.2 respectively in the affirmative.

Still fixing $\mu \in \mathbb{C}^{Q_0}$ the *deformed preprojective algebra*, introduced by Crawley–Boevey and Holland in [6], is defined by

$$
\Pi_{Q,\mu} := \mathbb{C}\overline{Q} \Big/ \Big\langle \sum_{a \in Q_1} [a, a^*] + \sum_{i \in Q_0} \mu_i e_i \Big\rangle,
$$

where e_i is the path of length zero beginning and ending at the vertex i. Via the methods used to prove Theorem A, we are able to calculate the Borel–Moore homology (along with its mixed Hodge structure) of all stacks of representations of deformed preprojective algebras, simultaneously generalising a result of Crawley–Boevey and Van den Bergh [7] from the case of generic μ and indivisible dimension vector $\mathbf{d}$, and the result from [9] which deals with the case $\mu = 0$ and arbitrary $\mathbf{d}$:

Theorem B. *For arbitrary* $\mu \in R$ *and* $\mathbf{d} \in \mathbb{N}^{Q_0}$ *there is an isomorphism of* $\mathbb{N}^{Q_0}$*-graded mixed Hodge structures*

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}_c\big(\mathfrak{M}_{\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q}\big) \otimes \mathscr{L}^{\chi_Q(\mathbf{d},\mathbf{d})} \cong \mathbf{Sym}\left(\bigoplus_{\substack{0 \neq \mathbf{d} \in \mathbb{N}^{Q_0} \\ \mathbf{d} \cdot \mu = 0}} \mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}^{\mathrm{hdg}, \vee} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}} \right), \qquad (1.17)$$

where

$$\mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}^{\mathrm{hdg}, \vee} \cong \bigoplus_{i \in \mathbb{Z}} \big(\mathscr{L}^{i+1/2}\big)^{\oplus \mathtt{a}_{Q, \mathbf{d}, i}}$$

and

$$\mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} = \bigoplus_{i \geq 0} \mathscr{L}^{-1/2 - i}.$$

In particular, the compactly supported cohomology of $\mathfrak{M}(\Pi_{Q,\mu})$*, the stack of representations of the deformed preprojective algebra, is pure, of Tate type.*

See Section 3.3 for the proof of Theorems A and B.

2. Cohomological DT theory for quivers with potential

2.1 Some algebras from quivers

A quiver is determined by a set of vertices Q_0, a set of edges Q_1, and two morphisms $s, t : Q_1 \to Q_0$ taking an arrow to its source and target respectively. We always assume that Q_0 and Q_1 are finite. We define the Euler form

$$\chi_Q(\cdot, \cdot) : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z} \qquad (2.1)$$

$$(\mathbf{d}, \mathrm{e}) \mapsto \sum_{a \in Q_1} \mathbf{d}_{s(a)} \mathbf{d}_{t(a)} - \sum_{i \in Q_0} \mathbf{d}_i \mathrm{e}_i.$$

Where there is no possibility of confusion, we drop the quiver Q from the notation and just write $\chi(\cdot, \cdot)$. We denote by $\overline{Q}$ the doubled quiver of Q, obtained by adding an arrow a^* for every arrow $a \in Q_1$, where a^* has the opposite orientation to a. We denote by $\tilde{Q}$ the tripled quiver, obtained from $\overline{Q}$ by adding a loop ω_i at each vertex $i \in Q_0$.

Given a ring A and a quiver Q, we denote by AQ the free path algebra of Q with coefficients in A. We denote by $R \subset \mathbb{C}Q$ the semisimple subalgebra spanned by length zero paths, so we may identify $R = \mathbb{C}^{Q_0}$. We denote by Π_Q the preprojective algebra for Q, defined to be the quotient of the free path algebra $\mathbb{C}\overline{Q}$ by the two-sided ideal generated by the element $\sum_{a \in Q_1} [a, a^*]$. As in the introduction, we denote by $\Pi_Q[\omega]$ the trivial extension obtained by adjoining a central element ω to the algebra Π_Q, i.e. $\Pi_Q[\omega] = \Pi_Q \otimes \mathbb{C}[\omega]$.

Let $\mu \in R$. We recall the central extension of Π_Q introduced by Etingof and Rains [16]:

$$\Pi_Q^\mu = \mathbb{C}[\omega]\overline{Q} / \left\langle \sum_{a \in Q_1} [a, a^*] + \mu\omega \right\rangle.$$

There is an obvious isomorphism

$$\Pi_Q^0 \cong \Pi_Q[\omega]$$

and natural isomorphisms

$$\Pi_{Q,\mu} \cong \Pi_Q^\mu / \langle \omega - 1 \rangle$$

$$\Pi_Q \cong \Pi_Q^\mu / \langle \omega \rangle,$$

where $\Pi_{Q,\mu}$ is the deformed preprojective algebra recalled in the introduction. The algebra Π_Q^μ provides an $\mathbb{A}^1$-family of algebras interpolating between the preprojective algebra Π_Q and the deformed preprojective algebra $\Pi_{Q,\mu}$.

Let Q be a quiver and let $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ be a potential, i.e. a linear combination of cyclic words, where cyclic words are considered to be equivalent if they can be cyclically permuted to each other. We will call the data of a quiver with potential (Q, W) a QP. Given $a \in Q_1$, if $W = a_1 \dots a_n$ is a single cyclic word we define

$$\partial W/\partial a = \sum_{a_m = a} a_{m+1}a_{m+2} \dots a_n a_1 \dots a_{m-1}$$

and define $\partial W/\partial a$ for general W by extending linearly. We define

$$\mathrm{Jac}(Q, W) = \mathbb{C}Q/\langle \partial W/\partial a \mid a \in Q_1 \rangle.$$

In this paper, we will study Jacobi algebras obtained from the tripled QP $\left(\tilde{Q}, \tilde{W}\right)$ defined in (1.5) by adding polynomials in the extra loops ω_i; we refer the reader to [18, Sect. 4] for general background on this construction, [36, 42] for the noncommutative geometry background in type ADE, and [4] for the physics perspective.

Proposition 2.1. [18, Ex. 4.3.5] *Let* $\mu = \sum_i \mu_i e_i$. *Set* $\tilde{W}^\mu = \tilde{W} + \frac{1}{2}\sum_{i \in Q_0} \mu_i \omega_i^2$. *Then there is an isomorphism*

$$\Pi_Q^\mu \cong \mathrm{Jac}\left(\tilde{Q}, \tilde{W}^\mu\right).$$

In particular, there is a natural isomorphism

$$\Pi_Q[\omega] \cong \mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right).$$

Proof. This follows more or less from the definitions. The noncommutative derivatives of $\tilde{W}^\mu$ with respect to the arrows a impose the relation that ω commutes with the arrows a^*, and vice versa, while the noncommutative derivatives with respect to the loops ω_i impose the defining relations of Π_Q^μ as a quotient of $\mathbb{C}[\omega]\overline{Q}$. $\qquad\square$

Example 2.2. *Let Q be defined as in (1.10), and set* $\mu = e_0 - e_1$. *Then*

$$\tilde{W}^\mu = \omega_0(a^*a - bb^*) - \omega_1(aa^* - b^*b) + \frac{1}{2}\left(\omega_0^2 - \omega_1^2\right).$$

After the noncommutative change of variables

$$\omega_0 \mapsto \omega_0 - a^*a + bb^*$$
$$\omega_1 \mapsto \omega_1 - aa^* + b^*b$$

the potential transforms to

$$W = \frac{1}{2}\left(\omega_0^2 - \omega_1^2\right) + b^*baa^* - bb^*a^*a.$$

Now the relations $\partial W/\partial \omega_i = \pm\omega_i$ mean that in the Jacobi algebra we may simply remove the loops ω_i. Thus, there is a natural isomorphism

$$\mathrm{Jac}\left(\tilde{Q}, W\right) \cong \mathrm{Jac}(Q_{\mathrm{con}}, W_{\mathrm{KW}}),$$

giving the isomorphism (1.12). In particular, the noncommutative conifold is isomorphic to one of the central extensions of Π_Q considered above, for Q as in (1.10).

Later we will use the following elementary result.

Proposition 2.3. *Let ρ be a $\mathbf{d}$-dimensional simple $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$-module. Then, the operator $\rho(\omega)$ acts on the underlying vector space of ρ by multiplication by some scalar $\lambda \in \mathbb{C}$, and if $\mu \cdot \mathbf{d} \neq 0$ then $\lambda = 0$.*

Proof. The first part follows from the fact that $\omega = \sum_{i \in Q_0} \omega_i$ is central in $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$, so each eigenspace of $\rho(\omega)$ is preserved by the action of $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$. For the second part, consider the relation in $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$

$$0 = \sum_{i \in Q_0} \partial \tilde{W}^{\mu} / \partial \omega_i = \sum_{i \in Q_0} \mu_i \omega_i + \sum_{a \in Q_1} [a, a^*].$$

Applying ρ and taking the trace, the final sum vanishes, and we find

$$0 = \sum_{i \in Q_0} \mu_i \lambda \mathrm{Tr}\left(\mathrm{Id}_{\mathbf{d}_i \times \mathbf{d}_i}\right)$$
$$= \lambda \mu \cdot \mathbf{d}$$

as required. $\qquad \square$

2.2 Moduli spaces of quiver representations

Given a quiver Q and a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ we set

$$\mathbb{A}_{\mathbf{d}}(Q) := \prod_{a \in Q_1} \mathrm{Hom}\left(\mathbb{C}^{\mathbf{d}_{s(a)}}, \mathbb{C}^{\mathbf{d}_{t(a)}}\right)$$
$$\mathrm{GL}_{\mathbf{d}} := \prod_{i \in Q_0} \mathrm{GL}_{\mathbf{d}_i}(\mathbb{C}).$$

The group $\mathrm{GL}_{\mathbf{d}}$ acts on $\mathbb{A}_{\mathbf{d}}(Q)$ via change of basis. We denote by $\mathfrak{M}_{\mathbf{d}}(Q)$ the stack of $\mathbf{d}$-dimensional $\mathbb{C}Q$-modules. There is an isomorphism of stacks

$$\mathfrak{M}_{\mathbf{d}}(Q) \cong \mathbb{A}_{\mathbf{d}}(Q)/\mathrm{GL}_{\mathbf{d}}.$$

We denote by $\mathcal{M}_{\mathbf{d}}(Q)$ the coarse moduli space of $\mathbf{d}$-dimensional $\mathbb{C}Q$-modules. Geometric K-points of $\mathcal{M}_{\mathbf{d}}(Q)$ are in natural bijection with semisimple KQ-modules. There is an isomorphism

$$\mathcal{M}_{\mathbf{d}}(Q) \cong \mathrm{Spec}\left(\Gamma(\mathbb{A}_{\mathbf{d}}(Q))^{\mathrm{GL}_{\mathbf{d}}}\right).$$

We denote by $\mathrm{JH}_{\mathbf{d}} : \mathfrak{M}_{\mathbf{d}}(Q) \to \mathcal{M}_{\mathbf{d}}(Q)$ the affinisation morphism. Although this morphism is not projective, it is *approximated by projective maps* in the sense of [11], meaning that JH_* and $\mathrm{JH}_!$ commute with vanishing cycle functors (introduced in the next section).

For spaces and morphisms involving a subscript $\mathbf{d}$, if we omit the subscript, the union over all dimension vectors is intended.

There is a finite morphism [33]

$$\oplus : \mathcal{M}(Q) \times \mathcal{M}(Q) \to \mathcal{M}(Q)$$

which at the level of geometric points takes a pair of KQ-modules to their direct sum. Since this morphism is invariant under swapping the two factors of $\mathcal{M}(Q)$ in the domain, and finite morphisms are exact with respect to the perverse t structure, we obtain an induced symmetric monoidal product on $\mathbf{Perv}(\mathcal{M}(Q))$, defined by

$$\mathcal{F}' \boxtimes_{\oplus} \mathcal{F}'' := \oplus_*\left(\mathcal{F}' \boxtimes \mathcal{F}''\right).$$

Given a potential $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$, we form the function $\mathrm{Tr}(W)_{\mathbf{d}}$ on $\mathbb{A}_{\mathbf{d}}(Q)$. This is well defined and $\mathrm{GL}_{\mathbf{d}}$-invariant by cyclic invariance of trace. As such, $\mathrm{Tr}(W)_{\mathbf{d}}$ induces functions on $\mathfrak{M}_{\mathbf{d}}(Q)$ and $\mathcal{M}_{\mathbf{d}}(Q)$, which we continue to denote by $\mathrm{Tr}(W)_{\mathbf{d}}$, or just $\mathrm{Tr}(W)$ if there is no risk of ambiguity.

2.3 Cohomological Donaldson–Thomas theory of quivers with potential

Given a function f on a smooth complex variety X, we define $X_0 = f^{-1}(0)$ and consider the diagram

$$
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \mathbb{A}^1 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \exp} \\
X_0 \overset{\kappa}{\hookrightarrow} X & \longrightarrow & \mathbb{A}^1,
\end{array}
$$

in which the square is Cartesian. Then, we define the nearby cycles functor $\psi_f : \mathcal{D}^b(\mathbf{Perv}(X)) \to \mathcal{D}^b(\mathbf{Perv}(X))$ via

$$
\psi_f = \kappa_* \kappa^* p_* p^*.
$$

The vanishing cycles functor ϕ_f is defined so that for $\mathcal{F} \in \mathrm{Ob}\big(\mathcal{D}^b(\mathbf{Perv}(X))\big)$ there is a distinguished triangle

$$
\kappa_* \kappa^* \mathcal{F} \to \psi_f \mathcal{F} \to \phi_f \mathcal{F}.
$$

Both ${}^{\mathfrak{p}}\psi_f := \psi_f[-1]$ and ${}^{\mathfrak{p}}\phi_f := \phi_f[-1]$ send perverse sheaves to perverse sheaves [23, Cor. 10.3.13] and (naturally) commute with Verdier duality [29].

We give a lightning account of the critical cohomological Hall algebra associated with a quiver with potential. More details can be found in [11, 26]. For a stack $\mathfrak{M}$ for which each connected component is irreducible and generically smooth, we define the intersection complex

$$
\mathcal{IC}_{\mathfrak{M}} := \coprod_{\mathfrak{N} \in \pi_0(\mathfrak{M})} \mathcal{IC}_{\mathfrak{N}}(\mathbb{Q}_{\mathfrak{N}^{\mathrm{sm}}}[\dim(\mathfrak{N})]),
$$

i.e. it is the intermediate extension of the constant perverse sheaf from the smooth locus. We define

$$
\mathcal{RA}_{Q,W} := \mathrm{JH}_* \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathcal{IC}_{\mathfrak{M}(Q)}.
$$

The morphism π_2 from (1.2) is proper, so that there is a natural integration map

$$
\alpha_{\mathbf{d}', \mathbf{d}''} : \pi_{2,*} \mathbb{Q}_{\mathfrak{E}\mathrm{xact}_{\mathbf{d}', \mathbf{d}''}(Q)} \to \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}'+\mathbf{d}''}(Q)} \big[-2\chi(\mathbf{d}', \mathbf{d}'')\big]
$$

(the shift is given by the relative dimension of π_2). Composing appropriate shifts of the morphisms

$$
\oplus_* \, \mathrm{JH}_* \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \big(\mathbb{Q}_{\mathfrak{M}(Q) \times \mathfrak{M}(Q)} \to (\pi_1 \times \pi_3)_* \mathbb{Q}_{\mathfrak{E}\mathrm{xact}} \big)
$$

and the sum of $\mathrm{JH}_* \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \alpha_{\mathbf{d}', \mathbf{d}''}$ over pairs $(\mathbf{d}', \mathbf{d}'')$, and using commutativity of vanishing cycle functors with proper and with smooth morphisms, we obtain the morphism

$$
\beta : \oplus_* (\mathrm{JH} \times \mathrm{JH})_* \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathcal{IC}_{\mathfrak{M}_{\mathbf{d}'}(Q) \times \mathfrak{M}_{\mathbf{d}''}(Q)} \to \mathrm{JH}_* \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathcal{IC}_{\mathfrak{M}_{\mathbf{d}}(Q)}.
$$

Finally, composing β with $\oplus_*(\mathrm{JH} \times \mathrm{JH})_* \mathtt{TS}$, where $\mathtt{TS}$ is (a shift of) the Thom–Sebastiani isomorphism [28]

$$
\mathtt{TS} : {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}'}(Q)} \boxtimes {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}''}(Q)} \overset{\cong}{\to} {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathbb{Q}_{\mathfrak{M}_{\mathbf{d}'}(Q) \times \mathfrak{M}_{\mathbf{d}''}(Q)},
$$

we define the (relative) Hall algebra multiplication

$$
\mathcal{RA}_{Q,W} \boxtimes_{\oplus} \mathcal{RA}_{Q,W} \to \mathcal{RA}_{Q,W}. \tag{2.2}
$$

The cohomology

$$
\begin{aligned}
\mathcal{A}_{Q,W} &:= \mathbf{H}\big(\mathfrak{M}(Q), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)} \mathcal{IC}_{\mathfrak{M}(Q)} \big) \\
&\cong \mathbf{H}\big(\mathcal{M}(Q), \mathcal{RA}_{Q,W} \big)
\end{aligned}
$$

has a $\mathbb{N}^{Q_0}$-grading by dimension vectors induced by the decomposition $\mathcal{M}(Q) = \bigsqcup_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathcal{M}_{\mathbf{d}}(Q)$, and the associative product induced by taking derived global sections of the morphism (2.2) respects this grading.

Assume that Q is symmetric[3]. For $\mathbf{d} \in \mathbb{N}^{Q_0}$, we define

$$\mathcal{BPS}_{Q,W,\mathbf{d}} := \begin{cases} {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)}\mathcal{IC}_{\mathcal{M}_{\mathbf{d}}(Q)} & \text{if there is a } \mathbf{d}\text{-dimensional simple } \mathbb{C}Q\text{-module} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

According to our conventions, $\mathcal{IC}_{\mathcal{M}_{\mathbf{d}}(Q)}$ is the intermediate extension of the constant perverse sheaf $\mathbb{Q}_{\mathcal{M}_{\mathbf{d}}^{simp}(Q)}[1 - \chi(\mathbf{d}, \mathbf{d})]$ on the (open, dense) subscheme of $\mathcal{M}_{\mathbf{d}}(Q)$ corresponding to simple modules. Since $\mathcal{IC}_{\mathcal{M}_{\mathbf{d}}(Q)}$ is Verdier self-dual, and vanishing cycle functors commute with Verdier duality [29], there are natural isomorphisms

$$\mathbb{D}\mathcal{BPS}_{Q,W,\mathbf{d}} \cong \mathcal{BPS}_{Q,W,\mathbf{d}}. \tag{2.4}$$

We recall the following version of the *cohomological integrality theorem* from [11]

Theorem 2.4. *There is an isomorphism of bounded above complexes of perverse sheaves*

$$\mathrm{JH}_! \, {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)}\mathcal{IC}_{\mathfrak{M}(Q)} \cong \mathbf{Sym}_{\boxtimes_{\oplus}}\left(\bigoplus_{\mathbb{N}^{Q_0} \ni \mathbf{d} \neq 0} \mathcal{BPS}_{Q,W,\mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right),$$

where

$$\mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \mathbb{Q}[1 + 2i].$$

We define the *cohomological BPS invariants*

$$\mathrm{BPS}_{Q,W,\mathbf{d}} := \mathbf{H}(\mathcal{M}_{\mathbf{d}}(Q), \mathcal{BPS}_{Q,W,\mathbf{d}}).$$

Applying the compactly supported cohomology functor to Theorem 2.4, and using self-Verdier duality (2.4) of $\mathcal{BPS}_{Q,W,\mathbf{d}}$, yields

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}_c(\mathfrak{M}_{\mathbf{d}}(Q), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(W)}\mathcal{IC}_{\mathfrak{M}_{\mathbf{d}}(Q)}) \cong \mathbf{Sym}\left(\bigoplus_{\mathbb{N}^{Q_0} \ni \mathbf{d} \neq 0} \mathrm{BPS}_{Q,W,\mathbf{d}}^{\vee} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right).$$

The *BPS invariants* of the Jacobi algebra *Jac(Q,W)* are defined via

$$\begin{aligned} \omega_{Q,W,\mathbf{d}} &= \chi\big(\mathbf{H}(\mathcal{M}_{\mathbf{d}}(Q), \mathcal{BPS}_{Q,W,\mathbf{d}})\big) \\ &:= \sum_{i \in \mathbb{Z}} (-1)^i \dim\big(\mathbf{H}^i\big(\mathcal{M}_{\mathbf{d}}(Q), \mathcal{BPS}_{Q,W,\mathbf{d}}\big)\big) \\ &= \chi\big(\mathbf{H}_c\big(\mathcal{M}_{\mathbf{d}}(Q), \mathcal{BPS}_{Q,W,\mathbf{d}}\big)\big), \end{aligned}$$

where the final identity again follows from Verdier self-duality of the BPS sheaf. Turning to Verdier duals, we have instead

Theorem 2.5. [11] *There is an isomorphism of unbounded complexes of perverse sheaves*

$$\mathcal{RA}_{Q,W} \cong \mathbf{Sym}_{\boxtimes_{\oplus}}\left(\bigoplus_{\mathbb{N}^{Q_0} \ni \mathbf{d} \neq 0} \mathcal{BPS}_{Q,W,\mathbf{d}} \otimes \mathbf{H}(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right),$$

---

[3]There is a version of the integrality theorem for non-symmetric quivers, concerning vanishing cycle cohomology of stacks of semistable $\mathbb{C}Q$-modules, but we won't need it in this paper. See [11, Thm. A Thm. C]

where

$$\mathbf{H}(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} \mathbb{Q}[-1-2i],$$

so that

$$^{\mathfrak{p}}\mathcal{H}^1\big(\mathcal{R}\mathcal{A}_{Q,W}\big) = \big(^{\mathfrak{p}}\tau^{\leq 1}\mathcal{R}\mathcal{A}_{Q,W}\big)[1] \cong \bigoplus_{\mathbb{N}^{Q_0} \ni \mathbf{d}\neq 0} \mathcal{BPS}_{Q,W,\mathbf{d}}.$$

Applying the natural transformation $^{\mathfrak{p}}\tau^{\leq 1} \to \mathrm{id}$ *to* $\mathcal{R}\mathcal{A}_{Q,W}$ *and taking hypercohomology, there is a natural inclusion*

$$\mathfrak{g}_{Q,W} := \mathrm{BPS}_{Q,W}[-1] = \mathbf{H}\big(\mathcal{M}(Q), \mathcal{BPS}_{Q,W}\big)[-1] \hookrightarrow \mathcal{A}_{Q,W}.$$

The image of this inclusion is closed under the commutator[4] *Lie bracket induced by the associative algebra structure on* $\mathcal{A}_{Q,W}$.

The resulting Lie algebra $\mathfrak{g}_{Q,W}$ is called the *BPS Lie algebra* for the pair (Q,W), see [11] for more details.

2.4 Hodge theoretic BPS invariants

We give another lightning introduction, this time to Hodge theoretic DT theory, via monodromic mixed Hodge modules. For more details, we refer the reader to [11, 26], and for a comparison with the treatment of monodromic mixed Hodge modules in [39], we refer the reader to [11, Sec. 2]. Mixed Hodge structures are important in the subject of refined DT theory, since the extra q-variable appearing in refined DT theory keeps track of the weight filtration on certain mixed Hodge structures. On the other hand, our main result states that all mixed Hodge structures appearing in this paper are pure, so that weight polynomials can be replaced by Poincaré polynomials. The takeaway is that this section can be skimmed by the reader that is happy to use the purity part of Theorem A to identify the refined BPS invariants of $\mathrm{Jac}\big(\tilde{Q}, \tilde{W}^{\mu}\big)$ with the Poincaré polynomials of the BPS cohomology.

For X a variety we denote by $\mathbf{MHM}(X)$ the category of mixed Hodge modules on X. There is an equivalence of categories between $\mathbf{MHM}(\mathrm{pt})$ and the category of graded-polarisable mixed Hodge structures. Let $\mathcal{B}_X$ denote the full subcategory of $\mathbf{MHM}(X \times \mathbb{A}^1)$ containing those mixed Hodge modules $\mathcal{F}$ such that for each $x \in X$ and $i \in \mathbb{Z}$ the mixed Hodge modules

$$\mathcal{H}^i\big(\big(\{x\} \times \mathbb{A}^1 \to X \times \mathbb{A}^1\big)^* \mathcal{F}\big) \tag{2.5}$$

are locally constant away from $x \times \{0\}$. We denote by $\mathcal{C}_X$ the full subcategory of $\mathcal{B}_X$ containing those $\mathcal{F}$ such that each (2.5) is constant. Equivalently, we may define $\mathcal{C}_X$ as the essential image of $\pi_X^*[1]$, for $\pi_X : X \times \mathbb{A}^1 \to X$ the projection. We denote by $\mathbf{MMHM}(X)$ the Serre quotient $\mathcal{B}_X/\mathcal{C}_X$. There is an embedding of categories $\mathbf{MHM}(X) \hookrightarrow \mathbf{MMHM}(X)$ defined via

$$\big(X \times \{0\} \hookrightarrow X \times \mathbb{A}^1\big)_* : \mathbf{MHM}(X) \to \mathbf{MHM}\big(X \times \mathbb{A}^1\big).$$

The direct image functor

$$\Theta := \big(X \times \mathbb{C}^* \hookrightarrow X \times \mathbb{A}^1\big)_*$$

induces an equivalence of categories between the category of mixed Hodge modules on $X \times \mathbb{C}^*$ with locally constant cohomology sheaves after restriction to each $\{x\} \times \mathbb{C}^*$ and the category of monodromic

---

[4]Strictly speaking, for this part of the theorem to be true, we need to twist the symmetric monoidal structure on $\mathbb{N}^{Q_0}$-graded, cohomologically graded vector spaces by a sign, over and above the Koszul sign rule (see [11, Sects. 1.6, 6.1]). Thankfully for the quiver $\tilde{Q}$ this sign is always $+$ (see [10, Rem. 2.3]).

mixed Hodge modules on X. Denoting by Θ^{-1} an inverse equivalence, there is a faithful forgetful functor

$$\mathbf{rat} \circ \left(X \xrightarrow{x \mapsto (x,1)} X \times \mathbb{C}^* \right)^* \circ \Theta^{-1}[-1]$$

taking a monodromic mixed Hodge module to its underlying perverse sheaf on X. We abuse notation by denoting this functor also by $\mathbf{rat}$. For X a variety, we denote by $\underline{\mathbb{Q}}_X$ the lift of the constant sheaf $\mathbb{Q}_X$ to a complex of mixed Hodge modules on X. For f a regular function on X, we define the vanishing cycles functor

$$\phi_f^{\mathrm{mon}} : \mathbf{MHM}(X) \to \mathbf{MMHM}(X)$$

$$\mathcal{F} \mapsto \Theta \underline{\phi}_{u \cdot f} \left(\mathcal{F} \boxtimes \underline{\mathbb{Q}}_{\mathbb{C}^*} \right)[1],$$

where u is the coordinate on $\mathbb{C}^*$ and $\underline{\phi}_{u \cdot f}$ is the lift of $^{\mathfrak{p}}\phi_{u \cdot f}$ to the categories of mixed Hodge modules. There is a natural isomorphism $\mathbf{rat}\phi_f^{\mathrm{mon}} \cong {}^{\mathfrak{p}}\phi_f \mathbf{rat}$.

An object $\mathcal{F} \in \mathrm{Ob}(\mathbf{MMHM}(X))$ inherits a weight filtration from the weight filtration on objects of $\mathbf{MHM}(X \times \mathbb{A}^1)$. We say that $\mathcal{F}$ is pure of weight n if the associated graded object with respect to this filtration is concentrated in degree n. We say that an object $\mathcal{F} \in \mathrm{Ob}(\mathcal{D}^b(\mathbf{MMHM}(X)))$ is *pure* if each $\mathcal{H}^i(\mathcal{F})$ is pure of weight i.

The cohomologically graded mixed Hodge structure $\mathscr{L} = \mathbf{H}_c(\mathbb{A}^1, \mathbb{Q})$ is pure: it is concentrated in cohomological degree two and is pure of weight two. This object has a tensor square root in $\mathcal{D}^b(\mathbf{MMHM}(\mathrm{pt}))$ provided by

$$\mathscr{L}^{1/2} := \mathrm{cone}\left(\underline{\mathbb{Q}}_{\mathbb{A}^1} \to d_* \underline{\mathbb{Q}}_{\mathbb{A}^1} \right),$$

where $d : \mathbb{A}^1 \to \mathbb{A}^1$ is the morphism $z \mapsto z^2$. We say a monodromic mixed Hodge structure is of Tate type if it is a direct sum of (possibly negative) tensor powers of the monodromic mixed Hodge structure $\mathscr{L}^{1/2}[1]$.

For X an irreducible variety, we denote by

$$\mathcal{IC}_X^{\mathrm{hdg}} := \mathcal{IC}_X\left(\underline{\mathbb{Q}}_{X^{\mathrm{sm}}}[\dim X] \right) \otimes \mathscr{L}^{-\dim(X)/2}[-\dim X]$$

the natural lift of $\mathcal{IC}_X$ to a pure weight zero monodromic mixed Hodge module.

We define

$$\mathcal{BPS}_{Q,W,\mathbf{d}}^{\mathrm{hdg}} := \begin{cases} \phi_{\mathrm{Tr}(W)}^{\mathrm{mon}} \mathcal{IC}_{\mathcal{M}_{\mathbf{d}}(Q)}^{\mathrm{hdg}} & \text{if there is a } \mathbf{d}\text{-dimensional simple } \mathbb{C}Q\text{-module} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

This is the natural lift of the BPS sheaf to a monodromic mixed Hodge module, i.e. $\mathbf{rat}(\mathcal{BPS}_{Q,W,\mathbf{d}}^{\mathrm{hdg}}) \cong \mathcal{BPS}_{Q,W,\mathbf{d}}$. Similarly, we define the monodromic mixed Hodge structure

$$\mathrm{BPS}_{Q,W,\mathbf{d}}^{\mathrm{hdg}} := \mathbf{H}\left(\mathcal{M}_{\mathbf{d}}(Q), \mathcal{BPS}_{Q,W,\mathbf{d}}^{\mathrm{hdg}} \right)$$

satisfying $\mathbf{rat}(\mathrm{BPS}_{Q,W,\mathbf{d}}^{\mathrm{hdg}}) \cong \mathrm{BPS}_{Q,W,\mathbf{d}}$. We define[5]

$$\mathcal{RA}_{Q,W}^{\mathrm{hdg}} = \mathrm{JH}_* \phi_{\mathrm{Tr}(W)}^{\mathrm{mon}} \mathcal{IC}_{\mathfrak{M}(Q)}^{\mathrm{hdg}}.$$

Since all of the natural transformations defining the multiplication on the Hall algebra $\mathcal{A}_{Q,W}$ lift to categories of monodromic mixed Hodge modules [8, 11, 26], as does the Thom–Sebastiani theorem [39], we may define a multiplication on $\mathcal{RA}_{Q,W}^{\mathrm{hdg}}$ that recovers the multiplication on $\mathcal{RA}_{Q,W}$ after applying the functor $\mathbf{rat}$. Likewise, taking derived direct image to a point we obtain the algebra object $\mathcal{A}_{Q,W}^{\mathrm{hdg}}$ in monodromic mixed Hodge structures. Then by [11], Theorems 2.3 and 2.4 lift to the categories of

---

[5] Since there is not a fully developed theory of mixed Hodge modules for stacks, some care has to be taken care with this definition. See [11] for the details.

monodromic mixed Hodge modules and monodromic mixed Hodge structures. In particular, the BPS Lie algebra $\mathfrak{g}_{Q,W} = \mathrm{BPS}_{Q,W}[-1]$ lifts to a Lie algebra object

$$\mathfrak{g}_{Q,W}^{\mathrm{hdg}} := \mathrm{BPS}_{Q,W}^{\mathrm{hdg}} \otimes \mathscr{L}^{1/2}$$

inside $\mathcal{D}^{\mathrm{b}}(\mathbf{MMHM}(\mathrm{pt}))$. This is a Lie subalgebra of $\mathcal{A}_{Q,W}^{\mathrm{hdg}}$, considered as a Lie algebra in the category of $\mathbb{N}^{Q_0}$-graded, cohomologically graded monodromic mixed Hodge structures, via the commutator Lie bracket. See [8, 11] for full details.

2.5 Cohomological Donaldson–Thomas theory for preprojective algebras

In this section, we restrict our attention to 'tripled' QPs of the form $\left(\tilde{Q}, \tilde{W}\right)$, as in the introduction. Firstly, we recall the following purity result on the BPS cohomology of the Jacobi algebra $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right)$:

Theorem 2.6. [9] *For an arbitrary quiver Q and dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, the mixed Hodge structure* $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}^{\mathrm{hdg}}$ *is pure, of Tate type. In addition (or as a consequence of the cohomological integrality theorem), the mixed Hodge structure on*

$$\mathbf{H}_c\left(\mathfrak{M}_{\mathbf{d}}\left(\tilde{Q}\right), \phi_{\mathrm{Tr}(\tilde{W})}^{\mathrm{mon}} \mathcal{IC}_{\mathfrak{M}_{\mathbf{d}}(\tilde{Q})}^{\mathrm{hdg}}\right)$$

is pure, of Tate type.

We denote by

$$\tilde{e} : \mathcal{M}\left(\Pi_Q\right) \times \mathbb{A}^1 \to \mathcal{M}\left(\tilde{Q}\right) \tag{2.7}$$

the closed embedding that sends a pair (ρ, t) to the $\mathbb{C}\tilde{Q}$-module ρ' for which the action of the arrows $a, a^* \in \overline{Q}$ are the same as for ρ, and the action of each $\rho'(\omega_i)$ is given by multiplication by t.

We will need the following result on the support and equivariance of the BPS sheaf itself:

Lemma 2.7. [9] *For a quiver Q and dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, there is a perverse sheaf*

$$\mathcal{BPS}_{\Pi_Q, \mathbf{d}} \in \mathbf{Perv}\left(\mathcal{M}_{\mathbf{d}}\left(\Pi_Q\right)\right)$$

such that there is an isomorphism

$$\mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}} \cong \tilde{e}_!\left(\mathcal{BPS}_{\Pi_Q, \mathbf{d}} \boxtimes \mathcal{IC}_{\mathbb{A}^1}\right).$$

The same result holds at the level of monodromic mixed Hodge modules.

In words, the theorem says that the BPS sheaf is supported on the subspace of $\mathbb{C}\tilde{Q}$-modules for which all of the generalised eigenvalues of the operators $\rho(\omega_i)$ are the same, and the sheaf is moreover equivariant for the $\mathbb{A}^1$-action that acts by adding scalar multiples of the identity to all of the operators $\rho(\omega_i)$ simultaneously.

We recall from [9] the description of the BPS cohomology of the Jacobi algebra $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}\right)$ in terms of Kac polynomials:

Theorem 2.8. *The Poincaré polynomials of the cohomological BPS invariants for the QP $\left(\tilde{Q}, \tilde{W}\right)$ satisfy*

$$\mathrm{p}\left(\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}, q^{1/2}\right) = q^{-1/2}\mathrm{a}_{Q,\mathbf{d}}\left(q^{-1}\right), \tag{2.8}$$

where

$$\mathsf{a}_{Q,\mathbf{d}}(q) = \sum_{i \in \mathbb{Z}_{\geq 0}} \mathsf{a}_{Q,\mathbf{d},i}\, q^i$$

is the Kac polynomial, counting the number of isomorphism classes of absolutely[6] indecomposable $\mathbb{F}_q Q$-modules for $\mathbb{F}_q$ a field of order q. Furthermore, the natural mixed Hodge structure on $\mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}$ is pure, of Tate type, so that we can write

$$\mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q},\tilde{W},\mathbf{d}} \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (\mathscr{L}^{i-1/2})^{\oplus \mathsf{a}_{Q,\mathbf{d},-i}} \tag{2.9}$$

and so

$$\mathfrak{g}^{\mathrm{hdg}}_{\tilde{Q},\tilde{W},\mathbf{d}} \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (\mathscr{L}^i)^{\oplus \mathsf{a}_{Q,\mathbf{d},-i}}. \tag{2.10}$$

3. Deformed dimensional reduction and proofs of main results

3.1 Deformed dimensional reduction

The main tool in proving Theorem A will be *deformed dimensional reduction*, as introduced in joint work with Tudor Pădurariu [13]. This is a geometric result about vanishing cycle functors for functions satisfying certain $\mathbb{C}^*$-equivariance properties. We state the version that we need below.

Theorem 3.1. [13, Thm. 1.3] *Let the algebraic group G act on a variety X and affine space $\mathbb{A}^n$. Assume that $\mathbb{A}^n$ is also given a $\mathbb{C}^*$-action, with non-negative weights, which commutes with the G-action. Let $\mathbb{C}^*$ act on $\overline{X} = X \times \mathbb{A}^n$ via the product of the given action on $\mathbb{A}^n$ with the trivial action on X. Let g be a function on $\overline{X}$ that is G-invariant and $\mathbb{C}^*$-semi-invariant, with strictly positive weight. Assume that we are given a $G \times \mathbb{C}^*$-equivariant decomposition $\mathbb{A}^n = \mathbb{A}^m \times \mathbb{A}^{n-m}$ and that we can write*

$$g = g_0 + \sum_{1 \leq j \leq m} g_j t_j,$$

where the functions $g_0, \ldots g_m$ are pulled back from $X \times \mathbb{A}^{n-m}$ and $t_1, \ldots, t_m$ are a system of coordinates for $\mathbb{A}^m$. Let $Z \subset X \times \mathbb{A}^{n-m}$ be the vanishing locus of the functions $g_1, \ldots, g_m$. Then, Z is G-invariant. Set $\overline{Z} = Z \times \mathbb{A}^m \subset \overline{X}$. We denote by

$$\pi : \overline{X} \to X$$
$$q : X \times \mathbb{A}^{n-m} \to X$$

the natural projections. Then, the natural transformation

$$\pi_! {}^{\mathfrak{p}}\phi_g \mathbb{Q}_{\overline{X}/G} \to \pi_! {}^{\mathfrak{p}}\phi_{g_0} \mathbb{Q}_{\overline{Z}/G} \cong q_! {}^{\mathfrak{p}}\phi_{g_0} \mathbb{Q}_{Z/G}[-2m] \tag{3.1}$$

is an isomorphism.

   Since the functor **rat** *is faithful, the same statement is true at the level of (monodromic) mixed Hodge modules: the natural transformation*

$$\pi_! \phi_g^{\mathrm{mon}} \underline{\mathbb{Q}}_{\overline{X}/G} \to \pi_! \phi_{g_0}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\overline{Z}/G} \cong q_! \phi_{g_0}^{\mathrm{mon}} \underline{\mathbb{Q}}_{Z/G} \otimes \mathscr{L}^m$$

is an isomorphism in[7] $\mathcal{D}^{\mathrm{b}}(\mathbf{MMHM}(X/G))$.

---

[6] A module is called *absolutely* indecomposable if it remains indecomposable after extending scalars to the algebraic closure $\overline{\mathbb{F}}_q$.
[7] The derived category of monodromic mixed Hodge modules on this global quotient stack is defined following, for example, [1].

3.2 Proof of Theorem A

We proceed by applying deformed dimensional reduction to the function $\text{Tr}(\tilde{W}^\mu)$.

Proof of Theorem A. Given a quiver Q, we denote by Q^+ the quiver obtained by adding a loop ω_i at each vertex $i \in Q_0$, or equivalently the quiver obtained by removing all of the arrows a^* from $\tilde{Q}$. We denote by $\mathfrak{I}(Q)$ the stack of pairs (ρ, f), consisting of a $\mathbb{C}Q$-module ρ and an endomorphism $f \in \text{Hom}_{\mathbb{C}Q}(\rho, \rho)$. Then, we consider the commutative diagram

$$
\begin{array}{ccccc}
& & \mathfrak{I}(Q) & & \\
& & \downarrow{\scriptstyle i} & & \\
\mathfrak{M}(\tilde{Q}) & \xrightarrow{p} & \mathfrak{M}(Q^+) & \xrightarrow{q} & \mathfrak{M}(Q) \\
\downarrow{\scriptstyle \tilde{\text{JH}}} & & \downarrow{\scriptstyle \text{JH}^+} & & \downarrow{\scriptstyle \text{JH}} \\
\mathcal{M}(\tilde{Q}) & \xrightarrow{p'} & \mathcal{M}(Q^+) & \xrightarrow{q'} & \mathcal{M}(Q) \\
\uparrow{\scriptstyle \tilde{e}} & & \uparrow{\scriptstyle e} & & \searrow{\scriptstyle \tau} \\
\mathcal{M}(\Pi_Q) \times \mathbb{A}^1 & \xrightarrow{h} & \mathcal{M}(Q) \times \mathbb{A}^1 & & \text{pt}.
\end{array}
$$

In the above diagram, i is the natural embedding of stacks, sending a pair (ρ, f) to the $\mathbb{C}Q^+$-module ρ' for which the underlying $\mathbb{C}Q$-module is ρ, and the action of the loops ω_i is given by f. All of the horizontal arrows are the natural forgetful maps. We set

$$
\pi = qp
$$
$$
\pi' = q'p'.
$$

We denote by Ω the quiver containing the same vertices as Q and for which the only arrows are the loops ω_i. We denote by Q^{op} the quiver containing the same vertices as Q, and only the arrows a^* for $a \in Q_1$, i.e. Q^{op} is the opposite quiver to Q. In the notation of Theorem 3.1, we set

$$
X = \mathbb{A}_{\mathbf{d}}(Q) \tag{3.2}
$$
$$
\mathbb{A}^m = \mathbb{A}_{\mathbf{d}}(Q^{\text{op}})
$$
$$
\mathbb{A}^{n-m} = \mathbb{A}_{\mathbf{d}}(\Omega)
$$
$$
G = \text{GL}_{\mathbf{d}}.
$$

We let $\mathbb{C}^*$ act on $\mathbb{A}^m$ and $\mathbb{A}^{n-m}$ with weight one.

We set

$$
L_\mu = \frac{1}{2} \sum_{i \in Q_0} \mu_i \omega_i^2
$$

so $\tilde{W}^\mu = \tilde{W} + L_\mu$. Then $\text{Tr}(\tilde{W} + L_\mu)$ satisfies the conditions of Theorem 3.1; for example, it is $\mathbb{C}^*$ semi-invariant with weight two. In the notation of that theorem $Z \subset X \times \mathbb{A}^{n-m} = \mathbb{A}_{\mathbf{d}}(Q^+)$ is the locus containing those $\mathbb{C}Q^+$-modules ρ such that the endomorphisms $\rho(\omega_i)$ determine an endomorphism of the underlying $\mathbb{C}Q$-module of ρ. It follows that $Z/G = \mathfrak{I}(G)$. By Theorem 3.1, there is an isomorphism[8]

$$
\pi_! {}^{\mathfrak{p}}\phi_{\text{Tr}(\tilde{W}^\mu)} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})} \cong q_! {}^{\mathfrak{p}}\phi_{\text{Tr}(L_\mu)} \mathbb{Q}_{\mathfrak{I}(Q)}. \tag{3.3}
$$

If instead we set

$$
X = \mathbb{A}_{\mathbf{d}}(Q^+)
$$
$$
\mathbb{A}^m = \mathbb{A}_{\mathbf{d}}(\Omega)
$$

---

[8] Note that the shift $2 \sum_{a \in Q_1} \mathbf{d}_{s(a)} \mathbf{d}_{t(a)}$ appearing in the definition of the intersection complex is equal to the $2m$ appearing in (3.1).

set $n = m$, and again let $\mathbb{C}^*$ act on $\mathbb{A}^m$ with weight one, then the function $\mathrm{Tr}(\tilde{W})$ still satisfies the conditions of Theorem 3.1 (now it is semi-invariant with weight one). So instead we arrive at the isomorphism

$$p_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})} \cong \mathbb{Q}_{\mathfrak{J}(Q)}. \tag{3.4}$$

Since JH^+ is approximated by proper maps, there is a natural isomorphism (see [11, Sec. 4.1])

$$\mathrm{JH}_!^{+\,\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} \mathbb{Q}_{\mathfrak{J}(Q)} \cong {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} \mathrm{JH}_!^+ \mathbb{Q}_{\mathfrak{J}(Q)}. \tag{3.5}$$

Combining all of the above, we can write[9]

$$\pi'_! \tilde{\mathrm{JH}}_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W}^\mu)} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})} \cong \mathrm{JH}_! \pi_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W}^\mu)} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})}$$

$$\cong \mathrm{JH}_! q_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} \mathbb{Q}_{\mathfrak{J}(Q)} \qquad\qquad \text{by (3.3)}$$

$$\cong \mathrm{JH}_! q_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})} \qquad\qquad \text{by (3.4)}$$

$$\cong q'_! \mathrm{JH}_!^{+\,\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})}$$

$$\cong q'_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} \mathrm{JH}_!^+ p_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})} \qquad\qquad \text{by (3.5)}$$

$$\cong q'_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p'_! \tilde{\mathrm{JH}}_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(\tilde{W})} \mathcal{IC}_{\mathfrak{M}(\tilde{Q})}$$

$$\cong q'_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p'_! \mathbf{Sym}_\oplus \left(\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \setminus 0} \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right) \qquad \text{by Theorem (2.4)}$$

$$\cong q'_! {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} \mathbf{Sym}_\oplus \left(\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \setminus 0} p'_! \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right) \qquad \text{since } p' \text{ is a monoid map}$$

$$\cong q'_! \mathbf{Sym}_\oplus \left(\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \setminus 0} {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p'_! \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right) \qquad \text{by Thom–Sebastiani.}$$

By Lemma 2.7, we can write

$$p'_! \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \cong e_! \left(\mathcal{F} \boxtimes \mathcal{IC}_{\mathbb{A}^1} \right), \tag{3.6}$$

where

$$\mathcal{F} = (\mathcal{M}(\Pi_Q) \to \mathcal{M}(Q))_! \mathcal{BPS}_{\Pi_Q, \mathbf{d}}.$$

Writing $f = \mathrm{Tr}(L_\mu) \mid_{\mathcal{M}_{\mathbf{d}}(Q) \times \mathbb{A}^1}$, we have

$$f : \mathcal{M}_{\mathbf{d}}(Q) \times \mathbb{A}^1 \to \mathbb{C}$$

$$(\rho, t) \mapsto \frac{1}{2}(\mu \cdot \mathbf{d})t^2.$$

It follows from (3.6) that

$${}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p'_! \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \cong \begin{cases} e_!(\mathcal{F} \boxtimes \mathbb{Q}_0) & \text{if } \mu \cdot \mathbf{d} \neq 0 \\ e_!(\mathcal{F} \boxtimes \mathcal{IC}_{\mathbb{A}^1}) & \text{if } \mu \cdot \mathbf{d} = 0, \end{cases}$$

so that

$$\mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(Q^+), {}^{\mathrm{p}}\phi_{\mathrm{Tr}(L_\mu)} p'_! \mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}) \cong \mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(\Pi_Q), \mathcal{BPS}_{\Pi_Q, \mathbf{d}})[-g(\mathbf{d})],$$

where we define

$$g(\mathbf{d}) = \begin{cases} 1 & \text{if } \mu \cdot \mathbf{d} = 0 \\ 0 & \text{if } \mu \cdot \mathbf{d} \neq 0. \end{cases}$$

---

[9] For the final isomorphism, see [11, Prop. 3.11] for the required compatibility with the symmetrising morphism.

Finally, we deduce that

$$\mathbf{H}_c\left(\mathfrak{M}\left(\tilde{Q}\right), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W}^{\mu})}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})}\right) = \tau_!\pi'_!\tilde{\mathrm{J}}\mathrm{H}_!{}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W}^{\mu})}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})}$$

$$\cong \mathbf{Sym}\left(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}\setminus 0}\left(\mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(\Pi_Q), \mathcal{BPS}_{\Pi_Q,\mathbf{d}}) \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right)[-g(\mathbf{d})]\right). \tag{3.7}$$

On the other hand, by Theorem 2.4 again, we have

$$\mathbf{H}_c(\mathfrak{M}\left(\tilde{Q}\right), {}^{\mathfrak{p}}\phi_{\mathrm{Tr}(\tilde{W}^{\mu})}\mathcal{IC}_{\mathfrak{M}(\tilde{Q})}) \cong \mathbf{Sym}\left(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}\setminus 0}\mathrm{BPS}^{\vee}_{\tilde{Q},\tilde{W}^{\mu},\mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right). \tag{3.8}$$

Comparing (3.7) and (3.8), we find

$$\begin{aligned}
\mathrm{BPS}_{\tilde{Q},\tilde{W}^{\mu},\mathbf{d}} &\cong \left(\mathbf{H}_c\left(\mathcal{M}_{\mathbf{d}}\left(\Pi_Q\right), \mathcal{BPS}_{\Pi_Q,\mathbf{d}}\right)\right)^{\vee}[g(\mathbf{d})] \\
&\cong \left(\mathbf{H}_c\left(\mathcal{M}_{\mathbf{d}}\left(\tilde{Q}\right), \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}\right)[1]\right)^{\vee}[g(\mathbf{d})] \qquad \text{by Theorem 2.7} \\
&\cong \mathbf{H}_c\left(\mathcal{M}_{\mathbf{d}}\left(\tilde{Q}\right), \mathcal{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}\right)^{\vee}[g(\mathbf{d})-1] \\
&\cong \mathrm{BPS}_{\tilde{Q},\tilde{W},\mathbf{d}}[g(\mathbf{d})-1]
\end{aligned}$$

and now (2.8) follows from Theorem 2.8.

Performing the same calculations in the category of monodromic mixed Hodge structures, isomorphisms (3.7) and (3.8) yield the isomorphism of $\mathbb{N}^{Q_0}$-graded complexes of monodromic mixed Hodge structures

$$\mathbf{Sym}\left(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}\setminus 0}\mathrm{BPS}^{\mathrm{hdg},\vee}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathscr{L}^{(g(\mathbf{d})-1)/2} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right) \cong \mathbf{Sym}\left(\bigoplus_{\mathbf{d}\in\mathbb{N}^{Q_0}\setminus 0}\mathrm{BPS}^{\mathrm{hdg},\vee}_{\tilde{Q},\tilde{W}^{\mu},\mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right), \tag{3.9}$$

where

$$\mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} = \bigoplus_{i\geq 0}\mathscr{L}^{-1/2-i}.$$

On the other hand, by Theorem 2.6 the mixed Hodge structure $\mathrm{BPS}^{\mathrm{hdg}}_{\Pi_Q,\mathbf{d}}$ is pure, so that $\mathrm{BPS}^{\mathrm{hdg},\vee}_{\Pi_Q,\mathbf{d}} \otimes \mathscr{L}^{n/2}$ is pure for all $n\in\mathbb{Z}$. It follows that both sides of (3.9) are pure. The isomorphism

$$\mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q},\tilde{W},\mathbf{d}} \otimes \mathscr{L}^{(g(\mathbf{d})-1)/2} \cong \mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q},\tilde{W}^{\mu},\mathbf{d}}$$

then follows from (3.9) and semisimplicity of the category of pure monodromic mixed Hodge structures as in [13, Cor. 7.1]. □

Example 3.2. *To recover the example that we started the paper with, consider the quiver $Q^{(0)}$ with one vertex and no arrows. We label the unique arrow of the one-loop quiver $Q^{(1)} = \widetilde{Q^{(0)}}$ by ω_0. For $\mu = \mu_0 e_0$ to be generic, we just have to pick $\mu_0 \neq 0$. Then, we find*

$$\begin{aligned}
\mathrm{Jac}\left(Q^{(1)}, \tilde{W}^{\mu}\right) &= \mathrm{Jac}\left(Q^{(1)}, \omega_0^2\right) \\
&\cong \mathbb{C}Q^{(0)}
\end{aligned}$$

and so the fermionic version of $\mathcal{A}_{\widetilde{Q^{(0)}},\tilde{W}} = \mathcal{A}_{Q^{(1)}}$ is indeed $\mathcal{A}_{Q^{(0)}}$.

Example 3.3. *We return once more to the noncommutative conifold. As observed in Example 2.2, this algebra is obtained by setting $\mu = (1, -1)$ for the quiver (1.10) and considering $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$. The cohomological BPS invariants of the QP $\left(\tilde{Q}, \tilde{W}\right)$ are given in (1.11), while the cohomological BPS*

invariants of the noncommutative conifold are given in (1.13). *Comparing the two, there is a cohomological shift between the respective (m,n)th cohomological BPS invariants precisely if $m \neq n$, i.e. precisely if $\mu \cdot (m, n) \neq 0$.*

Recall from the introduction that the BPS invariants for the resolved conifold are only a partially *fermionised version of the BPS invariants of $Y_0 \times \mathbb{A}^1$. In view of the main result, we see that the generic deformation in the commutative algebraic geometry context corresponds to the deformation $\mu = (1, -1)$. To fully fermionise the DT theory, we are obliged to work in the fully noncommutative[10] context provided by generic μ. Note that deformations within algebraic geometry of the Kleinian singularity were parameterised by $\mathfrak{h}$, the Cartan subalgebra of the (reduced) McKay graph Γ', whereas noncommutative deformations are parameterised by the Cartan subalgebra of the full McKay graph.*

Remark 3.4. *Along the course of the proof of Theorem A, we have shown that*

$$\pi'_* \mathcal{BPS}_{\tilde{Q}, \tilde{W}^\mu, \mathbf{d}} \cong \begin{cases} \pi'_* \mathcal{BPS}_{\tilde{Q}, \tilde{W}}[-1] & \text{if } \mu \cdot \mathbf{d} \neq 0 \\ \pi'_* \mathcal{BPS}_{\tilde{Q}, \tilde{W}} & \text{otherwise.} \end{cases}$$

With a little effort, one may lift this statement to the level of monodromic mixed Hodge module complexes.

3.3 Proof of Theorem B

Given a quiver Q, an element $\mu \in R$, and a number $n \in \mathbb{Z}_{\geq 1}$, define

$$\tilde{W}_n^\mu = \tilde{W} + \frac{1}{n} \mu \omega^n.$$

Then in the decomposition (3.2), we let $\mathbb{C}^*$ act with weight 1 on $\mathbb{A}_\mathbf{d}(\Omega)$, weight $n - 1$ on $\mathbb{A}_\mathbf{d}(Q^{\mathrm{op}})$, and trivially on $\mathbb{A}_\mathbf{d}(Q)$, so $\mathrm{Tr}(\tilde{W}_\mathbf{d}^\mu)$ satisfies the conditions of Theorem 3.1 (now it is a weight n function). Then, the argument of Section 3.2 gives that

$$\mathrm{BPS}_{\tilde{Q}, \tilde{W}_n^\mu, \mathbf{d}}^{\mathrm{hdg}} \cong \mathrm{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}^{\mathrm{hdg}} \otimes \mathscr{L}^{1/2} \otimes \mathbf{H}\left(\mathbb{A}^1, \phi_{(\mu \cdot \mathbf{d}) x^n}^{\mathrm{mon}} \mathcal{IC}_{\mathbb{A}^1}\right). \tag{3.10}$$

Theorem A follows from the special case $n = 2$ and Theorem 2.8, observing that $\mathbf{H}\left(\mathbb{A}^1, \phi_{x^2}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\mathbb{A}^1}\right) = \mathscr{L}^{1/2}$.

In this section, we consider instead the special case $n = 1$;, i.e., we consider the quiver $\tilde{Q}$ with the potential

$$\tilde{W}_1^\mu = \sum_{a \in Q_1} [a, a^*] + \mu \omega.$$

This potential is linear in the loops ω_i, so that the following proposition is a straightforward application of (undeformed) dimensional reduction:

Proposition 3.5. *There is an isomorphism in the derived category of complexes of mixed Hodge modules*

$$\pi_! \phi_{\mathrm{Tr}(\tilde{W}_1^\mu)}^{\mathrm{mon}} \underline{\mathbb{Q}}_{\mathfrak{M}_\mathbf{d}(\tilde{Q})} \to \underline{\mathbb{Q}}_{\mathfrak{M}_\mathbf{d}\left(\Pi_{Q, \mu}\right)} \otimes \mathscr{L}^{\mathbf{d} \cdot \mathbf{d}}.$$

Proof. In the setup of Theorem 3.1, we put $X = \mathbb{A}_\mathbf{d}(\overline{Q})$, $n = m$, $\mathbb{A}^m = \mathbb{A}_\mathbf{d}(\Omega)$ and $G = \mathrm{GL}_\mathbf{d}$. Then, g_0 is the zero function, so that there is a natural isomorphism $\phi_{g_0}^{\mathrm{mon}} \to \mathrm{id}$, and the result follows from Theorem 3.1 and $\dim(\mathbb{A}_\mathbf{d}(\Omega)) = \mathbf{d} \cdot \mathbf{d}$. $\square$

---

[10] I.e. with a Jacobi algebra that is not derived equivalent to a threefold.

Proof of Theorem B. We have isomorphisms of $\mathbb{N}^{Q_0}$-graded mixed Hodge structures

$$\bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}_c\left(\mathfrak{M}_{\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q}\right) \otimes \mathscr{L}^{\chi_Q(\mathbf{d},\mathbf{d})} \cong \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}_c\left(\mathfrak{M}_{\mathbf{d}}\left(\tilde{Q}\right), \phi^{\mathrm{mon}}_{\mathrm{Tr}(W_1^\mu)} \mathcal{IC}_{\mathfrak{M}_{\mathbf{d}}(\tilde{Q})}\right)$$

$$\cong \mathbf{Sym}\left(\bigoplus_{\mathbb{N}^{Q_0} \ni \mathbf{d} \neq 0} \mathrm{BPS}^{\mathrm{hdg},\vee}_{\tilde{Q}, W_1^\mu, \mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}\right),$$

where the first isomorphism is dimensional reduction (as in Proposition 3.5) and the second is the cohomological integrality theorem. So the theorem follows from (2.9) and the claim that

$$\mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q}, W_1^\mu, \mathbf{d}} \cong \begin{cases} \mathrm{BPS}^{\mathrm{hdg}}_{\tilde{Q}, \tilde{W}, \mathbf{d}} & \text{if } \mathbf{d} \cdot \mu = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This follows from (3.10) and the observation that for $\lambda \in \mathbb{C}$

$$\mathbf{H}\left(\mathbb{A}^1, \phi^{\mathrm{mon}}_{\lambda x} \mathcal{IC}_{\mathbb{A}^1}\right) \cong \begin{cases} \mathscr{L}^{-1/2} & \text{if } \lambda = 0 \\ 0 & \text{if } \lambda \neq 0. \end{cases} \qquad \square$$

Assume that the dimension vector $\mathbf{d}$ is indivisible, meaning that there is no $n \in \mathbb{Z}_{\geq 2}$ such that $\frac{1}{n}\mathbf{d} \in \mathbb{N}^{Q_0}$. Assume also that μ is chosen to be generic, subject to the constraint that $\mathbf{d} \cdot \mu = 0$. Equivalently, $\mathbf{d}$ and μ are chosen so that $\mathbf{d}' \cdot \mu = 0$ implies that $\mathbf{d}'$ is an integer multiple of $\mathbf{d}$. Then (1.17) simplifies to

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbf{H}_c\left(\mathfrak{M}_{n\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q}\right) \otimes \mathscr{L}^{n^2 \chi_Q(\mathbf{d},\mathbf{d})} \cong \mathbf{Sym}\left(\bigoplus_{n \in \mathbb{Z}_{>0}} \mathrm{BPS}^{\mathrm{hdg},\vee}_{\tilde{Q}, \tilde{W}, n\mathbf{d}} \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q})_{\mathrm{vir}}\right). \tag{3.11}$$

Our assumptions imply that $\mathcal{M}_{\mathbf{d}}(\Pi_Q)$ is a fine moduli scheme, and $\mathfrak{M}_{\mathbf{d}}(\Pi_Q)$ is a trivial $B\mathbb{C}^*$-gerbe over it, meaning that

$$\mathbf{H}_c\left(\mathfrak{M}_{\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q}\right) \cong \mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(\Pi_Q), \mathbb{Q}) \otimes \mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q}).$$

Recall that there is an isomorphism

$$\mathbf{H}_c(\mathrm{pt}/\mathbb{C}^*, \mathbb{Q}) \cong \bigoplus_{i \geq 0} \mathscr{L}^{-i-1}.$$

From the $n = 1$ piece of (3.11), we deduce the following slightly stronger version of a result of Crawley–Boevey and Van den Bergh [7]:

Corollary 3.6. *Let* $\mathbf{d}$ *be indivisible, and* μ *be generic. Then, there is an isomorphism of mixed Hodge structures*

$$\mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q}) \cong \bigoplus_{i \geq 0} (\mathscr{L}^{1+i-\chi_Q(\mathbf{d},\mathbf{d})})^{\oplus \mathfrak{a}_{Q,\mathbf{d},i}}.$$

In particular, $\mathbf{H}_c(\mathcal{M}_{\mathbf{d}}(\Pi_{Q,\mu}), \mathbb{Q})$ *is pure of Tate type.*

4. Further directions

4.1 Calculating the BPS sheaves

While our main theorem gives a way to calculate the BPS cohomology $\mathrm{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$ for arbitrary $Q, \mathbf{d} \in \mathbb{N}^{Q_0}$ and $\mu \in R$, the actual BPS sheaf $\mathcal{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$ remains a little mysterious. We can at least generalise (part of) the support lemma (Lemma 2.7) from the case $\mu = 0$.

Lemma 4.1. *For a quiver Q, and dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$, we have*

$$\mathrm{supp}(\mathcal{BPS}_{\Pi_Q^{\mu}, \mathbf{d}}) \subset \begin{cases} \tilde{e}(\mathcal{M}_{\mathbf{d}}(\Pi_Q) \times \mathbb{A}^1) & \text{if } \mu \cdot \mathbf{d} = 0 \\ \tilde{e}(\mathcal{M}_{\mathbf{d}}(\Pi_Q) \times \{0\}) & \text{if } \mu \cdot \mathbf{d} \neq 0, \end{cases}$$

where $\tilde{e}$ is defined in (2.7).

Proof. The proof is very similar to [9, Lem. 4.1]. Let ρ be a $\mathrm{Jac}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$-module lying in the support of $\mathcal{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$. By definition, ρ is semisimple, and so since $\rho\left(\sum_{i \in Q_0} \omega_i\right)$ is central, it acts via a diagonal matrix. Arguing as in [9, Lem. 4.1] and using centrality of $\rho\left(\sum_{i \in Q_0} \omega_i\right)$ along with the cohomological integrality theorem, we deduce that the generalised eigenvalues of $\rho(\omega)$ are all the same, i.e. $\rho(\omega)$ acts via scalar multiplication, establishing the lemma in the case $\mu \cdot \mathbf{d} \neq 0$. Arguing as in Proposition 2.3, if $\mu \cdot \mathbf{d} \neq 0$, we must have $\lambda = 0$. This establishes the lemma in the case $\mu \cdot \mathbf{d} \neq 0$. □

Even in the case $\mu \cdot \mathbf{d} = 0$, this is a slightly weaker statement than Lemma 2.7; it is harder to show that $\mathcal{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$ is $\mathbb{A}^1$-equivariant (where $\mathbb{A}^1$ acts by adding a scalar multiple of the identity matrix to $\rho(\omega)$) since the function $\mathrm{Tr}(\tilde{W}^{\mu})$ is not $\mathbb{A}^1$-invariant unless $\mu = 0$.

It would be interesting to compare $\mathcal{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$ with $\mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}} \cong \tilde{e}_! \left(\mathcal{BPS}_{\Pi_Q, \mathbf{d}} \boxtimes \mathcal{IC}_{\mathbb{A}^1}\right)$ in the case $\mu \cdot \mathbf{d} = 0$, and with $\tilde{e}_! \left(\mathcal{BPS}_{\Pi_Q, \mathbf{d}} \boxtimes \mathbb{Q}_0\right)$ in the case $\mu \cdot \mathbf{d} \neq 0$. In particular, we may ask the following question:

Question 4.2. Is the monodromic mixed Hodge module $\mathcal{BPS}_{\tilde{Q}, \tilde{W}^{\mu}, \mathbf{d}}$ pure?

By [10, Thm. A], we know that $\mathcal{BPS}_{\tilde{Q}, \tilde{W}, \mathbf{d}}$ is a pure monodromic mixed Hodge module, i.e. we know that for $\mu = 0$ the answer to Question 4.2 is yes.

4.2 The BPS algebra

Identifying the BPS cohomology of the Jacobi algebra $\mathbb{C}\left(\tilde{Q}, \tilde{W}^{\mu}\right)$ is only part of understanding its cohomological DT theory; it tells us the size of the graded pieces of $\mathcal{A}_{\tilde{Q}, \tilde{W}^{\mu}}$, but nothing about the algebra structure above what we already know from [11] regarding general quivers with potential (e.g. the PBW theorem). At least for $\mu = 0$, the algebra $\mathcal{A}_{\tilde{Q}, \tilde{W}^{\mu}}$ has been studied from various points of view (see e.g. [14, 22, 31, 35, 44]) while for the $\mu \neq 0$ case, it would be interesting (at least for noncommutative resolutions of toric CY3s) to relate these algebras to the Yangians defined in terms of (generalised) McMahon modules and crystal melting in [17, 27].

For general Q and μ, the shifted BPS cohomology $\mathfrak{g}_{\tilde{Q}, \tilde{W}^{\mu}} := \mathrm{BPS}_{\tilde{Q}, \tilde{W}^{\mu}}[-1]$ carries a Lie algebra structure (by Theorem 2.5). Even in the case $\mu = 0$, we do not yet fully understand this Lie algebra for general Q. We can at least try to relate the case of general μ to the case $\mu = 0$, via the following construction: Fix Q and μ, and define

$$\mathfrak{g}_{\mathrm{even}} := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \mid \mathbf{d} \cdot \mu = 0} \mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathbf{d}}$$

$$\mathfrak{g}_{\mathrm{odd}} := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0} \mid \mathbf{d} \cdot \mu \neq 0} \mathfrak{g}_{\tilde{Q}, \tilde{W}, \mathbf{d}}.$$

Then, $\mathfrak{g}_{\text{even}}$ is a Lie subalgebra of $\mathfrak{g}_{\tilde{Q},\tilde{W}}$, and $\mathfrak{g}_{\text{odd}}$ is a Lie module for it, and we consider the extension

$$\mathfrak{g}^{\mu} = \mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}[-1],$$

with the Lie bracket

$$[(\alpha, \beta), (\alpha', \beta')] = ([\alpha, \alpha'], [\alpha, \beta'] - [\alpha', \beta]).$$

Question 4.3. Is there is an isomorphism of Lie algebras

$$\mathfrak{g}^{\mu} \cong \mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}}?$$

Proposition 4.4. *The answer to Question* 4.3 *is yes in the case* $\mu = 0$*, and for generic* μ*.*

Proof. The case $\mu = 0$ is trivial, since then

$$\mathfrak{g}^{\mu} = \mathfrak{g}_{\text{even}}$$
$$= \mathfrak{g}_{\tilde{Q},\tilde{W}}$$

and $\tilde{W}^{\mu} = \tilde{W}$. In the case of generic μ, Theorem A tells us that there is an isomorphism

$$\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}} \cong \mathfrak{g}_{\tilde{Q},\tilde{W}}[-1] \tag{4.1}$$

as cohomologically graded vector spaces, and so it is sufficient to show that the Lie bracket on $\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}}$ vanishes, since the Lie bracket on $\mathfrak{g}^{\mu} := \mathfrak{g}_{\text{odd}}[-1]$ does by definition. Combining (4.1) and (2.10), we deduce that $\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}}$ lies entirely in odd cohomological degree. Since the Hall algebra multiplication, and hence the commutator Lie bracket, preserves cohomological degree, the Lie bracket vanishes as required. □

In this paper, we have ignored completely the question of whether there is a natural double to the Lie algebra $\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}}$. Despite Proposition 4.4 the expected answer to Question 4.3, once extended to the double of $\mathfrak{g}_{\tilde{Q},\tilde{W}^{\mu}}$ is *no*; one of the motivations for this paper is work of Kevin Costello [5], in which he conjectures that in the case of, for example, the resolved conifold, the Lie bracket does not vanish on the fermionic part of the (doubled) BPS Lie algebra. As we saw in Section 1.2, and Example 3.3 the (noncommutative) resolved conifold is a *partial* fermionisation of the QP $\left(\tilde{Q}, \tilde{W}\right)$ where $\tilde{Q}$ is the affine A_1 quiver; so the conifold is precisely the kind of case that Proposition 4.4 does not cover.

4.3 Representation theory

The boson-fermion correspondence is typically considered (by mathematicians) to be a part of representation theory, and so it would be remiss to finish the paper without saying *anything* about representations of $\mathcal{A}_{\tilde{Q},\tilde{W}^{\mu}}$.

Fix a quiver Q, and a framing dimension vector $\mathbf{f} \in \mathbb{N}^{Q_0}$. We form the quiver $Q_{\mathbf{f}}$ by adding one extra vertex (labelled ∞) to Q_0, and $\mathbf{f}_i$ arrows from ∞ to i for each $i \in Q_0$. Set $\mathcal{Q} = \tilde{Q}_{\mathbf{f}}$. In other words, this is the usual doubled framed quiver that one uses to define Nakajima quiver varieties, but with an extra loop at every vertex (including a loop ω_{∞} at the framing vertex).

For $\mu = \sum_{i \in Q_0} \mu_i e_i$ we define

$$\tilde{W}_{\mathbf{f}}^{\mu} = \left(\sum_{i \in \mathcal{Q}_0} \omega_i\right)\left(\sum_{a \in (Q_{\mathbf{f}})_1} [a, a^*]\right) + \sum_{i \in Q_0} \mu_i \omega_i^2.$$

We write dimension vectors for $\mathcal{Q}$ as $\mathbf{e} = (\mathbf{d}, n)$ where $\mathbf{d} \in \mathbb{N}^{Q_0}$ and $n = \mathbf{e}_{\infty} \in \mathbb{N}$. We define

$$\mathbb{A}_{(\mathbf{d},1)}^{st}(\mathcal{Q}) \subset \mathbb{A}_{(\mathbf{d},1)}(\mathcal{Q})$$

to be the subvariety containing those ρ such that ρ_∞ generates ρ as a $\mathbb{C}\mathscr{Q}$-module. We call such $\mathbb{C}\mathscr{Q}$-modules stable. Then, we define

$$\mathscr{N}_{\mathbf{f},\mathbf{d}}(Q) := \mathbb{A}^{st}_{(\mathbf{d},1)}(\mathscr{Q})/\mathrm{GL}_{\mathbf{d}}.$$

This is a smooth variety and is naturally isomorphic to the stack of stable $(\mathbf{d},1)$-dimensional $\mathbb{C}\mathscr{Q}$-modules along with a trivialisation $\rho_\infty \cong \mathbb{C}$. We define

$$\mathcal{F}l_{(\mathbf{d},1),\mathbf{d}'}(\mathscr{Q})$$

to be the stack of pairs of a stable $(\mathbf{d},1)$-dimensional $\mathbb{C}\mathscr{Q}$-module ρ, along with a $(\mathbf{d}',0)$-dimensional submodule $\rho' \subset \rho$ and a trivialisation $\rho_\infty \cong \mathbb{C}$. Set $\mathbf{d}'' = \mathbf{d} - \mathbf{d}'$. In the correspondence diagram,

$$\mathfrak{M}_{\mathbf{d}'}(\widetilde{Q}) \times \mathscr{N}_{\mathbf{f},\mathbf{d}''}(Q) \xleftarrow{\;\pi_1 \times \pi_3\;} \mathcal{F}l_{(\mathbf{d},1),\mathbf{d}'}(\mathscr{Q}) \xrightarrow{\;\pi_2\;} \mathscr{N}_{\mathbf{f},\mathbf{d}}(Q)$$

the morphism π_2 is proper, and so via push-forward and pull-back in vanishing cycle cohomology, we obtain an action of $\mathcal{A}_{\tilde{Q},\tilde{W}^\mu}$ on

$$\mathbf{M}^\mu_{\mathbf{f}}(Q) := \bigoplus_{\mathbf{d} \in \mathbb{N}^{Q_0}} \mathbf{H}\left(\mathscr{N}_{\mathbf{f},\mathbf{d}''}(Q), \phi^{\mathrm{mon}}_{\mathrm{Tr}(\tilde{W}^\mu_{\mathbf{f}})}\underline{\mathbb{Q}}_{\mathscr{N}_{\mathbf{f},\mathbf{d}''}(Q)}\right) \otimes \mathscr{L}^{\chi_{\mathscr{Q}}((1,\mathbf{d}),(1,\mathbf{d}))/2}.$$

Given a quiver Q, we denote by

$$\mu_{Q,\mathbf{d}} : \mathbb{A}_{\mathbf{d}}(\overline{Q}) \to \mathfrak{gl}_{\mathbf{d}}$$
$$\rho \mapsto \sum_{a \in Q_1} [\rho(a), \rho(a^*)].$$

the usual moment map. We define

$$Z(\mathbf{f},\mathbf{d}) \subset \mathbb{A}^{st}_{(\mathbf{d},1)}\left(\overline{Q_{\mathbf{f}}}\right)$$

the vanishing locus of the composition of $\mu_{Q_{\mathbf{f}},(\mathbf{d},1)}$ with the natural projection $\mathfrak{gl}_{(\mathbf{d},1)} \to \mathfrak{gl}_{\mathbf{d}}$. Here, the stability condition is the same as above: we restrict to the open locus of those ρ such that ρ_∞ generates. Then, the Nakajima quiver variety is defined to be the smooth variety

$$X(\mathbf{f},\mathbf{d}) = Z(\mathbf{f},\mathbf{d})/\mathrm{GL}_{\mathbf{d}}.$$

Proposition 4.5. *Denote by g the restriction of the function $\mathrm{Tr}(\tilde{W}^\mu_{\mathbf{f}})$ to $\mathscr{N}_{\mathbf{f},\mathbf{d}}(Q)$, then*

$$\mathrm{crit}(g) \cong \begin{cases} X(\mathbf{f},\mathbf{d}) & \text{if } \mu \cdot \mathbf{d} \neq 0 \\ X(\mathbf{f},\mathbf{d}) \times \mathbb{A}^1 & \text{if } \mu \cdot \mathbf{d} = 0. \end{cases}$$

Moreover, the monodromic mixed Hodge module

$$\phi^{\mathrm{mon}}_g \underline{\mathbb{Q}}_{\mathscr{N}_{\mathbf{f},\mathbf{d}}(Q)} \otimes \mathscr{L}^{\chi_{\mathscr{Q}}((\mathbf{d},1),(\mathbf{d},1))/2}$$

is analytically locally isomorphic to the constant mixed Hodge module on the critical locus of g.

Proof. The second part follows from the first: the holomorphic Bott–Morse lemma tells us that since the critical locus of g is scheme-theoretically smooth, g can be written analytically locally around $\mathrm{crit}(g)$ in the form

$$g = x_1^2 + \ldots + x_c^2,$$

where c is the codimension of $\mathrm{crit}(g)$ inside $\mathscr{N}_{\mathbf{f},\mathbf{d}}(Q)$. The first part follows by the same argument as Proposition 2.3.

In the special case $\mu = 0$, it is possible to show that the monodromy of the rank one local system ${}^{\mathfrak{p}}\phi_g \underline{\mathbb{Q}}_{\mathcal{N}_{\mathbf{f},\mathbf{d}''}(Q)}$ is trivial [10, Prop. 6.3], and we conjecture that this is always true. Assuming the conjecture, Proposition 4.5 implies that there are isomorphisms

$$\mathbf{M}_{\mathbf{f}}^{\mu}(Q) \cong \left(\bigoplus_{\substack{\mathbf{d} \in \mathbb{N}^{Q_0} \\ \mu \cdot \mathbf{d} \neq 0}} \mathbf{H}(X(\mathbf{f},\mathbf{d}), \mathbb{Q})_{\mathrm{vir}} \right) \oplus \left(\bigoplus_{\substack{\mathbf{d} \in \mathbb{N}^{Q_0} \\ \mu \cdot \mathbf{d} = 0}} \mathbf{H}(X(\mathbf{f},\mathbf{d}), \mathbb{Q})_{\mathrm{vir}} \otimes \mathscr{L}^{-1/2} \right). \tag{4.2}$$

Note that the vacuum vector, spanning $\mathbf{H}(X(\mathbf{f}, 0), \mathbb{Q})$, lies in the second summand.

Let Q' be the full subquiver of Q obtained by removing all vertices that support loops, as well as arrows to or from them, and let $\mathfrak{n}_{Q'}^{-}$ be the negative piece of the Kac–Moody Lie algebra associated with Q'. By [10, Thm. 6.6], there is an inclusion of Lie algebras

$$\mathfrak{n}_{Q'}^{-} \hookrightarrow \mathfrak{g}_{\bar{Q},\bar{W}}$$

as the part of the BPS Lie algebra lying in cohomological degree zero, so that (4.2) in case $\mu = 0$ allows us to reconstruct (one half of) Nakajima's action of $\mathfrak{g}_{Q'}$ on the cohomology of quiver varieties [34]. The conjecture suggests that the cohomology of Nakajima quiver varieties should be as crucial to the representation theory of partially fermionised BPS Lie algebras as they are to the representation theory of their bosonic counterparts.

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