# ON NON-STATIONARY PROBLEMS OF CELESTIAL MECHANICS

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Some non-stationary problems of celestial mechanics can be described in an inertial system of right-angled coordinates $0x_1x_2x_3$ with gravitational potential of the form:

$$\mathcal{U}(x_1, x_2, x_3, t) = \frac{\gamma(t)}{\gamma_0}\gamma_0 \tilde{\mathcal{U}}(x_1, x_2, x_3) \quad , \tag{I}$$

where $\gamma(t)$ is a sufficiently arbitrary function of time and $\gamma_0$ is the meaning of $\gamma(t)$ in the initial epoch $t_0$. For example, in a two-body problem of variable mass $M(t)$ we have:

$$\mathcal{U} = -\frac{GM(t)}{r} \, , \qquad r^2 = x_1^2 + x_2^2 + x_3^2 \quad . \tag{2}$$

We can also remember a generalized problem of two immovable centres with the variable gravitational constant $G(t)$, when

$$\begin{aligned} \mathcal{U} &= -\frac{G(t)m}{2}\left[\frac{1+zi}{r_1} + \frac{1-zi}{r_2}\right] \quad , \\ i &= \sqrt{-1} \quad , \\ r_1^2 &= x_1^2 + x_2^2 + [x_3 - c(z+i)]^2 \quad , \\ r_2^2 &= x_1^2 + x_2^2 + [x_3 - c(z-i)]^2 \quad , \end{aligned} \tag{3}$$

where $m, z, c$ - some constants. It is of interest for analysis of effects of variable gravitation in an orbital motion of earth artificial satellites [I]. By a transformation of the time constant we can easily ascertain the following result: if the stationary problem with the potential $\gamma_0 \tilde{\mathcal{U}}(x_1, x_2, x_3)$ is integrated, then the following system of equations is also integrated:

$$\frac{d^2 x_i}{dt^2} = -\frac{\partial \mathcal{U}}{\partial x_i} + \frac{1}{2\gamma}\cdot\frac{d\gamma}{dt}\,\frac{dx_i}{dt} \, , \qquad (i = 1, 2, 3) \, . \tag{4}$$



*R. L. Duncombe (ed.), Dynamics of the Solar System, 49–52.*

When in the gravitational potential (I) the value $\gamma(t)$ is changed sufficiently slowly, the solution of this system can be considered as an unperturbed motion in a corresponding non-stationary problem. Specifically, for the gravitational potential (2) we have the aperiodic motion along a conic section, described by equation [2]

$$\frac{d^2x_i}{dt^2} = -\frac{G\mathcal{M}(t)}{r^3}x_i + \frac{1}{2\mathcal{M}}\cdot\frac{d\mathcal{M}}{dt}\frac{dx_i}{dt}, \quad (i=1,2,3). \tag{5}$$

With the purpose to use the well worked out canonical theory of perturbations we can try to construct Lagrangian of the non-conservative system (4). Multiply both parts of equations (4) by some function $R(x_1, x_2, x_3, t)$ :

$$R\ddot{x}_i = -R\left(\frac{\partial \mathcal{U}}{\partial x_i} - \frac{\dot{\gamma}}{2\gamma}\dot{x}_i\right), \qquad (i=1,2,3) \quad , \tag{6}$$

where the point above means a differentiation in time. Take $R$ so as the equalities to be identically fulfilled:

$$\begin{aligned} R\left(\ddot{x}_i + \frac{\partial \mathcal{U}}{\partial x_i} - \frac{\dot{\gamma}}{2\gamma}\dot{x}_i\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = \frac{\partial^2 L}{\partial \dot{x}_i \partial t} + \\ &+ \sum_{j=1}^{3}\left(\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\ddot{x}_j + \frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\dot{x}_j\right) - \frac{\partial L}{\partial x_i}\ ; \qquad (i=1,2,3)\ , \end{aligned} \tag{7}$$

where

$$L = R\left(\frac{1}{2}\sum \dot{x}_i^2 - \mathcal{U}\right). \tag{8}$$

Comparing in the expression (7) members, contaning the arbitraries of the same order, we shall receive:

$$R \equiv m(t) = \mu_0 \gamma^{-\frac{1}{2}} \quad , \qquad \mu_0 = const \ . \tag{9}$$

With due regard for expression (9) in the formula (6), we have the following system of equations:

$$m\frac{d^2x_i}{dt^2} = -m\frac{\partial \mathcal{U}}{\partial x_i} - \frac{dm}{dt}\cdot\frac{dx_i}{dt}\ , \quad (i=1,2,3), \tag{I0}$$

which is equivalent to the system (4) and has Lagrangian (8).

Describe the Lagrangian, which we have found, in a spherical coordinates:

$$L = m\left[\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2 + r^2\cos^2\varphi\,\dot{\lambda}^2\right) - U\right] \quad . \tag{I1}$$

Turn to Humilton's function:

$$H = \frac{1}{2m}\left(P_1^2 + P_2^2/r^2 + P_3^2/r^2\cos^2\varphi + 2m^2 U\right) \quad , \tag{I2}$$

where $P_1, P_2, P_3$ - are generalized impulses:

$$P_1 = \frac{\partial L}{\partial \dot{r}} \quad , \quad P_2 = \frac{\partial L}{\partial \dot{\varphi}} \quad , \quad P_3 = \frac{\partial L}{\partial \dot{\lambda}} \quad . \tag{I3}$$

The corresponding Humilton-Jakoby's equation has the form:

$$\frac{\partial \Psi}{\partial t} + \frac{1}{2m}\left[\left(\frac{\partial \Psi}{\partial t}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \Psi}{\partial \varphi}\right)^2 + \frac{1}{r^2\cos^2\varphi}\left(\frac{\partial \Psi}{\partial \lambda}\right)^2 + 2m^2 U\right] = 0 \tag{I4}$$

For the case of the system of equations (5) variable values $t, r, \varphi, \lambda$ in equation (I4) can be divided [3]. Suppose

$$\Psi = \Psi_0(t) + \Psi_1(r) + \Psi_2(\varphi) + \Psi_3(\lambda) \quad , \tag{I5}$$

we can find:

$$\begin{aligned}
\Psi_3 &= \alpha_3 \lambda \quad , \\
\Psi_2 &= \int_0^{\varphi} \sqrt{\alpha_2^2 - \frac{\alpha_3^2}{\cos^2\varphi}}\, d\varphi \quad , \\
\Psi_1 &= \int_{r_0}^{r} \sqrt{\frac{2G\mu_0}{r} - \frac{\alpha_2^2}{r^2} - 2\alpha}\, dr \quad , \\
\Psi_0 &= \alpha_1 \int_{t_0}^{t} \frac{dt}{m(t)} \equiv \alpha_1 F(t) \quad ,
\end{aligned} \tag{I6}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants of integration. The general solution of the system of equations (5), which have been received by this method, is anelogous to the solution of classical two-body problem, but with one exclusion: instead of time $t$ in the corresponding expression the function of time $F(t)$ takes part. This circumstance permits us to write at once the analogy of different systems of canonical elements [3].

## R E F E R E N C E S

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DISCUSSION

Szebehely: Is the unsteadiness always represented by the mass as a function of time?

Omarov: The mass and/or the constant of gravity may be functions of the time.

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