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Short Geodesics of Unitaries in the L^2 Metric

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Abstract. Let $\mathcal{M}$ be a type II_1 von Neumann algebra, τ a trace in $\mathcal{M}$, and $L^2(\mathcal{M},\tau)$ the GNS Hilbert space of τ. We regard the unitary group $U_{\mathcal{M}}$ as a subset of $L^2(\mathcal{M},\tau)$ and characterize the shortest smooth curves joining two fixed unitaries in the L^2 metric. As a consequence of this we obtain that $U_{\mathcal{M}}$, though a complete (metric) topological group, is not an embedded riemannian submanifold of $L^2(\mathcal{M},\tau)$

1 Introduction

Let $\mathcal{M}$ be a type II_1 von Neumann algebra with a faithful and normal tracial state τ. Let $L^2(\mathcal{M},\tau)$ be the Hilbert space obtained by completion of $\mathcal{M}$ with the norm $\|x\|_2 = \tau(x^*x)^{1/2}$. Denote by $U_{\mathcal{M}}$ the group of unitaries of $\mathcal{M}$. Then $U_{\mathcal{M}}$, as a subset of $L^2(\mathcal{M},\tau)$, is a complete metric space and a topological group. The Hilbert space norm induces on $U_{\mathcal{M}}$ the strong operator topology. These are well-known facts (see [10]). In a previous note [1], we showed that $U_{\mathcal{M}}$ cannot be embedded as a differentiable submanifold in a way which makes the product of unitaries a differentiable map. Here we show that the same question, dropping the requirement for the product, again has a negative answer: $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ is not an embedded riemannian submanifold.

Hence, it makes sense to study the following: are there curves of unitaries of $\mathcal{M}$ which have minimal length measured in the L^2 metric? We measure the length of a curve of unitaries in the following way: let $\mu(t)$ be a curve in $U_{\mathcal{M}}$, with $\mu(0) = v$ and $\mu(1) = u$, which is piecewise C^1 as a curve in $L^2(\mathcal{M},\tau)$, then the length of μ is

$$\ell(\mu) = \int_0^1 \|\dot{\mu}(t)\|_2 \, dt,$$

where, as is the usual notation, $\|x\|_2 = \tau(x^*x)^{1/2}$. The usual norm of $\mathcal{M}$ is denoted by $\| \ \|$.

Suppose that we fix u and v. Is there a shortest curve joining u and v inside $U_{\mathcal{M}}$? We obtain the following answer (Theorem 3.4):

There exists $x = x^* \in \mathcal{M}$ with $\|x\| \leq \pi$ such that $v^*u = e^{ix}$. The curve

$$\delta(t) = ve^{itx}$$

has minimal length among piecewise C^1 curves of unitaries joining u and v.

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1. *If $\|x\| < \pi$, then such x is uniquely determined and the curve δ is unique among piecewise C^∞ minimizing curves.*
2. *Otherwise ($\|x\| = \pi$), δ is non unique. Other minimizing piecewise C^2 curves are of the form $\gamma(t) = ve^{itL_\xi}$, with $\xi = J\xi \in L^4(\mathcal{M}, \tau)$.*

In both cases, the shortest (piecewise C^1) curve has length $\|x\|_2$.

The first condition defines a set of unitaries, namely:

$$\{u \in U_\mathcal{M} : v^*u = e^{ix} \text{ for } x^* = x \text{ with } \|x\| < \pi\},$$

which is an open neighbourhood of v in the norm topology, but not in the *strong operator* topology. In [7] Popa and Takesaki found what E. Michael [6] calls a geodesic structure for the unitary group of certain type II_1 factors. Such a structure has strong topological implications, leading for example to a complete elucidation of the homotopy type of the unitary group for such factors, in the strong operator topology. We wanted to know if the naive "geodesic" curves, of the form $\delta(t) = ve^{itx}$, could be used to obtain a geodesic structure for all type II_1 von Neumann algebras in the strong operator topology, as is the case in the norm topology for arbitrary C^*-algebras [2]. The result above proves that one cannot.

We call these curves δ geodesics, because they are the geodesics of a covariant derivative defined in $U_\mathcal{M}$ in a natural way. If $U_\mathcal{M}$ were an embedded submanifold of $L^2(\mathcal{M}, \tau)$, this covariant derivative would be the Levi–Civita derivative. Therefore the result above also shows that $U_\mathcal{M}$ is not a submanifold of $L^2(\mathcal{M}, \tau)$.

This study was inspired by the paper by Durán, Mata-Lorenzo and Recht [4] which studied minimal curves of projections for the p-norms.

2 Geodesics in $U_\mathcal{M}$

Let us first define the tangent spaces of $U_\mathcal{M}$ in the L^2 topology. Let $J\colon L^2(\mathcal{M}, \tau) \to L^2(\mathcal{M}, \tau)$ be the involution, *i.e.,* the extension to $L^2(\mathcal{M}, \tau)$ of the usual involution $*$ of $\mathcal{M}$. Clearly $J^2 = I$. Let $L^2(\mathcal{M}, \tau)_+ = \{\xi \in L^2(\mathcal{M}, \tau) : J\xi = \xi\}$ and $L^2(\mathcal{M}, \tau)_- = \{\xi \in L^2(\mathcal{M}, \tau) : J\xi = -\xi\}$, which are *real* Hilbert spaces. $L^2(\mathcal{M}, \tau)_-$ is the completion in the L^2 norm of the set of antihermitian elements of $\mathcal{M}$ ($x^* = -x$), which is the tangent space of $U_\mathcal{M}$ at the identity 1 in the norm topology. Let us postulate $T(U_\mathcal{M})_1 := L^2(\mathcal{M}, \tau)_-$. For $u \in U_\mathcal{M}$, the map $L_u\colon L^2(\mathcal{M}, \tau) \to L^2(\mathcal{M}, \tau)$, defined on $\mathcal{M} \subset L^2(\mathcal{M}, \tau)$ as $L_u(x) = ux$ (*i.e.,* the GNS representation of u as an operator in $L^2(\mathcal{M}, \tau)$) is a unitary operator. Then we choose $T(U_\mathcal{M})_u = L_u(L^2(\mathcal{M}, \tau)_-)$. Also, right multiplication $R_u(x) = xu$ extends to a unitary operator in $L^2(\mathcal{M}, \tau)$. For brevity, we shall write $u\xi$ and $u(L^2(\mathcal{M}, \tau))$ (resp. ξu and $(L^2(\mathcal{M}, \tau)_-)u$) instead of $L_u\xi$ and $L_u(L^2(\mathcal{M}, \tau)_-)$ (resp. $R_u(\xi)$ and $R_u(L^2(\mathcal{M}, \tau)_-)$).

Let μ be a curve of unitaries which is C^1 as a curve in the Hilbert space $L^2(\mathcal{M}, \tau)$, and let X be a differentiable vector field in a neighbourhood of $\{\mu(t) : t \in [0, 1]\}$, which takes values in $TU_\mathcal{M}$ when restricted to $U_\mathcal{M}$, *i.e.,* $X_{\mu(t)} \in \mu(t)L^2(\mathcal{M}, \tau)_-$. For obvious reasons, such a field will be called a *tangent* vector field along μ. The covariant derivative of X along μ is given by:

$$\frac{DX}{dt} = \frac{1}{2}\{\dot{X} - \mu J(\dot{X})\mu\},$$

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where $\dot{X}$ denotes the usual derivative with respect to t in the Hilbert space $L^2(\mathcal{M},\tau)$. This formula is obtained simply by projecting $\dot{X}$ orthogonally (with respect to the inner product given by the real part of τ) onto $T(U_{\mathcal{M}})_\mu$. Note that if $\mu(t)$ is a C^2 curve in $U_{\mathcal{M}}$, then $\dot{\mu}$ is a tangent vector field along μ as usual. In particular, μ is a geodesic if

$$0 \equiv \frac{D\dot{\mu}}{dt}$$

or equivalently

$$\ddot{\mu} = \mu J(\dot{\mu})\mu. \tag{1}$$

It is straightforward to verify that if $x \in \mathcal{M}$ with $x^* = x$, and $v \in U_{\mathcal{M}}$, then $\mu(t) = ve^{itx}$ is a C^∞ curve with $\dot{\mu}(t) = ivxe^{itx}$.

There are other exponentials which give curves in $U_{\mathcal{M}}$. If $\xi \in L^2(\mathcal{M},\tau)_+$, then ξ induces a possibly unbounded selfadjoint operator L_ξ on $L^2(\mathcal{M},\tau)$, affiliated to $\mathcal{M}$ (see [3, 9]). Namely, L_ξ is the closure of the linear map $L_\xi\colon \mathcal{M} \subset L^2(\mathcal{M},\tau) \to L^2(\mathcal{M},\tau)$ given by $L_\xi(m) = Jm^*J\xi$. Therefore $\mu(t) = e^{itL_\xi}$ is a continuous curve in the L^2 topology, which is differentiable in $L^2(\mathcal{M},\tau)$. Indeed, the topological embedding $U_{\mathcal{M}} \subset L^2(\mathcal{M},\tau)$ can be regarded as evaluation at the vector $1 \in L^2(\mathcal{M},\tau)$. Strictly speaking, one should write $\mu(t) = e^{itL_\xi}1$. Since 1 lies in the domain of the operator L_ξ [9], by Stone's theorem $\mu(t)$ can be differentiated, and the derivative equals (see [8])

$$\dot{\mu}(t) = ie^{itL_\xi}\xi.$$

However, this curve $\dot{\mu}(t)$ cannot be differentiated again (in $L^2(\mathcal{M},\tau)$) if ξ^2 does not belong to $L^2(\mathcal{M},\tau)$. It could be differentiated in $L^1(\mathcal{M},\tau)$. Clearly it is not in general a C^∞ curve of $L^2(\mathcal{M},\tau)$.

Lemma 2.1 *Let $\xi \in L^2(\mathcal{M},\tau)_+$, then the curve $\mu(t) = e^{itL_\xi}$ is C^∞ if and only if L_ξ is bounded,* i.e., $\xi \in \mathcal{M}$.

Proof The "if" part is clear. Suppose that μ has derivatives of any order. This implies that all the powers L_ξ^k, $k \geq 1$ lie in $L^2(\mathcal{M},\tau)$. Denote by m the probability measure on $\mathbb{R}$ given by the trace of the spectral measure of L_ξ. Then

$$\infty > \|L_\xi^k 1\|_2^2 = \int_{\mathbb{R}} \lambda^{2k}\, dm(\lambda), \quad \text{for all } k \geq 1.$$

The above statement means that the map $\mathbb{R} \to \mathbb{R}$, $\lambda \mapsto \lambda$ lies in $L^\infty(\mathbb{R}, m)$, *i.e.,* m has support contained in a bounded interval $[-K, K]$. This implies that L_ξ is bounded by K, and therefore lies in $\mathcal{M}$. ∎

Note that if ξ lies in $L^2(\mathcal{M},\tau)$ but not in $L^4(\mathcal{M},\tau)$, then $\mu(t) = ve^{itL_\xi}$ is C^1 but not C^2, *etc.* Indeed, $\dot{\mu}(t) = iL_\xi e^{itL_\xi}$ is continuous in the L^2 norm: if $t \to t_0$, then

$$\|\dot{\mu}(t) - \dot{\mu}(t_0)\|_2 = \|e^{i(t-t_0)L_\xi}\xi - \xi\|_2 \to 0.$$

Let us call a C^2 curve a *geodesic* in $U_{\mathcal{M}}$ if it is a solution of the differential equation (1).

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Proposition 2.2 *The C^∞ geodesics in $U_{\mathcal{M}}$ are of the form $\delta(t) = ve^{itx}$, for $x^* = x \in \mathcal{M}$.*

Proof First note that if $x^* = x$, then $\delta(t) = ve^{itx}$ satisfies (1). Let μ be a C^∞ curve in $L^2(\mathcal{M}, \tau)$ with values in $U_{\mathcal{M}}$, which is a solution of (1), parametrized in the interval $[0, 1]$, with $\mu(0) = v$. Let $i\xi = \dot\mu(0)$ and $\xi' = \ddot\mu(0)$, which lie in $L^2(\mathcal{M}, \tau)$ because μ is C^∞.

If ν is a solution of (1), then $v^*\nu$ is another solution. Since $J(v^*\ddot\nu) = J(\ddot\nu)v$,

$$v^*\nu J(v^*\ddot\nu)v^*\nu = v^*\nu J(\ddot\nu)\nu = v^*\ddot\nu = v^*\ddot\nu.$$

Therefore we may suppose $v = 1$ without loss of generality.

Differentiating the identity $\mu(t)\mu^*(t) = 1$, one obtains (we omit the parameter t)

$$\dot\mu\mu^* + \mu J(\dot\mu) = 0$$

($\dot\mu$ may lie outside $\mathcal{M}$, so we find more appropriate to write $J(\dot\mu)$ instead of $\dot\mu^*$). Differentiating again,

$$\ddot\mu\mu^* + 2\dot\mu J(\dot\mu) + \mu J(\ddot\mu) = 0.$$

At $t = 0$ one obtains the relations

$$i\xi + J(i\xi) = 0, \quad \text{i.e. } \xi \in L^2(\mathcal{M}, \tau)_+$$

and

$$2\xi' + 2i\xi J(i\xi) = 0, \quad \text{i.e. } \xi' = -\xi J(\xi) = -\xi^2.$$

Consider the curve $\gamma(t) = e^{itL_\xi}$. Then $\dot\gamma(t) = ie^{itL_\xi}\xi$ and $\ddot\gamma(t) = e^{itL_\xi}\xi'$. Therefore γ is C^2 ($\xi' \in L^2(\mathcal{M}, \tau)$), and the relations above show that it is a solution of (1), satisfying

$$\dot\gamma(0) = i\xi = \dot\mu(0) \text{ and } \ddot\gamma(0) = \xi' = \ddot\mu(0).$$

We claim that these facts imply that $\mu = \gamma$. To prove this claim, one needs a result on uniqueness of solutions of second order differential equations on Banach spaces. Let us first obtain a new form for equation (1). Consider again the identity $\ddot\mu\mu^* + 2\dot\mu J(\dot\mu) + \mu J(\ddot\mu) = 0$ and multiply it on the right by μ

$$\ddot\mu + 2\dot\mu J(\dot\mu)\mu + \mu J(\ddot\mu)\mu = 0.$$

Then the identity (1) $\ddot\mu = \mu J(\ddot\mu)\mu$, replaced above gives

$$\ddot\mu = -\dot\mu J(\dot\mu)\mu, \tag{2}$$

which we shall adopt. We need a Banach space on which this equation will be considered. Our $L^2(\mathcal{M}, \tau)$ is not appropriate, since the right-hand side of the equation does not make sense for arbitrary $\mu(t)$ with derivatives in $L^2(\mathcal{M}, \tau)$, because $\dot\mu J(\dot\mu)$ may lie outside $L^2(\mathcal{M}, \tau)$. We are not worried about existence—we already know the solutions—we need a uniqueness result. Let us consider $L^4(\mathcal{M},\tau)$. The map $L^4(\mathcal{M},\tau) \to L^2(\mathcal{M},\tau)$, $\xi \mapsto \xi J(\xi)$ is differentiable. It follows that the function

$$F(x,\xi) = -\xi J(\xi) x$$

with variables $x \in \mathcal{M}$ and $\xi \in L^4(\mathcal{M},\tau)$ and values in $L^2(\mathcal{M},\tau)$, satisfies a Lipschitz condition. Therefore the differential equation (2), $\ddot{\mu} = F(\mu,\dot{\mu})$ has unique local solutions for any given set of initial conditions. Note that any solution μ of (2) should satisfy $\dot{\mu} \in L^4(\mathcal{M},\tau)$ anyway.

Therefore $\mu(t) = e^{itL_\xi}$. The fact that μ is C^∞ implies, by the lemma above, that $\xi = x$ is a selfadjoint element of $\mathcal{M}$. ■

Remark 2.3 The same argument can be used to prove that the C^2 geodesics are of the form $\delta(t) = ve^{itL_\xi}$, with $\xi \in L^4(\mathcal{M},\tau)$.

Our next result is borrowed and adapted from [4]. There it is stated for variations of geodesics of the grassmannian manifold (*i.e.,* manifold of selfadjoint projections) of a C^*-algebra with trace. Also, there the p-length functionals are considered (induced by the p-norms $\|x\|_p = \tau((x^*x)^{p/2})^{1/p}$), for $p = 2n$. We are interested only in the case $p = 2$. Our exposition in the rest of this section follows [4] with slight modifications. We want to compute the extremals of the functional

$$\ell(\mu) = \int_0^1 \|\dot{\mu}(t)\|_2 \, dt.$$

Let $U(t,s)\colon [0,1] \times (-\epsilon,\epsilon) \to U_{\mathcal{M}}$ be a variation of a curve $\mu\colon [0,1] \to U_{\mathcal{M}}$, with fixed endpoints, *i.e.,*

$$U(t,0) = \mu(t) \quad \text{for all } t \in [0,1],$$

and

$$U(0,s) = \mu(0), \quad U(1,s) = \mu(1) \quad \text{for all } s \in [0,1].$$

The variation is through piecewise C^2 curves, *i.e.,* for each fixed s, the curve $U(t,s)$ is piecewise C^2 in the parameter t, and vice versa. Denote by $\delta\ell(s)$ the *variation*

$$\delta\ell(s) = \frac{\partial}{\partial s} \int_0^1 \Big\| \frac{\partial U}{\partial t} \Big\|_2 \, dt.$$

The extremals of ℓ are the curves μ such that $\delta\ell(0) = 0$ for any $U(t,s)$ as above. Denote $V = \frac{\partial U}{\partial t}$ and $W = \frac{\partial U}{\partial s}$. Let us compute

$$\delta\ell(s) = \frac{\partial}{\partial s} \int_0^1 \Big\| \frac{\partial U}{\partial t} \Big\|_2 \, dt = \int_0^1 \frac{\partial}{\partial s} \tau\Big(J\Big(\frac{\partial U}{\partial t}\Big) \frac{\partial U}{\partial t}\Big)^{1/2} dt.$$

An easy computation shows that if $\xi(s) \neq 0$ is differentiable in $L^2(\mathcal{M},\tau)$, then

$$\frac{d}{ds}\tau\big(J(\xi(s))\xi(s)\big)^{1/2} = \frac{1}{2\|\xi(s)\|_2} \tau\Big(J\Big(\frac{dx(s)}{ds}\Big) x(s) + J(x(s))\frac{dx(s)}{ds}\Big).$$

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In our case this gives

$$\delta\ell(s) = \int_0^1 \frac{1}{2\|V\|_2}\tau\Big(\Big[\frac{\partial}{\partial s}J(V)\Big]V + J(V)\frac{\partial}{\partial s}V\Big)\,dt.$$

We shall assume that the curve μ is parametrized by a multiple of arc length. In other words, $\|V\|_2$ is constant for $s = 0$. One should make the further assumption that V does not vanish for all s, t, in order that the above expression makes sense. Let us point out that at the final stages of this computation we put $s = 0$. Therefore it suffices to have that $V(t,s)$ does not vanish for all t and small s (which is attained if we suppose μ with constant speed).

Since U is (piecewise) C^2 we may interchange

$$\frac{\partial}{\partial s}V = \frac{\partial}{\partial s}\Big(\frac{\partial U}{\partial t}\Big) = \frac{\partial}{\partial t}\Big(\frac{\partial U}{\partial s}\Big) = \frac{\partial}{\partial t}W.$$

Therefore the variation formula equals

$$\frac{1}{2}\int_0^1 \tau\Big(J\Big(\frac{\partial}{\partial t}W\Big)\frac{V}{\|V\|_2} + J\Big(\frac{V}{\|V\|_2}\Big)\frac{\partial}{\partial t}W\Big)\,dt.$$

Fix s, and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of $[0,1]$ such that $U(t,s)$ is C^2 in the interior of the smaller intervals. We may integrate the above formula by parts in each interval $[t_{i-1}, t_i]$ to obtain

$$\begin{aligned}\frac{1}{2}\int_{t_{i-1}}^{t_i} \tau\Big(J\Big(\frac{\partial}{\partial t}W\Big)\frac{V}{\|V\|_2} + J\Big(\frac{V}{\|V\|_2}\Big)\frac{\partial}{\partial t}W\Big)\,dt &= \\ \frac{1}{2}\Big\{\tau\Big(J(W)\frac{V}{\|V\|_2} + WJ\Big(\frac{V}{\|V\|_2}\Big)\Big)\Big\}\Big|_{t_{i-1}}^{t_i}& \\ -\frac{1}{2}\int_{t_{i-1}}^{t_i}\tau\Big(J(W)\frac{\partial}{\partial t}\Big(\frac{V}{\|V\|_2}\Big) + W\frac{\partial}{\partial t}J\Big(\frac{V}{\|V\|_2}\Big)\Big)\,dt.&\end{aligned}$$

Recall from the beginning of this section the definition of the covariant derivative of a tangent vector field X along a curve μ of unitaries:

$$\frac{DX}{dt} = \frac{1}{2}\{\dot{X} - \mu J(\dot{X})\mu\}.$$

In our case, for each fixed s, the field $\frac{V}{\|V\|_2}$ is tangent along the curve $U(t,s)$, so we have

$$\frac{D}{dt}\frac{V}{\|V\|_2} = \frac{1}{2}\Big\{\frac{\partial}{\partial t}\frac{V}{\|V\|_2} - UJ\Big(\frac{\partial}{\partial t}\frac{V}{\|V\|_2}\Big)U\Big\}.$$

Now we differentiate the identity $U^*U = 1$ with respect to t. It was pointed out in the introduction that the product of unitaries is not a differentiable map of the arguments in the L^2 topology. However a product $u(t)v(t)$ of C^2 curves of unitaries

$u(t)$ and $v(t)$ can be differentiated twice with respect to t. Indeed, the first derivative yields $\dot{u}v + u\dot{v}$. Since u and v are C^2, the norms $\|\dot{v}(t)\|_2$ and $\|\dot{u}(t)\|_2$ are uniformly bounded, and the second derivative can be computed. In our case, the derivative of the identity $U^*U = 1$ gives

$$V = -UJ(V)U,$$

i.e.,

$$\frac{V}{\|V\|_2} = -UJ\Big(\frac{V}{\|V\|_2}\Big)U.$$

Before computing the second derivative we put $s = 0$

$$\frac{\dot{\mu}}{\|\dot{\mu}\|_2} = -\mu J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\mu.$$

Differentiating this expression with respect to t (recall that we assume that μ is parametrized proportionally to arc length, *i.e.*, $\|\dot{\mu}\|_2$ is constant)

$$\frac{\partial}{\partial t}\frac{\dot{\mu}}{\|\dot{\mu}\|_2} = -\dot{\mu}J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\mu - \mu J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\dot{\mu} - \mu J\Big(\frac{\partial}{\partial t}\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\mu.$$

Combining these one obtains

$$2\frac{\partial}{\partial t}\frac{\dot{\mu}}{\|\dot{\mu}\|_2} = 2\frac{D}{dt}\frac{\dot{\mu}}{\|\dot{\mu}\|_2} - \frac{\dot{\mu}J(\dot{\mu})}{\|\dot{\mu}\|_2}\mu - \mu\frac{J(\dot{\mu})\dot{\mu}}{\|\dot{\mu}\|_2},$$

with an analogous expression for $2J\big(\frac{\partial}{\partial t}\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\big)$. We add the integrals over the intervals $[t_{i-1}, t_i]$, and use these relations to obtain,

$$\begin{aligned}\delta\ell(s) = \frac{1}{2}\sum_{1=1}^{n}\Big\{\tau\Big(J(W)\frac{\dot{\mu}}{\|\dot{\mu}\|_2} + WJ\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\Big)\Big\}\Big|_{t_{i-1}}^{t_i} \\ + \frac{1}{2}\int_0^1 \tau\Big(J(W)(\mu\dot{\mu}J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big) - 2J(W)\frac{D}{dt}\frac{\dot{\mu}}{\|\dot{\mu}\|_2} \\ + W(\mu^*\dot{\mu}J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big) + J\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\dot{\mu}\mu^*\Big) - 2J\Big(\frac{D}{dt}\frac{\dot{\mu}}{\|\dot{\mu}\|_2}\Big)\Big)\,dt.\end{aligned}$$

We can deal better with this expression if we relate it to the second differential of the map $x \mapsto \tau(x^*x)$, which is the (real) bilinear form

$$H\colon L^2(\mathcal{M},\tau)\times L^2(\mathcal{M},\tau)\to\mathbb{R}, \quad H(\xi,\eta) = \tau(\xi J(\eta) + J(\xi)\eta).$$

Then the expression for the variation of ℓ becomes

$$\begin{aligned}\delta\ell(0) = \frac{1}{2}\sum_{i=1}^{n} H\Big(\frac{\dot{\mu}}{\|\dot{\mu}\|_2}, W\Big)\Big|_{t_{i-1}}^{t_i} \\ + \int_0^1 H\Big(\mu^*W, \frac{1}{2\|\dot{\mu}\|_2}(J(\dot{\mu})\dot{\mu} - \dot{\mu}J(\dot{\mu}))\Big) - H\Big(\frac{D}{dt}\frac{\dot{\mu}}{\|\dot{\mu}\|_2}, W\Big)\,dt.\end{aligned}$$

A fact used here is that the field W satisfies relations analogous as V, *i.e.*, $U^*W = -J(W)U$. A remark is in order. The element $\dot\mu J(\dot\mu)$ (resp. $\dot\mu J(\dot\mu)$) lies in $L^2(\mathcal{M},\tau)$. This is a consequence of μ being (piecewise) C^2, namely, its second derivatives, which involve such terms, lie in $L^2(\mathcal{M},\tau)$.

Note that $\frac{1}{\|\dot\mu\|_2}(J(\dot\mu)\dot\mu - \dot\mu J(\dot\mu))$ lies in $L^2(\mathcal{M},\tau)_+$ (is "hermitian") and μ^*W lies in $L^2(\mathcal{M},\tau)_-$ ("antihermitian"). Indeed, the latter has just been remarked. The former holds because $\dot\mu$ can be approximated by elements x of $\mathcal{M}$, and therefore $J(\dot\mu)\dot\mu - \dot\mu J(\dot\mu)$ can be approximated by $x^*x - xx^*$. Now if $\xi \in L^2(\mathcal{M},\tau)_-$ and $\eta \in L^2(\mathcal{M},\tau)_+$, it is clear that $H(\xi,\eta) = 0$. Therefore we arrive at our final expression for the variation

$$\delta\ell(0) = -\frac{1}{2}\sum_{i=1}^{n} H\Big(\frac{\dot\mu}{\|\dot\mu\|_2}, W\Big)\Big|_{t_{i-1}}^{t_i} - \int_0^1 H\Big(\frac{D}{dt}\frac{\dot\mu}{\|\dot\mu\|_2}, W\Big)\,dt. \tag{3}$$

Let us transcribe Theorem 3.3 by Durán, Mata-Lorenzo and Recht [4], which applies to our situation, with minor adaptations, once we have (3) analogous to their expression for the variation.

If a piecewise C^2 curve μ has minimal length among all the piecewise C^2 curves of unitaries joining the same endpoints, then clearly $\delta\ell(0)$ vanishes for any variation U of μ. As is standard use, let us call a curve for which all variations make $\delta\ell(0)$ vanish, an extremal of ℓ.

Theorem 2.4 *The extremals of ℓ (among piecewise C^2-curves) are precisely the geodesics of $U_{\mathcal{M}}$.*

Proof Clearly a geodesic is an extremal of ℓ. Suppose now that μ is a piecewise C^2 curve of unitaries. The converse is proven as in [4], by means of the following facts:

1. If μ is an extremal of ℓ, then for all $t \in [0,1]$ and every vector field W along μ

$$H\Big(W(t), \frac{D}{dt}\frac{\dot\mu(t)}{\|\dot\mu(t)\|_2}\Big) = 0.$$

2. If μ is an extremal of ℓ, then μ is C^2.
3. If μ is C^2 and satisfies that for any vector field W along μ

$$H\Big(W(t), \frac{D}{dt}\frac{\dot\mu(t)}{\|\dot\mu(t)\|_2}\Big) = 0$$

then μ is a geodesic.

For the first assertion, suppose that for some t_0 (a point where μ is C^2) one has

$$H\Big(W(t_0), \frac{D}{dt}\frac{\dot\mu(t_0)}{\|\dot\mu(t_0)\|_2}\Big) > 0$$

for some variation U. Let us consider another variation

$$\tilde U(t,s) = U(t, \varphi(t)s),$$

where φ is a scalar function satisfying

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1. $0 \le \varphi(t) \le 1$, with $\varphi(0) = 1$ and $\varphi(1) = 1$.
2. $\varphi(t_0) = 1$ and φ vanishes on small intervals around the points $t_1, \dots, t_n$ where the derivative of μ is not continuous.

Note that $\tilde{U}(t, 0) = U(t, 0) = \mu(t)$. Also the first condition above implies that $\tilde{U}(0, s) = U(s, 0) = \mu(0)$ and $\tilde{U}(1, s) = U(1, s) = \mu(1)$. In other words, $\tilde{U}$ is another variation of μ with fixed endpoints. Moreover

$$\tilde{W}(t, s) = \frac{\partial \tilde{U}}{\partial s} = \frac{\partial U}{\partial s}(t, \varphi(t)s) = \varphi(t)W(t, \varphi(t)s),$$

and therefore $\tilde{W}(t) = \tilde{W}(t, 0) = \varphi(t)W(t)$. Note that since $\varphi(t_0) = 1$,

$$H\Big(\frac{D}{dt}\frac{\dot{\mu}(t_0)}{\|\dot{\mu}(t_0)\|_2}, \tilde{W}(t_0)\Big) > 0.$$

We can further choose φ in order that

$$H\Big(\frac{D}{dt}\frac{\dot{\mu}(t)}{\|\dot{\mu}(t)\|_2}, \tilde{W}(t)\Big) = \varphi(t)H\Big(\frac{D}{dt}\frac{\dot{\mu}(t)}{\|\dot{\mu}(t)\|_2}, W(t)\Big) \ge 0.$$

Since $\tilde{W}(t) = \varphi(t)W(t)$ vanishes at the points $t_1, \dots, t_n$, it follows that for $\tilde{U}$ the variation is

$$\delta\ell(0) = -\frac{1}{2}\int_0^1 H\Big(\frac{D}{dt}\frac{\dot{\mu}(t)}{\|\dot{\mu}(t)\|_2}, \tilde{W}(t)\Big)\, dt > 0,$$

and therefore μ is not an extremal.

To prove the second assertion, suppose that μ is an extremal of ℓ, and that t_0 is a point where $\dot{\mu}$ is not continuous. Denote by V_0^+ and V_0^- the lateral limits of $\frac{D}{dt}\frac{\dot{\mu}(t)}{\|\dot{\mu}(t)\|_2}$ at $t = t_0$. Note that V_0^+ and V_0^- are unit vectors. Put

$$U(t, s) = e^{is\varphi(t)V_0^+},$$

where $\varphi(t)$ is a smooth scalar function, which satisfies that $0 \le \varphi(t) \le 1$, $\varphi(t_0) = 1$ and φ vanishes on the other points where $\dot{\mu}$ is not continuous. By the first assertion, the integral term in the expression of the variation of μ vanishes. Moreover, by the choice of φ, one has

$$\delta\ell(0) = H(W(t_0), V_0^+) - H(W(t_0), V_0^-) = H(V_0^+, V_0^+) - H(V_0^+, V_0^-).$$

Now

$$H(V_0^+, V_0^+) = \tau(V_0^+ J(V_0^+) + J(V_0^+)V_0^+) = 2\|V_0^+\|_2^2 = 2.$$

On the other hand, the fact that $\frac{\dot{\mu}(t)}{\|\dot{\mu}(t)\|_2}$ has a jump at $t = t_0$ implies that the unit vectors V_0^+ and V_0^- do not point in the same direction, *i.e.*, the Cauchy–Schwarz inequality is strict:

$$\tau(V_0^+ J(V_0^-)) < \|V_0^+\|_2 \|V_0^-\|_2 = 1,$$

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and analogously $\tau(J(V_0^+)V_0^-) < 1$. It follows that

$$\delta\ell(0) > 0$$

for this U, and μ is not an extremal.

The third assertion is straightforward. Since in our case, the form H is nondegenerate, the identity

$$H\Big(W(t), \frac{D}{dt}\frac{\dot\mu(t)}{\|\dot\mu(t)\|_2}\Big) = 0$$

for any field W implies that

$$\frac{D}{dt}\frac{\dot\mu(t)}{\|\dot\mu(t)\|_2} = 0$$

i.e., μ is a geodesic. ■

3 Short Curves

The key to our main result is the following:

Lemma 3.1 *Let x be a selfadjoint element of $\mathcal{M}$ with finite spectrum and $\|x\| < \pi$. Then $\delta(t) = e^{itx}$ has minimal length amongst piecewise C^1 curves joining 1 and e^{ix}, in the L^2 metric.*

Proof The element x is of the form $x = \sum_{i=1}^k \alpha_i p_i$, where $p_1, \dots, p_k$ are pairwise orthogonal projections and $\alpha_1, \dots, \alpha_k$ are real numbers with $|\alpha_i| < \pi$. The length of the geodesic δ is $\|x\|_2 = (\sum_{i=1}^k \alpha_i^2 r_i)^{1/2}$, where $r_i = \tau(p_i)$. Suppose that μ is another piecewise C^1 curve of unitaries with $\mu(0) = 1$ and $\mu(1) = e^{ix}$. Then

$$\ell(\mu) = \int_0^1 \big(\tau(J(\dot\mu)\dot\mu)\big)^{1/2}\, dt = \int_0^1 \Big(\sum_{i=1}^k \tau(p_i J(\dot\mu)\dot\mu p_i)\Big)^{1/2} dt.$$

For each $1 \le i \le k$ denote by $S_{r_i^{1/2}}$ the sphere of radius $r_i^{1/2}$ in $L^2(\mathcal{M}, \tau)$,

$$S_{r_i^{1/2}} = \{\xi \in L^2(\mathcal{M}, \tau) : \langle \xi, \xi\rangle = r_i\}.$$

Note that the curves $p_i\delta$ and $p_i\mu$ are curves in $S_{r_i^{1/2}}$. Indeed, for example

$$\langle p_i\mu, p_i\mu\rangle = \tau((p_i\mu)^* p_i\mu) = \tau(p_i) = r_i.$$

Moreover, $p_i\delta$ is a geodesic of $S_{r_i^{1/2}}$ with length strictly less than $\pi r_i^{1/2}$. An elementary spectral argument shows that

$$p_i\delta(t) = p_i e^{itx} = p_i e^{it\alpha_i},$$

which is clearly a geodesic of the sphere $S_{r_i^{1/2}}$. The length of $p_i\delta$ is

$$\ell(p_i\delta) = \|\alpha_i p_i\|_2 = |\alpha_i| r_i^{1/2} < r_i^{1/2}\pi.$$

In other words, $p_i\delta$ is the shortest curve in $S_{r_i^{1/2}}$ joining its endpoints.

Consider the riemannian submanifold of $L^2(\mathcal{M}, \tau)^k$

$$\mathcal{S} = S_{r_1^{1/2}} \times \cdots \times S_{r_k^{1/2}}$$

with its Levi–Civita connection. The curve $\Delta(t) = (p_1\delta(t), \ldots, p_k\delta(t))$ is a geodesic of $\mathcal{S}$, since it is a k-tuple of geodesics of the coordinates. Moreover, it is the shortest curve of $\mathcal{S}$ joining its endpoints. Indeed, none of its coordinates could be replaced by a shorter curve. Therefore it is shorter than the curve $M(t) = (p_1\mu(t), \ldots, p_k\mu(t))$. Now the length of M in $\mathcal{S}$ is measured as follows:

$$\int_0^1 \langle \dot{M}(t), \dot{M}(t)\rangle^{1/2}\, dt = \int_0^1 \Big(\sum_{i=1}^k \tau(p_i J(\dot{\mu}(t))\dot{\mu}(t)) \Big)^{1/2} dt = \ell(\mu).$$

Analogously, the length of Δ coincides with $\ell(\delta)$. It follows that

$$\ell(\mu) \geq \ell(\delta). \qquad \blacksquare$$

Lemma 3.2 *Let $x \in \mathcal{M}$ be a selfadjoint element with $\|x\| < \pi$, and $v \in U_{\mathcal{M}}$. Then the geodesic $\delta(t) = ve^{itx}$ has minimal length among piecewise C^1 curves of unitaries joining its endpoints. It is unique among piecewise C^∞ curves with this property.*

Proof There is no loss in generality if we suppose $v = 1$. Indeed, for any curve μ of unitaries, $\ell(\mu) = \ell(v^*\mu)$. Suppose that there exists a piecewise C^1 curve of unitaries μ which is strictly shorter than δ, $\ell(\mu) < \ell(\delta) - \epsilon = \|x\|_2 - \epsilon$. The element x can be approximated in the norm topology of $\mathcal{M}$ by selfadjoint elements of $\mathcal{M}$, say z, with finite spectrum and the following conditions:

1. $\|z\| \leq \|x\| < \pi$.
2. $\|x\|_2 - \epsilon/2 < \|z\|_2 \leq \|x\|_2$.
3. $\|e^{ix} - e^{iz}\| < 2$.
4. There exists a C^∞ curve of unitaries joining e^{ix} and e^{iz} of length less than $\epsilon/2$.

The first three are clear. The fourth condition can be obtained as follows. By the third condition $e^{-ix}e^{iz} = e^{iy}$, with $y^* = y \in \mathcal{M}$. Moreover z can be adjusted so as to obtain y of arbitrarily small norm. Then the curve of unitaries $\gamma(t) = e^{ix}e^{ity}$ is C^∞, joins e^{ix} and e^{iz}, with length $\|y\|_2 \leq \|y\| < \epsilon/2$.

Consider now the curve μ', which is the curve μ followed by the curve $e^{ix}e^{ity}$ above. Then clearly

$$\ell(\mu') \leq \ell(\mu) + \|y\|_2 < \ell(\mu) + \epsilon/2.$$

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Therefore $\ell(\mu') < \|x\|_2 - \epsilon/2$. On the other hand, since μ' joins 1 and e^{iz}, by the lemma above, it must have length greater than or equal to $\|z\|_2$. It follows that

$$\|z\|_2 \leq \|x\|_2 - \epsilon/2,$$

a contradiction.

Let us now show that δ is unique. Let δ' be another piecewise C^∞ curve joining the same endpoints, parametrized proportional to arc length, with $\ell(\delta) = \ell(\delta')$. The minimality of δ' implies, by Theorem 2.4, that it is a C^∞ geodesic. Then $\delta'(t) = e^{itx'}$ for some $x'^* = x' \in \mathcal{M}$. We claim that $x' = x$.

Since $\|x\| < \pi$, ix can be obtained as an analytic logarithm of $e^{ix} = e^{ix'}$. It follows that x and x' commute. Then $e^{i(x-x')} = 1$ and therefore $x - x'$ is a selfadjoint element with finite spectrum, contained in the discrete set $\{2n\pi : n \in \mathbb{Z}\}$. Then $x' = x + \sum_{i=1}^k 2n_i\pi p_i$ with $n_i \in \mathbb{Z}$ and p_i pairwise orthogonal projections in $\mathcal{M}$, $i = 1, \dots, k$. Note that $xp_i = 0$. Therefore

$$\|x'\|_2^2 = \|x\|_2^2 + \sum_{i=1}^k 4n_i^2\pi^2\tau(p_i).$$

Now, since $\|x\|_2 = \ell(\delta) = \ell(\delta') = \|x'\|_2$, it follows that $\tau(p_i) = 0$, for each $i = 1, \dots, k$, *i.e.*, $x = x'$. ■

Lemma 3.3 *Let x be a selfadjoint element of $\mathcal{M}$ with $\|x\| = \pi$. Then $\delta = ve^{itx}$ is the shortest curve joining its endpoints.*

Proof The proof is the same as the first part of the above lemma, approximating x with z of finite spectrum and $\|z\| < \pi$. Note that any unitary $u \in U_{\mathcal{M}}$ is of the form $u = e^{ix}$ with $x^* = x$ and $\|x\| \leq \pi$. This element x is non unique. ■

We may summarize these lemmas in our main result.

Theorem 3.4 *Let u, v be unitaries in $\mathcal{M}$, and $x = x^* \in \mathcal{M}$ with $\|x\| \leq \pi$, such that $v^*u = e^{ix}$.*

1. *If $\|x\| < \pi$, then there exists a geodesic joining u and v, which has minimal length among piecewise C^1 curves with these endpoints. It is unique with this property among piecewise C^∞ curves.*
2. *If $\|x\| = \pi$, there exist many minimal C^∞ geodesics joining u and v.*

Remark 3.5 In case 2, the multiple C^∞ geodesics are of the form $\delta(t) = ve^{itx}$ for diverse $x = x^* \in \mathcal{M}$ with $\|x\| = \pi$ such that $v^*u = e^{ix}$. If one only requires that the curves be C^2, other minimizing curves appear. Namely, by Remark 2.3 they are of the form $\gamma(t) = ve^{itL_\xi}$, where ξ lies in $L^4(\mathcal{M}, \tau)$, and satisfies $J\xi = \xi$ and $v^*u = e^{iL_\xi}$.

The following corollary might be obtained in a more straightforward way.

Corollary 3.6 *Let $x, y \in \mathcal{M}$ be selfadjoint elements of norm less than or equal to π such that $e^{ix} = e^{iy}$. Then $\tau(x^2) = \tau(y^2)$.*

Proof Both $\delta(t) = e^{itx}$ and $\gamma(t) = e^{ity}$ are minimizing geodesics joining 1 and e^{ix}, therefore $\ell(\delta) = \ell(\gamma)$, *i.e.*, $\tau(x^2) = \tau(y^2)$. ■

4 Non Embeddability of $U_{\mathcal{M}}$ in $L^2(\mathcal{M}, \tau)$

In this section we show that $U_{\mathcal{M}}$ is not a riemannian submanifold of $L^2(\mathcal{M}, \tau)$. By this we mean that $U_{\mathcal{M}}$ is not a riemannian manifold with the inner product of $L^2(\mathcal{M}, \tau)$ at each tangent space. We also consider other aspects of the local structure of $U_{\mathcal{M}}$.

Lemma 4.1 *There exists a sequence of selfadjoint elements $a_n \in \mathcal{M}$ such that $\|a_n\|_2 = \epsilon$ for a given $\epsilon > 0$ and $\|e^{ia_n} - 1\|_2$ tends to zero.*

Proof For each $n \geq 1$ pick a projection p_n in $\mathcal{M}$ such that $\tau(p_n) = \frac{\epsilon^2}{n^2}$. Put $a_n = np_n$. Note that $\|a_n\|_2 = n\tau(p_n)^{1/2} = \epsilon$. On the other hand

$$\|e^{ia_n} - 1\|_2^2 = 2 - \tau(e^{ia_n}) - \tau(e^{-ia_n}).$$

Clearly

$$\tau(e^{ia_n}) = 1 + \frac{\epsilon^2}{n^2}(e^{in} - 1),$$

which tends to 1. Analogously for $\tau(e^{-ia_n})$. ■

Corollary 4.2 *$U_{\mathcal{M}}$ is not a riemannian submanifold of $L^2(\mathcal{M}, \tau)$.*

Proof Consider $u_n = e^{ia_n} \in U_{\mathcal{M}}$ as above. Then the sequence u_n tends to 1 in the L^2 metric. If $U_{\mathcal{M}}$ were a riemannian submanifold, then $\delta_n(t) = e^{ita_n}$ would be a geodesic. If one adjusts ϵ smaller than the radius of a normal neighbourhood around $1 \in U_{\mathcal{M}}$, then δ_n would be a minimizing geodesic. It follows that the geodesic distance between 1 and e^{ia_n} equals ϵ for all n. This leads to contradiction: in a riemannian manifold the topology given by the geodesic distance and the underlying topology are equivalent. ■

Note that δ_n above is in fact not a minimizing geodesic, according to our discussion of the previous section. Indeed, $\|a_n\| = n$. If one tries to compute minimizing geodesics joining 1 and e^{ia_n}, one must replace the exponent $a_n = np_n$ by $x_n = (n - 2k_n\pi)p_n$, where k_n is an integer such that $|n - 2k_n\pi| \leq \pi$ (in this case it will be strictly smaller than π). Such x_n satisfy

$$\|x_n\|_2^2 = (n - 2k_n\pi)^2 \frac{\epsilon^2}{n^2} \to 0 \text{ as } n \to \infty.$$

In other words, these minimizing geodesics have lengths which tend to 0.

Let us denote by d_g the geodesic distance in $U_{\mathcal{M}}$, *i.e.,*

$$d_g(u,v) = \inf\{\ell(\mu) : \mu \text{ piecewise } C^1 \text{ curve of unitaries with } \mu(0) = u, \mu(1) = v\}.$$

Since $U_{\mathcal{M}}$ is not a riemannian manifold, we must prove the following:

Proposition 4.3 *d_g is a metric in $U_{\mathcal{M}}$.*

Proof Clearly $d_g(u,v) \geq 0$ and $d_g(u,v) = 0$ imply $u = v$. Also it is clear that $d_g(u,v) = d_g(v,u)$. Let us verify that the triangle inequality holds. Let $u, v, w \in U_{\mathcal{M}}$. We need to show that

$$d_g(u,v) \leq d_g(u,w) + d_g(w,v).$$

By Theorem 3.4, u and w are joined by a minimizing geodesic δ and w and u are joined by a minimizing geodesic δ', with both curves realizing the geodesic distance. The curve δ followed by the curve δ' is a piecewise C^1 curve of unitaries joining u and v, with length $d_g(u,w) + d_g(w,v)$. Therefore $d_g(u,v) \leq d_g(u,w) + d_g(w,v)$. ■

Proposition 4.4 *The metrics d_g and $\| \ \|_2$ are equivalent in $U_{\mathcal{M}}$.*

Proof Both metrics are invariant by left translation with elements of $U_{\mathcal{M}}$, *i.e.,* $d_g(u,v) = d_g(v^*u, 1)$ and $\|u - v\|_2 = \|v^*u - 1\|_2$. Therefore it suffices to compare $d_g(u,1)$ and $\|u-1\|_2$, for $u \in U_{\mathcal{M}}$. Let $x = x^* \in \mathcal{M}$ with $\|x\| \leq \pi$ and $u = e^{ix}$. Then by Theorem 3.4

$$d_g(u,1) = \|x\|_2 = \tau(x^2)^{1/2}.$$

On the other hand

$$\|u-1\|_2^2 = 2 - \tau(e^{ix} + e^{-ix}) = 2\Big[\frac{\tau(x^2)}{2} - \frac{\tau(x^4)}{4!} + \frac{\tau(x^6)}{6!} - \cdots\Big].$$

Note that for all $n \geq 1$,

$$\frac{\tau(x^{2n})}{(2n)!} - \frac{\tau(x^{2n+2})}{(2n+2)!} \geq 0.$$

Indeed, it is apparent that this inequality is equivalent to $(2n+2)(2n+1) \geq \frac{\tau(x^{2n+2})}{\tau(x^{2n})}$. Since $x^2 \leq \pi^2$,

$$\frac{\tau(x^{2n+2})}{\tau(x^{2n})} = \frac{\tau(x^n x^2 x^n)}{\tau(x^{2n})} \leq \frac{\tau(x^n \pi^2 x^n)}{\tau(x^{2n})} = \pi^2,$$

and the above claim holds. First, note that with this inequality one has

$$\|u-1\|_2^2 = 2\Big[\frac{1}{2}\tau(x^2) - \Big(\frac{\tau(x^4)}{4!} - \frac{\tau(x^6)}{6!}\Big) - \cdots\Big] \leq \tau(x^2),$$

i.e., $\|u-1\|_2 \leq d_g(u,1)$.

On the other hand, the same inequality proves that

$$\|u-1\|_2^2 = 2\Big[\frac{1}{2}\tau(x^2) - \frac{1}{4!}\tau(x^4) + \Big(\frac{\tau(x^6)}{6!} - \frac{\tau(x^8)}{8!}\Big) + \cdots\Big] \geq 2\Big[\frac{1}{2}\tau(x^2) - \frac{1}{4!}\tau(x^4)\Big].$$

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Since $1-\frac{x^2}{12} \geq 1-\frac{\pi^2}{12} > 0$, it follows that

$$\frac{1}{2}\tau(x^2) - \frac{1}{4!}\tau(x^4) = \frac{1}{2}\tau(x^2(1-\frac{1}{12}x^2)) \geq \frac{1}{2}\left(1-\frac{\pi^2}{12}\right)\tau(x^2).$$

In other words,

$$\|u-1\|_2 \geq C d_g(u,1),$$

for $C = \sqrt{1-\frac{\pi^2}{12}}$. ∎

Further properties of this metric d_g will be studied elsewhere.

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