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Remarks on Naimark dilation theorem

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Abstract. Although Naimark dilation theorem was originally stated in 1940, it still finds many important applications in various areas. The objective of this paper is to introduce a method for explicitly constructing the vectors of complementary frames in the Naimark dilation theorem, specifically in cases where the initial Parseval frame contains a Riesz basis as a subset. These findings serve as a foundation for the construction of dual frames.

1 Introduction

Various versions of the Naimark dilation theorem have garnered steady interest in recent research, with applications in operator theory, signal processing, computer science, engineering, and quantum information theory (see, e.g., [5, 7, 9, 11, 12]). The original result, established by Naimark in 1940 for the case of a generalized resolution of identity [10], was analogized for a Parseval frame (PF) by Han and Larson in 2000 [5] and it states the following theorem.

Theorem 1.1 *Let $\mathscr{F}_e = \{e_j,\ j \in \mathbb{J}\}$ be a PF in a Hilbert space $\mathscr{K}$. Then there exist a Hilbert space $\mathscr{M}$ and a complementary PF $\mathscr{F}_m = \{m_j,\ j \in \mathbb{J}\}$ in $\mathscr{M}$ such that the set of vectors*

$$\mathscr{F}_{e\oplus m} = \{e_j \oplus m_j,\ j \in \mathbb{J}\} \tag{1.1}$$

is an orthonormal basis of $\mathscr{H} = \mathscr{K} \oplus \mathscr{M}$. The extension of $\mathscr{F}_e$ to an orthonormal basis $\mathscr{F}_{e\oplus m}$ described above is unique up to unitary equivalence.

Generalizations of the Naimark dilation theorem for frames and representation systems can be found in [4, 14]. It is also worth mentioning that a complementary PF $\mathscr{F}_m$ was described in [5] through the identification of $\mathscr{H}$ with $\ell_2(\mathbb{J})$. Such a description of $\mathscr{F}_m$ is not always adequate. Since Theorem 1.1 holds numerous significant applications, it becomes crucial to discover a relatively simple representation of the complementary frame $\mathscr{F}_m$ using the original PF $\mathscr{F}_e$. In this paper, we present a method for explicitly constructing the vectors of $\mathscr{F}_m$ in scenarios where the initial frame $\mathscr{F}_e$ includes a Riesz basis as a subset (Section 3). These results are subsequently utilized in the construction of dual frames in Section 4.

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Throughout the paper, all operators are assumed to be linear and bounded, $\mathcal{R}(A)$ and $\ker A$ denote the *range* and the *null-space* of an operator A, respectively, while $A|_{\mathcal{D}}$ stands for the restriction of A to the set $\mathcal{D}$. An operator A acting in a Hilbert space with scalar product $(\cdot,\cdot)$ is called *nonnegative* (*positive*) if $(Af,f)\geq 0$ ($(Af,f)>0,\ f\neq 0$).

Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be closed subspaces of Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. An operator of $\mathfrak{H}_1$ into $\mathfrak{H}_2$ that maps $\mathfrak{L}_1$ isometrically onto $\mathfrak{L}_2$ and annihilates $\mathfrak{H}_1\ominus\mathfrak{L}_1$ is called a *partial isometry*. Then $\mathfrak{L}_1$ is called the *initial space*, and $\mathfrak{L}_2$ is called the *final space* of the partial isometry.

2 Preliminaries

Here, all necessary facts about frames and PFs are presented in a form convenient for our exposition. More details can found in [2, 3, 6].

Let $\mathcal{K}$ be a separable Hilbert space with scalar product $(\cdot,\cdot)$ linear in the first argument. Denote by $\mathbb{J}$ a generic countable (finite) index set and by $|\mathbb{J}|$ its cardinality.

PF is a family of vectors $\mathcal{F}_e=\{e_j,\ j\in\mathbb{J}\}$ in $\mathcal{K}$ for which

$$\|f\|^2=\sum_{j\in\mathbb{J}}|(f,e_j)|^2,\quad f\in\mathcal{K}.$$

The above equality is an analogue to the Parseval equality known for orthonormal bases. *A frame* is a family of vectors $\mathcal{F}_\varphi=\{\varphi_j,\ j\in\mathbb{J}\}$ in $\mathcal{K}$ which satisfies

$$A\|f\|^2\leq\sum_{j\in\mathbb{J}}|(f,\varphi_j)|^2\leq B\|f\|^2,\quad f\in\mathcal{K},$$

where $0<A\leq B$. PFs are frames with $A=B=1$. A family $\mathcal{F}_\varphi$ is called a *frame sequence* if $\mathcal{F}_\varphi$ is a frame for $\overline{\text{span}}\,\mathcal{F}_\varphi$.

Each frame $\mathcal{F}_\varphi$ determines a bounded mapping $\theta_\varphi:\mathcal{K}\to\ell_2(\mathbb{J})$

$$\theta_\varphi f=\{(f,\varphi_j)\}_{j\in\mathbb{J}},\qquad f\in\mathcal{K}, \tag{2.1}$$

which is called the *analysis operator*. By the construction, the image set $\mathcal{R}(\theta_\varphi)$ of θ_φ is a subspace of $\ell_2(\mathbb{J})$.

Following [2], we recall that the *excess* $\mathbf{e}[\mathcal{F}_\varphi]$ of a frame $\mathcal{F}_\varphi$ is the greatest integer n such that n elements can be deleted from the frame $\mathcal{F}_\varphi$ and still leave a complete set, or ∞ if there is no upper bound to the number of elements that can be removed. In view of [2, Lemma 4.1],

$$\mathbf{e}[\mathcal{F}_\varphi]=\dim[\ell_2(\mathbb{J})\ominus\mathcal{R}(\theta_\varphi)].$$

This means that

$$|\mathbb{J}|=\mathbf{e}[\mathcal{F}_\varphi]+\dim\mathcal{K}. \tag{2.2}$$

The excess $\mathbf{e}[\mathcal{F}_e]$ of a PF $\mathcal{F}_e$ coincides with the dimension of the complementary Hilbert space $\mathcal{M}$ in Theorem 1.1. The zero excess of a frame $\mathcal{F}_\varphi$ (of a PF $\mathcal{F}_e$) means that $\mathcal{F}_\varphi$ is a Riesz basis ($\mathcal{F}_e$ is an orthonormal basis) of $\mathcal{K}$.

For a frame $\mathcal{F}_\varphi$, the corresponding *frame operator* $Sf=\sum_{j\in\mathbb{J}}(f,\varphi_j)\varphi_j$ is bounded, positive, and invertible in $\mathcal{K}$. The frame operators for PFs coincide with the identity operator. If S is a frame operator for a frame (for a Riesz basis) $\mathscr{F}_\varphi$, then the set $\{S^{-1/2}\varphi_j, j \in \mathbb{J}\}$ is a PF (is an orthonormal basis) of $\mathscr{K}$.

3 The main results

Assume that a PF $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ contains a Riesz basis $\mathscr{F}_e^0 = \{e_j, j \in \mathbb{J}_0\}$ ($\mathbb{J}_0 \subset \mathbb{J}$) as a subset and let S_0 be the frame operator of $\mathscr{F}_e^0$.

The operator $I - S_0$ is nonnegative in $\mathscr{K}$ since

(3.1) $$((I-S_0)f, f) = \|f\|^2 - \sum_{j\in\mathbb{J}_0} |(f, e_j)|^2 = \sum_{j\in\mathbb{J}_1} |(f, e_j)|^2 \geq 0, \quad f \in \mathscr{K}.$$

Denote $\mathscr{M}_1 = \overline{\text{span}}\ \mathscr{F}_e^1$, where $\mathscr{F}_e^1 = \{e_j, j \in \mathbb{J}_1\}$ and $\mathbb{J}_1 = \mathbb{J} \setminus \mathbb{J}_0$. The relation (3.1) implies that

(3.2) $$\ker(I - S_0) = \mathscr{K} \ominus \mathscr{M}_1.$$

Hence, $\mathscr{M}_1$ coincides with $\overline{\mathscr{R}(I - S_0)}$ and it is a reducing subspace for $I - S_0$. Denote by $(I - S_0)|_{\mathscr{M}_1}$ the restriction of $I - S_0$ onto $\mathscr{M}_1$. The operator $(I - S_0)|_{\mathscr{M}_1}$ is a positive self-adjoint operator acting in $\mathscr{M}_1$. Therefore, the inverse operator

$$((I - S_0)|_{\mathscr{M}_1})^{-1} : \mathscr{M}_1 \to \mathscr{M}_1$$

exists.

Lemma 3.1 *The following are equivalent:*

(i) *The range $\mathscr{R}(I - S_0)$ is a closed set.*
(ii) *The inverse operator $((I - S_0)|_{\mathscr{M}_1})^{-1}$ is bounded.*
(iii) *The family $\mathscr{F}_e^1$ is a frame sequence.*

Proof Items (i) and (ii) are equivalent due to the inverse mapping theorem [6, p. 75].

(iii) → (ii). The relation

(3.3) $$(I - S_0)f = \sum_{j\in\mathbb{J}} (f, e_j)e_j - \sum_{j\in\mathbb{J}_0} (f, e_j)e_j = \sum_{j\in\mathbb{J}_1} (f, e_j)e_j$$

implies that $(I - S_0)|_{\mathscr{M}_1}$ is a frame operator of the frame $\mathscr{F}_e^1$ in the Hilbert space $\mathscr{M}_1 = \overline{\mathscr{R}(I - S_0)}$. Hence, the inverse operator $((I - S_0)|_{\mathscr{M}_1})^{-1}$ is bounded.

(ii) → (iii). Since $(I - S_0)|_{\mathscr{M}_1}$ is positive, there exists $((I - S_0)_{\mathscr{M}_1})^{1/2}$ and, for all $f \in \mathscr{M}_1$,

$$\sum_{j\in\mathbb{J}_1} |(f, e_j)|^2 = ((I - S_0)f, f) \leq \|((I - S_0)_{\mathscr{M}_1})^{1/2}\|^2 \|f\|^2.$$

Similarly, taking into account that $((I - S_0)|_{\mathscr{M}_1})^{-1}$ is bounded, we get

$$\sum_{j\in\mathbb{J}_1} |(f, e_j)|^2 = \|((I - S_0)|_{\mathscr{M}_1})^{1/2} f\|^2 \geq \frac{1}{\|((I - S_0)|_{\mathscr{M}_1})^{-1/2}\|^2} \|f\|^2,$$

which completes the proof. ∎

Lemma 3.2 *Assume that the index set of a PF* $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ *can be decomposed* $\mathbb{J} = \mathbb{J}_0 \cup \mathbb{J}_1$ *in such a way that* $\mathscr{F}_e^0 = \{e_j, j \in \mathbb{J}_0\}$ *is a Riesz basis of* $\mathscr{K}$*, while* $\mathscr{F}_e^1 = \{e_j, j \in \mathbb{J}_1\}$ *is a frame sequence. Let* S_0 *be a frame operator of* $\mathscr{F}_e^0$*. Then the family of vectors*

$$\mathscr{F}_e^{ext} = \left\{ \begin{array}{ll} e_j \oplus (I - S_0)^{1/2} S_0^{-1/2} e_j, & j \in \mathbb{J}_0 \\ e_j \oplus -((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} e_j, & j \in \mathbb{J}_1 \end{array} \right\}, \tag{3.4}$$

is a PF of the Hilbert space $\mathscr{K} \oplus \mathscr{M}_1$*, where* $\mathscr{M}_1 = \overline{\text{span}}\mathscr{F}_e^1 = \mathscr{R}(I - S_0)$*. The excess* $\mathbf{e}[\mathscr{F}_e^{ext}]$ *satisfies the relation* $|\mathbb{J}_1| = \mathbf{e}[\mathscr{F}_e^{ext}] + \dim \mathscr{M}_1$*.*

Proof By virtue of Lemma 3.1, $\mathscr{M}_1 = \overline{\text{span}}\mathscr{F}_e^1 = \mathscr{R}(I - S_0)$.

Since S_0 is a frame operator of $\mathscr{F}_e^0$, the operators $S^{1/2}$ and $S_0^{-1/2}$ exist and are bounded in $\mathscr{K}$. Furthermore, $I - S_0$ is nonnegative in $\mathscr{K}$ and, therefore, there exists $(I - S_0)^{1/2}$. This means that $(I - S_0)^{1/2} S_0^{-1/2}$ is a well-defined operator in $\mathscr{K}$ and the vectors

$$\{(I - S_0)^{1/2} S_0^{-1/2} e_j, \ j \in \mathbb{J}_0\}$$

belong to $\mathscr{M}_1$.

Similarly, in view of Lemma 3.1, there exists the bounded operator $((I - S_0)|_{\mathscr{M}_1})^{-1/2}$. This means that the operator $((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} : \mathscr{M}_1 \to \mathscr{M}_1$ is well defined and the vectors $\{((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} e_j, \ j \in \mathbb{J}_1\}$ belong to $\mathscr{M}_1$. Hence, the right-hand part of (3.4) is well defined.

Denote by $\mathscr{L}_0$ and $\mathscr{L}_1$ the subspaces of $\mathscr{K} \oplus \mathscr{M}_1$ generated by the vectors

$$\{l_j^0 = e_j \oplus (I - S_0)^{1/2} S_0^{-1/2} e_j, \ j \in \mathbb{J}_0\},$$

$$\{l_j^1 = e_j \oplus -((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} e_j, \ j \in \mathbb{J}_1\},$$

respectively. The subspaces $\mathscr{L}_0$ and $\mathscr{L}_1$ are orthogonal since

$$\begin{aligned}(l_j^0, l_i^1) &= (e_j, e_i) - ((I - S_0)^{1/2} S_0^{-1/2} e_j, ((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} e_i) = \\ &= (e_j, e_i) - (e_j, e_i) = 0, \quad j \in \mathbb{J}_0, \quad i \in \mathbb{J}_1.\end{aligned}$$

Assume that $h = k \oplus m \in \mathscr{K} \oplus \mathscr{M}_1$ is orthogonal to $\mathscr{L}_0 \oplus \mathscr{L}_1$. Then, for every $l_j^0 \in \mathscr{L}_0$,

$$0 = (l_j^0, h) = (e_j, k) + ((I - S_0)^{1/2} S_0^{-1/2} e_j, m) = (e_j, k + S_0^{-1/2} (I - S_0)^{1/2} m).$$

Hence, $k = -S_0^{-1/2} (I - S_0)^{1/2} m$ (since $\{e_j, j \in \mathbb{J}_0\}$ is a complete set in $\mathscr{K}$). The last relation means that $k \in \mathscr{M}_1$ and $m = -((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} k$. Therefore, the vector

$$h = k \oplus m = k \oplus -((I - S_0)|_{\mathscr{M}_1})^{-1/2} S_0^{1/2} k \tag{3.5}$$

belongs to $\mathscr{L}_1$ and, simultaneously, h is orthogonal to $\mathscr{L}_1$. This means that $h = 0$ and $\mathscr{L}_0 \oplus \mathscr{L}_1 = \mathscr{K} \oplus \mathscr{M}_1$.

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Further, we analyze the sets $\{l_j^0, j \in \mathbb{J}_0\}$ and $\{l_j^1, j \in \mathbb{J}_1\}$ in detail. Since $\{S_0^{-1/2}e_j, j \in \mathbb{J}_0\}$ is an orthonormal basis of $\mathscr{K}$,

$$(l_j^0, l_i^0) = (e_j, e_i) + ((I - S_0)S_0^{-1/2}e_j, S_0^{-1/2}e_i) = (S_0^{-1/2}e_j, S_0^{-1/2}e_i) = \delta_{ji},$$

for $j, i \in \mathbb{J}_0$. Therefore, $\{l_j^0, j \in \mathbb{J}_0\}$ is an orthonormal basis of $\mathscr{L}_0$.

On the other hand, the family $\{l_j^1, j \in \mathbb{J}_1\}$ is a PF in $\mathscr{L}_1$. Indeed, each vector $h \in \mathscr{L}_1$ has the form (3.5), where $k \in \mathscr{M}_1$. Hence,

$$(h, l_j^1) = (k, e_j) + ((I - S_0)|_{\mathscr{M}_1})^{-1/2}S_0^{1/2}k, ((I - S_0)|_{\mathscr{M}_1})^{-1/2}S_0^{1/2}e_j) =$$

$$= (k, [I + S_0^{1/2}((I - S_0)|_{\mathscr{M}_1})^{-1}S_0^{1/2}]e_j) = (k, [I + S_0((I - S_0)|_{\mathscr{M}_1})^{-1}]e_j) =$$

$$= (k, ((I - S_0)|_{\mathscr{M}_1})^{-1}e_j) = (((I - S_0)|_{\mathscr{M}_1})^{-1/2}k, ((I - S_0)|_{\mathscr{M}_1})^{-1/2}e_j).$$

Since $\{((I - S_0)|_{\mathscr{M}_1})^{-1/2}e_j, j \in \mathbb{J}_1\}$ is a PF for $\mathscr{M}_1$, we get

$$\sum_{j\in\mathbb{J}_1} |(h, l_j^1)|^2 = \sum_{j\in\mathbb{J}_1} |(((I - S_0)|_{\mathscr{M}_1})^{-1/2}k, ((I - S_0)|_{\mathscr{M}_1})^{-1/2}e_j)|^2 =$$

$$= \|((I - S_0)|_{\mathscr{M}_1})^{-1/2}k\|^2 = \|h\|^2,$$

for all $h \in \mathscr{L}_1$ (see (3.5)). Hence, $\{l_j^1\}$ is a PF of the Hilbert space $\mathscr{L}_1$.

Summing up the above results: the set $\mathscr{F}_e^{ext}$ defined by (3.4) consists of the orthonormal basis $\{l_j^0, j \in \mathbb{J}_0\}$ of $\mathscr{L}_0$ and the PF $\{l_j^1, j \in \mathbb{J}_1\}$ of $\mathscr{L}_1$. Here, $\mathscr{L}_0$ and $\mathscr{L}_1$ are orthogonal subspaces of $\mathscr{K} \oplus \mathscr{M}_1$ and $\mathscr{L}_0 \oplus \mathscr{L}_1 = \mathscr{K} \oplus \mathscr{M}_1$. This means that $\mathscr{F}_e^{ext}$ is a PF in the Hilbert space $\mathscr{K} \oplus \mathscr{M}_1$ and its excess $\mathbf{e}[\mathscr{F}_e^{ext}]$ coincides with the excess of the PF $\{l_j^1, j \in \mathbb{J}_1\}$ in $\mathscr{L}_1$. Using (2.2) with $\mathscr{K} = \mathscr{L}_1$, $\mathbb{J} = \mathbb{J}_1$, $\mathscr{F}_\varphi = \{l_j^1, j \in \mathbb{J}_1\}$ and taking into account that $\dim \mathscr{L}_1 = \dim \mathscr{M}_1$ by the definition of $\mathscr{M}_1$, we obtain $|\mathbb{J}_1| = \mathbf{e}[\{l_j^1\}] + \dim \mathscr{M}_1 = \mathbf{e}[\mathscr{F}_e^{ext}] + \dim \mathscr{M}_1$ that completes the proof. ■

Remark 3.3 A similar result for a particular case of finite excess was proved in [9] by other methods.

Theorem 3.4 *Assume that the assumptions of Lemma 3.2 are satisfied and, additionally, $\mathscr{F}_e^1$ is a basis of $\mathscr{M}_1$. Then the Hilbert space $\mathscr{M}$ in the Naimark dilation theorem can be chosen as $\mathscr{M}_1$, the PF $\mathscr{F}_e^{ext}$ in (3.4) coincides with the orthonormal basis $\mathscr{F}_{e\oplus m}$ of $\mathscr{H} = \mathscr{K} \oplus \mathscr{M}$, and vectors of the complementary PF $\mathscr{F}_m = \{m_j, j \in \mathbb{J}\}$ are defined as follows:*

$$m_j = \begin{cases} (I - S_0)^{1/2}S_0^{-1/2}e_j, & j \in \mathbb{J}_0, \\ -((I - S_0)|_{\mathscr{M}_1})^{-1/2}S_0^{1/2}e_j, & j \in \mathbb{J}_1. \end{cases} \tag{3.6}$$

Proof If $\mathscr{F}_e^1 = \{e_j, j \in \mathbb{J}_1\}$ is a basis of $\mathscr{M}_1$, then the PF $\{l_j^1\}_{j\in\mathbb{J}_1}$ of $\mathscr{L}_1$ turns out be an orthonormal basis of the Hilbert space $\mathscr{L}_1$. In this case, the PF $\mathscr{F}_e^{ext}$ in (3.4) is an orthonormal basis of $\mathscr{K} \oplus \mathscr{M}_1$. Setting $\mathscr{M} = \mathscr{M}_1$ in Theorem 1.1, we complete the proof. ■

Assume now that a PF $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ has a finite excess $\mathbf{e}[\mathscr{F}_e]$. Then $\mathscr{F}_e$ contains a Riesz basis [8] and assumptions of Lemma 3.2 are satisfied. By virtue of Lemma 3.2, $|\mathbb{J}_1| = \mathbf{e}[\mathscr{F}_e^{ext}] + \dim \mathscr{M}_1$. Moreover, it follows from definition of excess that $|\mathbb{J}_1| = \mathbf{e}[\mathscr{F}_e]$. Therefore,

$$\mathbf{e}[\mathscr{F}_e] = \mathbf{e}[\mathscr{F}_e^{ext}] + \dim \mathscr{M}_1.$$

This means that $0 \le \mathbf{e}[\mathscr{F}_e^{ext}] \le \mathbf{e}[\mathscr{F}_e]$. Let us consider two edge cases:

(i) If $\mathbf{e}[\mathscr{F}_e^{ext}] = 0$, then a PF $\mathscr{F}_e^{ext}$ in $\mathscr{K} \oplus \mathscr{M}_1$ turns to be an orthonormal basis.

(ii) If $\mathbf{e}[\mathscr{F}_e] = \mathbf{e}[\mathscr{F}_e^{ext}]$, then $\dim \mathscr{M}_1 = 0$. In this case, $\mathscr{F}_e$ coincides with $\mathscr{F}_e^{ext}$ and it has a trivial structure: the orthonormal basis $\{e_j, j \in \mathbb{J}_0\}$ of $\mathscr{K}$ and the zero part $\{e_j = 0, j \in \mathbb{J}_1\}$. The complementary PF $\mathscr{F}_m$ in Theorem 1.1 is constructed in a trivial manner: $m_j = 0$ for $j \in \mathbb{J}_0$ and $\{m_j, j \in \mathbb{J}_1\}$ is an arbitrary orthonormal basis in a Hilbert space $\mathscr{M}$ with $\dim \mathscr{M} = |\mathbb{J}_1|$.

It is important that if $0 < \mathbf{e}[\mathscr{F}_e^{ext}] < \mathbf{e}[\mathscr{F}_e]$, then applying Lemma 3.2 finite times, we obtain one of the previous cases (i) or (ii).

Summing up: for each PF with finite excess, Lemma 3.2 allows one to determine a complementary PF $\mathscr{F}_m$ in the Naimark dilation theorem.

4 Construction of dual frames

A frame $\mathscr{F}_\psi = \{\psi_j, j \in \mathbb{J}\}$ is called a *dual frame* for a PF $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ if

$$f = \sum_{j \in \mathbb{J}} (f, e_j)\psi_j = \sum_{j \in \mathbb{J}} (f, \psi_j) e_j, \qquad f \in \mathscr{K}.$$

There are multiple techniques for constructing dual frames, as outlined in [3]. In particular, the method proposed in [9] uses the concept of a complementary PF $\mathscr{F}_m = \{m_j, j \in \mathbb{J}\}$ from the Naimark dilation theorem. The next statement refines the results derived in [9].

Theorem 4.1 *Each dual frame $\mathscr{F}_\psi = \{\psi_j, j \in \mathbb{J}\}$ of a PF $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ consists of the elements*

(4.1) $$\psi_j = e_j + (S - I)^{1/2}\Omega m_j, \qquad j \in \mathbb{J},$$

where m_j are the elements of a Hilbert space $\mathscr{M}$ that form the PF $\mathscr{F}_m$ in the Naimark dilation theorem, Ω is a partial isometry of $\mathscr{M}$ into the Hilbert space $\mathscr{K}$ with final space $\overline{\mathscr{R}(S - I)}$, and a self-adjoint operator S in $\mathscr{K}$ satisfies the conditions

(4.2) $$S - I \ge 0, \qquad \dim \mathscr{R}(S - I) \le \dim \mathscr{M}.$$

Proof An arbitrary frame $\mathscr{F}_\psi$ in $\mathscr{K}$ can be represented as follows (see, e.g., [9, Proposition 1]):

(4.3) $$\mathscr{F}_\psi = S^{1/2}\mathscr{F}_{e'},$$

where S is a positive self-adjoint operator with bounded inverse (the frame operator of $\mathscr{F}_\psi$) and $\mathscr{F}_{e'}$ is a PF in $\mathscr{K}$. The operator S and the PF $\mathscr{F}_{e'}$ are determined uniquely by the frame $\mathscr{F}_\psi$. If $\mathscr{F}_\psi$ is dual for $\mathscr{F}_e$, one should specify S and $\mathscr{F}_{e'}$. In particular, as was shown in [9, Theorem 6], the operator S must satisfy the additional conditions:

$$S - I \geq 0 \quad \text{and} \quad \dim \mathscr{R}(S - I) \leq \mathbf{e}[\mathscr{F}_e] = \dim \mathscr{M}. \tag{4.4}$$

Let us analyze $\mathscr{F}_{e'}$ in (4.3) assuming that $\mathscr{F}_\psi$ is a dual frame for $\mathscr{F}_e$. First of all, we note that the excess of $\mathscr{F}_{e'}$ coincides with $\mathbf{e}[\mathscr{F}_e]$ (it follows from [1, Theorem 2.2] and [9, Lemma 3]). This means that the complementary Hilbert spaces $\mathscr{M}$ and $\mathscr{M}'$ for PFs $\mathscr{F}_e$ and $\mathscr{F}_{e'}$ in the Naimark dilation theorem have the same dimension and, therefore, these spaces can be identified.

Denote by $\mathscr{F}_{e\oplus m} = \{e_j \oplus m_j,\ j \in \mathbb{J}\}, \mathscr{F}_{e'\oplus m'} = \{e' j \oplus m' {}_j,\ j \in \mathbb{J}\}$ the corresponding orthonormal bases of $\mathscr{H} = \mathscr{K} \oplus \mathscr{M}$ and consider the unitary operator W in $\mathscr{H}$ acting on $\mathscr{F}_{e\oplus m}$ as $W : e \oplus m \to e' \oplus m'$. It follows from the definition of W that

$$\mathscr{F}_{e'} = P\{e'_j \oplus m'_j,\ j \in \mathbb{J}\} = PW\{e_j \oplus m_j,\ j \in \mathbb{J}\}, \tag{4.5}$$

where P is an orthogonal projection in $\mathscr{H}$ onto $\mathscr{K}$.

With respect to the decomposition $\mathscr{H} = \mathscr{K} \oplus \mathscr{M}$, the operator W admits the matrix presentation

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \tag{4.6}$$

where $W_{11} : \mathscr{K} \to \mathscr{K}$, $W_{22} : \mathscr{M} \to \mathscr{M}$, $W_{21} : \mathscr{K} \to \mathscr{M}$, and $W_{12} : \mathscr{M} \to \mathscr{K}$. By virtue of (4.5) and (4.6),

$$\mathscr{F}_{e'} = \{W_{11}e_j + W_{12}m_j,\ j \in \mathbb{J}\}. \tag{4.7}$$

In other words: if $\mathscr{F}_\psi$ in (4.3) is a dual frame, then S satisfies (4.2) and the corresponding PF $\mathscr{F}_{e'}$ coincides with (4.7), where W_{11} and W_{12} are parts of (4.6).

Since $\mathscr{F}_\psi$ defined by (4.3) is dual for $\mathscr{F}_e$, for all $f \in \mathscr{K}$,

$$f = \sum_{j\in\mathbb{J}} (f, e_j)\psi_j = S^{1/2} \sum_{j\in\mathbb{J}} (f, e_j)e'_j = S^{1/2} PW \sum_{j\in\mathbb{J}} (f, h_j)h_j = S^{1/2} W_{11} f,$$

where $\{h_j = e_j \oplus m_j\}$ is an orthonormal basis of $\mathscr{H}$. Hence, $W_{11} = S^{-1/2}$. In this case, the unitarity of W and (4.6) imply that $W_{21}W_{21}^* = I - W_{11}^* W_{11} = I - S^{-1}$ and $W_{12}W_{12}^* = I - W_{11}W_{11}^* = I - S^{-1}$. Hence, the polar decomposition of operators $W_{21}, W_{12}^* : \mathscr{K} \to \mathscr{M}$ have the form

$$W_{21} = U_2(I - S^{-1})^{1/2}, \qquad W_{12}^* = U_1(I - S^{-1})^{1/2},$$

where $U_1, U_2 : \mathscr{K} \to \mathscr{M}$ are partial isometries with initial space[1] $\overline{\mathscr{R}(I - S^{-1})} = \overline{\mathscr{R}(S - I)}$ [13, Theorem 7.2].

A simple analysis other relations between counterparts W_{ij} of the unitary operator W leads to the conclusion that $W_{22} = -U_2 S^{-1/2} U_1^*$. Therefore, the unitary operator W in (4.6) has the form

[1]Such kinds of isometries exist because $\dim \mathscr{R}(S - I) \leq \dim \mathscr{M}$.

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$$W = \begin{bmatrix} S^{-1/2} & (I - S^{-1})^{1/2} U_1^* \\ U_2 (I - S^{-1})^{1/2} & -U_2 S^{-1/2} U_1^* \end{bmatrix}, \tag{4.8}$$

where U_1^* is a partial isometry of $\mathscr{M}$ into $\mathscr{K}$ with final space $\overline{\mathscr{R}(S - I)}$.

By substituting the derived expressions of W_{11}, W_{12} into (4.5), one gets

$$\mathscr{F}_{e'} = \{S^{-1/2} e_j + (I - S^{-1})^{1/2} U_1^* m_j, \ j \in \mathbb{J}\}.$$

Recalling (4.3),

$$\mathscr{F}_\psi = S^{1/2} \mathscr{F}_{e'} = \{\psi_j = e_j + S^{1/2} (I - S^{-1})^{1/2} U_1^* m_j, \ j \in \mathbb{J}\}.$$

Denoting U_1^* as Ω, taking into account that $S^{1/2}(I - S^{-1})^{1/2} = (S - I)^{1/2}$, we derive (4.1). ∎

The operator S and the partial isometry Ω play a role of parameters describing the set of all dual frames $\mathscr{F}_\psi$ for $\mathscr{F}_e$. The parameter S coincides with the frame operator of $\mathscr{F}_\psi$. If S is given, then partial isometries Ω describe all possible dual frames having the same frame operator S.

The dual-frame formula (4.1) requires knowledge of the vectors m_j from the complementary PF $\mathscr{F}_m$. The results presented in Section 3 offer a solution to address this problem. Specifically, the following statement holds true.

Corollary 4.1 *Assume that the index set of a PF $\mathscr{F}_e = \{e_j, j \in \mathbb{J}\}$ can be decomposed $\mathbb{J} = \mathbb{J}_0 \cup \mathbb{J}_1$ in such a way that $\mathscr{F}_e^0 = \{e_j, j \in \mathbb{J}_0\}$ is a Riesz basis of $\mathscr{K}$ with the frame operator S_0 and $\mathscr{F}_e^1 = \{e_j, j \in \mathbb{J}_1\}$ is a Riesz basis of $\overline{\text{span}}\ \mathscr{F}_e^1$. Each dual frame $\mathscr{F}_\psi$ of $\mathscr{F}_e$ is described by the formula*[2]

$$\psi_j = \begin{cases} (I + (S - I)^{1/2} \Omega (S_0^{-1} - I)^{1/2}) e_j, & j \in \mathbb{J}_0, \\ (I - (S - I)^{1/2} \Omega (S_0^{-1} - I)^{-1/2}) e_j, & j \in \mathbb{J}_1, \end{cases}$$

where Ω is a partial isometry of $\mathscr{M} = \mathscr{R}(S_0 - I)$ into the Hilbert space $\mathscr{K}$ with final space $\overline{\mathscr{R}(S - I)}$ and a self-adjoint operator S in $\mathscr{K}$ satisfies the conditions

$$S - I \geq 0, \qquad \dim \mathscr{R}(S - I) \leq \dim \mathscr{R}(S_0 - I).$$

Proof Follows immediately from Corollary 3.2 and Theorems 3.4 and 4.1. ∎

For instance, if $S = S_0^{-1}$, then $\mathscr{R}(S - I) = \mathscr{R}(S_0 - I)$ and the partial isometry Ω turns out to be an unitary operator on $\mathscr{R}(S_0 - I)$. In this case, choosing $\Omega = I$, we obtain the dual frame

$$\mathscr{F}_\psi = \begin{Bmatrix} S_0^{-1} e_j, & j \in \mathbb{J}_0 \\ 0, & j \in \mathbb{J}_1 \end{Bmatrix}$$

that corresponds to the biorthogonal Riesz basis for $\mathscr{F}_e^0 = \{e_j, j \in \mathbb{J}_0\}$.

[2] for simplicity of notation, we consider $(S_0^{-1} - I)^{-1/2}$ as an operator on $\mathscr{M}$ i.e., $(S_0^{-1} - I)^{-1/2} e_j = (I - S_0)|_{\mathscr{M}})^{-1/2} S_0^{1/2}) e_j$ for $e_j \in \mathscr{M}$.

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